Syzygies among reduction operators

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Plan

I. **Motivations**
   - Various notions of syzygy
   - Computation of syzygies

II. **Reduction operators**
   - Linear algebra, syzygies and useless reductions
   - Reduction operators and labelled reductions

III. **Lattice description of syzygies**
   - Lattice structure of reduction operators
   - Construction of a basis of syzygies
   - A lattice criterion for rejecting useless reductions
I. Motivations
Various notions of syzygy

Consider the following questions:

- **Standardisation problems**: given two vertices in an abstract rewriting system

\[ \begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \\
V \\
\uparrow \uparrow \uparrow \uparrow \\
V'
\end{array} \]

- **Construction of free resolutions**: given an augmented algebra

\[ A \langle X | R \rangle \rightarrow A \langle S \rangle \rightarrow A \langle X \rangle \rightarrow \epsilon \rightarrow K \rightarrow 0, \]

- **Detecting useless critical pairs**: how to obtain a criterion for rejecting useless critical pairs during the completion procedure?

A method for studying these problems:

compute a generating set for the associated notion of syzygy (two-dimensional cell, homological syzygy, identity among relations, ...).
Various notions of syzygy

Consider the following questions:

- **Standardisation problems**: given two vertices in an abstract rewriting system how to choose a "standard" path between them?
Various notions of syzygy

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- **Standardisation problems**: given two vertices in an abstract rewriting system

  ![Diagram of two vertices](image)

  how to choose a "standard" path between them?

- **Construction of free resolutions**: given an augmented algebra $A \langle X \mid R \rangle$ and

  $$A[R] \xrightarrow{d} A[X] \xrightarrow{d} A \xrightarrow{\varepsilon} K \rightarrow 0,$$

  how to extend the beginning of a resolution of the ground field?

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  \]

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Various notions of syzygy

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Various notions of syzygy

▶ Consider the following questions:

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![Diagram of a graph with vertices v and v’ connected by paths](image)

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▶ **Construction of free resolutions**: given an augmented algebra $A \langle X | R \rangle$ and

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A[S] \xrightarrow{d} A[R] \xrightarrow{d} A[X] \xrightarrow{d} A \xrightarrow{c} K \rightarrow 0,$$

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  - **Detecting useless critical pairs**: how to obtain a criterion for rejecting useless critical pairs during the completion procedure?

- A method for studying these problems:
  - compute a generating set for the associated notion of syzygy (two-dimensional cell, homological syzygy)
Various notions of syzygy

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  v \rightarrow v' \rightarrow v' \rightarrow v' \rightarrow v' \rightarrow v' \rightarrow v
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Syzygies among reduction operators
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Computation of syzygies

- Consider an algebra \( A \) presented by \( \langle X \mid R \rangle \).
Computation of syzygies

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  - Question: how are the syzygies of $\langle X \mid R \rangle$ generated?
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If \( A \) is homogeneous, a candidate is the Koszul dual \( A^! \langle X^* \mid R^\perp \rangle \).
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- Methods from rewriting theory

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Motivations

Reduction operators

Lattice description of syzygies

Computation of syzygies

Consider an algebra $A$ presented by $\langle X \mid R \rangle$.

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Methods from rewriting theory:

If $R$ is a Gröbner basis, the syzygies are spanned by critical pairs of $R$. 
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- Methods from rewriting theory:
  - If $R$ is a Gröbner basis, the syzygies are spanned by critical pairs of $R$.
  - If $R$ is a not a Gröbner basis, apply the completion-reduction procedure.
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    - Complete $R$ into a Gröbner basis $\overline{R}$.
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    - Reduce $\overline{S}$ and $\overline{R}$ (algebraic Morse theory, homotopical reduction, ···).
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- Our goals:
Consider an algebra $A$ presented by $\langle X \mid R \rangle$.

Question: how are the syzygies of $\langle X \mid R \rangle$ generated?

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Methods from rewriting theory:

- If $R$ is a Gröbner basis, the syzygies are spanned by critical pairs of $R$.
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  - Let $\bar{S}$ be the set of critical pairs of $\bar{R}$.
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Our goals:

- Compute syzygies of abstract rewriting systems using the lattice structure of reduction operators.
Computation of syzygies

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  - Question: how are the syzygies of $\langle X \mid R \rangle$ generated?

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- Methods from rewriting theory:
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- Our goals:
  - Compute syzygies of abstract rewriting systems using the lattice structure of reduction operators.
  - Deduce a lattice criterion for rejecting useless reductions during the completion procedure.
II. Reduction operators
Linear algebra, syzygies and useless reductions

- We consider abstract rewriting systems \((V, \rightarrow)\) such that \(V\) is a vector space and every reduction is labelled.
Linear algebra, syzygies and useless reductions

▶ We consider abstract rewriting systems \((V, \rightarrow)\) such that \(V\) is a vector space and every reduction is labelled.

