Syzygies among reduction operators

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Plan

I. Motivations

- ▷ Various notions of syzygy
- ▷ Computation of syzygies

II. Reduction operators

- ▷ Linear algebra, syzygies and useless reductions
- ▷ Reduction operators and labelled reductions

III. Lattice description of syzygies

- ▷ Lattice structure of reduction operators
- ▷ Construction of a basis of syzygies
- ▷ A lattice criterion for rejecting useless reductions

Plan

I. Motivations

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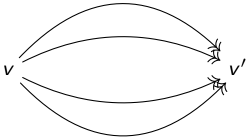

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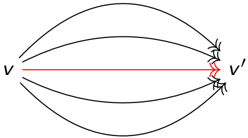

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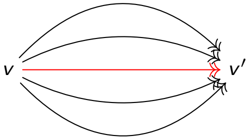

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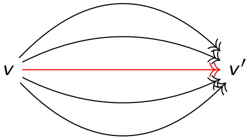

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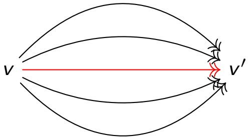

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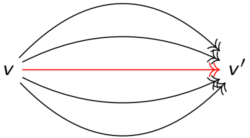

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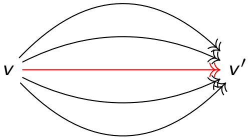

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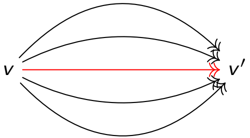

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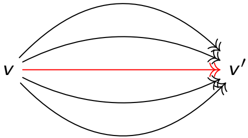

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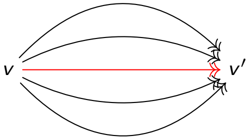

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  - Deduce a **lattice criterion** for rejecting **useless reductions** during the completion procedure.

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II. Reduction operators

Linear algebra, syzygies and useless reductions

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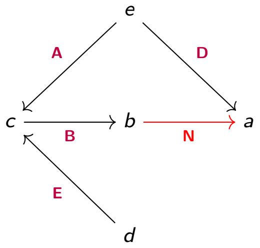

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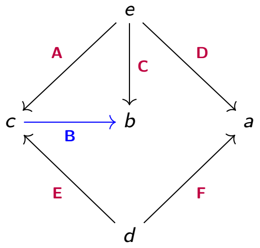

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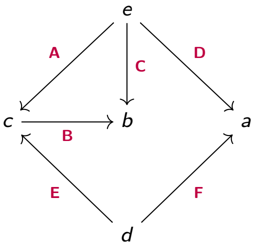

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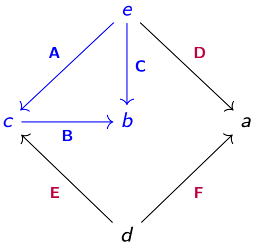

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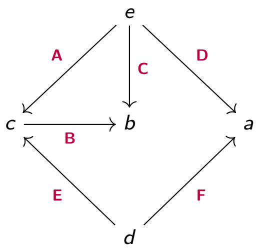

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Plan

III. Lattice description of syzygies

Definition of syzygies

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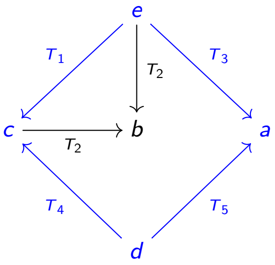

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Proposition i. *Let* $P = (T, T') \subset \mathbf{RO}(G,<)$. *We have a linear isomorphism*

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Proposition i. *Let* $P = (T,\ T') \subset \mathbf{RO}(G,<)$. *We have a linear isomorphism*

$$\ker\left(T \vee T'\right) \xrightarrow{\sim} \mathbf{syz}(P)\,,\ v \longmapsto (-v,\ v).$$

Lattice structure on $\mathbf{RO}(G,<)$ and syzygies

Lattice structure on $\mathbf{RO}(G,<)$.

- The map $\mathbf{RO}(G,<) \longrightarrow \text{Subspaces}(V)$, $T \longmapsto \ker(T)$ is a bijection.
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Proposition ii. *Let $F = \{T_1, \cdots, T_n\} \subset \mathbf{RO}(G,<)$. For every integer $2 \leq i \leq n$, we have a short exact sequence*

$$0 \longrightarrow \mathbf{syz}(T_1, \cdots, T_{i-1}) \xrightarrow{\iota_i} \mathbf{syz}(T_1, \cdots, T_i) \xrightarrow{\pi_i} \mathbf{syz}(T_1 \wedge \cdots \wedge T_{i-1}, T_i) \longrightarrow 0.$$

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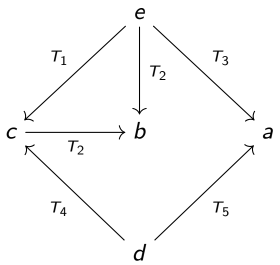

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Step 4. We have
$\ker\left((T_1 \wedge T_2 \wedge T_3 \wedge T_4) \vee T_5\right) = \mathbb{K}\{d - a\}$.

- ▷ $d - a = (d - T_4(d)) + (e - T_3(e)) - (e - T_1(e))$.
- ▷ $d - a = d - T_5(d)$.
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$$s_2 = u_{5,d} - u_{4,d} - u_{3,e} + u_{1,e}.$$

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The criterion. Let $F = \{T_1, \cdots, T_n\} \subset \mathbf{RO}(G, <)$.

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Thank you for listening!