▶ e.g., \(V\) is spanned by the letters \(\{a, b, c, d, e\}\) submitted to the reductions

\[
\begin{align*}
A' &= e - c, \\
B' &= c - b, \\
C' &= e - b, \\
\vdots \\
F' &= C - B - A + A'.
\end{align*}
\]

iii. Compute a row echelon basis of the syzygies.
iv. Remove the reductions corresponding to leading terms of syzygies.

The reductions \(C\) and \(F\) are useless!
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\[
\begin{align*}
A & \rightarrow C & D \\
B & \rightarrow b & a \\
C & \rightarrow e & D \\
E & \rightarrow e & B \\
F & \rightarrow F & \leftarrow E \\
\end{align*}
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\[
\begin{array}{c}
e \\
A & C & D \\
\downarrow & \downarrow & \downarrow \\
c & b & a \\
\downarrow & \downarrow & \downarrow \\
B & N & e \\
\downarrow & \downarrow & \downarrow \\
d & E & F \\
\end{array}
\]

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\begin{align*}
A' &= e - c, \\
B' &= c - b, \\
C' &= e - b, \\
D' &= e - a, \\
E' &= e - d, \\
F' &= e - b + A'
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How to detect useless reductions using syzygies and linear algebra?
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D &\rightarrow F
\end{align*}
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How to detect useless reductions using syzygies and linear algebra?

i. Replace each reduction by the difference between its source and its target.
Linear algebra, syzygies and useless reductions

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A & \rightarrow C \\
B & \rightarrow a \\
C & \rightarrow d \\
D & \rightarrow \downarrow \\
E & \rightarrow \downarrow \\
F & \rightarrow \downarrow
\end{align*}
\]

\text{i. } A' = e - c

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\begin{array}{c}
e \\
\downarrow \\
A \\
\downarrow \\
c \\
\downarrow \\
B \\
\downarrow \\
b \\
\downarrow \\
a \\
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\begin{align*}
\text{i. } A' &= e - c, & B' &= c - b, & C' &= e - b \\
\text{ii. } A' = e - c, & B' = c - b, & C' = e - b \\
\text{iii. } C' - B' - A' & \text{ and } F' - E' - D' + A'.
\end{align*}
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\downarrow & \downarrow & \downarrow \\
B & e & D \\
\end{array}
\]

\[
\begin{array}{ccc}
C & D & A \\
\downarrow & \downarrow & \downarrow \\
B & c & e \\
\end{array}
\]

\[
\begin{array}{ccc}
B & e & C \\
\downarrow & \downarrow & \downarrow \\
A & b & d \\
\end{array}
\]

\[
\begin{array}{ccc}
E & F & C \\
\downarrow & \downarrow & \downarrow \\
D & a & f \\
\end{array}
\]

\[
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D & F & E \\
\downarrow & \downarrow & \downarrow \\
C & b & c \\
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- The reductions \(C\) and \(F\) are useless!

How to detect useless reductions using syzygies and linear algebra?

1. Replace each reduction by the difference between its source and its target.
2. Introduce a terminating order on labels.
We consider abstract rewriting systems \((V, \rightarrow)\) such that \(V\) is a vector space and every reduction is labelled.

- e.g., \(V\) is spanned by the letters \(\{a, b, c, d, e\}\) submitted to the reductions

\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{C} \\
\downarrow \\
\text{D} \\
\downarrow \\
\text{E} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{B} \\
\downarrow \\
\text{c} \\
\downarrow \\
\text{d} \\
\downarrow \\
\text{a} \\
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\text{b} \\
\downarrow \\
\text{e} \\
\end{array}
\]

- i. \(A' = e - c\), \(B' = c - b\), \(C' = e - b\), \ldots
- ii. \(A \sqsubseteq B \sqsubseteq \cdots \sqsubseteq F\).

- The reductions \(C\) and \(F\) are useless!

How to detect useless reductions using syzygies and linear algebra?

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Motivations
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Lattice description of syzygies

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\begin{array}{c}
\text{e} \\
\text{A} \Downarrow \downarrow \Downarrow \text{D} \\
\text{c} \rightarrow \text{b} \rightarrow \text{a} \\
\text{B} \rightarrow \rightarrow \rightarrow \text{F} \\
\text{d} \\
\text{E} \rightarrow \rightarrow \rightarrow \text{F}
\end{array}
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  i. \(A' = e - c, \quad B' = c - b, \quad C' = e - b, \cdots\)
  ii. \(A \sqsubseteq B \sqsubseteq \cdots \sqsubseteq F.\)

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- How to detect useless reductions using syzygies and linear algebra?
  i. Replace each reduction by the difference between its source and its target.
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  iii. Compute a row echelon basis of the syzygies.
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    i. \quad A &\sqsubset B \sqsubset \cdots \sqsubset F. \\
    ii. \quad C' - B' - A' \quad \text{and} \quad F' - E' - D' + A'.
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\text{iii. } & \quad C' - B' - A' \quad \text{and} \quad F' - E' - D' + A'.
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How to detect useless reductions using syzygies and linear algebra?

- i. Replace each reduction by the difference between its source and its target.
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We consider abstract rewriting systems \((V, \rightarrow)\) such that \(V\) is a vector space and every reduction is labelled.

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\text{i. } A' & = e - c, \\
\text{ii. } A & \sqsubseteq B \sqsubseteq \cdots \sqsubseteq F.
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- \(C' - B' - A'\) and \(F' - E' - D' + A'\).

- \(C\) and \(F\).

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Reduction operators and labelled reductions

- We fix $V$ is a vector space equipped with a well-ordered basis $(G, <)$. 

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**Definition.**

- An endomorphism $T$ of $V$ is a **reduction operator** if
  - $T$ is a projector,
  - $\forall g \in G$, we have either $T(g) = g$ or $\text{lt}(T(g)) < g$. 

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Syzgies among reduction operators
We fix $V$ is a vector space equipped with a well-ordered basis $(G, <)$.

**Definition.**

- An endomorphism $T$ of $V$ is a reduction operator if
  - $T$ is a projector,
  - $\forall g \in G$, we have either $T(g) = g$ or $\text{lt}(T(g)) < g$.
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Reduction operators and labelled reductions

- We fix $V$ is a vector space equipped with a well-ordered basis $(G, <)$.

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- A reduction operator $T$ induces the labelled reductions $v \xrightarrow{\ell_{T,v}} T(v)$.
Motivations

Reduction operators

Lattice description of syzygies

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  \[
  \ell_{T_i,u} \sqsubseteq \ell_{T_j,v} := (i < j) \lor (i = j \land \text{lt} (u) < \text{lt} (v)).
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Reduction operators and labelled reductions

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Example

\[ G = \left\{ a < b < c < d < e \right\} \text{ and} \]

\[ ∵ \]

\[ \begin{array}{c}
  e \\
  \downarrow \quad \downarrow \quad \downarrow \\
  c \quad \quad \quad b \\
  \quad \quad \quad \quad \uparrow \quad \quad \quad \quad \uparrow \\
  d \quad \quad \quad a \\
  \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quatre
Example

$G = \left\{ a < b < c < d < e \right\}$ and

We obtain the following order on labels:
Example

- $G = \{a < b < c < d < e\}$ and

- We obtain the following order on labels:

  $\ell_{T_1,e}$
Example

\[ G = \{ a < b < c < d < e \} \text{ and} \]

\[
\begin{array}{c}
\text{\uparrow} e \\
\ell_{T_1,e} \\
\downarrow T_2 \\
b \\
\ell_{T_2,c} \\
\downarrow T_4 \\
c \\
\downarrow T_5 \\
d \\
\uparrow a
\end{array}
\]

- We obtain the following order on labels:

\[ \ell_{T_1,e} \sqsubseteq \ell_{T_2,c} \]
Example

Let $G = \{ a < b < c < d < e \}$ and

\[
\begin{array}{c}
\ell_{T_1,e} & \ell_{T_2,e} \\
\downarrow & \downarrow \\
T_4 & T_5 \\
\ell_{T_2,c} & \ell_{T_1,e} \\
\downarrow & \downarrow \\
c & b \\
a & d
\end{array}
\]

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Example

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Example

\[ G = \{ a < b < c < d < e \} \text{ and } \]

\[
\begin{array}{c}
\text{c} \\
\ell_{T2,c}
\end{array}
\quad
\begin{array}{c}
\text{b} \\
\ell_{T2,e}
\end{array}
\quad
\begin{array}{c}
\text{a}
\end{array}
\quad
\begin{array}{c}
\text{d}
\ell_{T5,d}
\end{array}
\]

\[
\begin{array}{c}
\text{e}
\ell_{T1,e}
\end{array}
\quad
\begin{array}{c}
\text{e}
\ell_{T3,e}
\end{array}
\quad
\begin{array}{c}
\text{c}
\ell_{T2,c}
\end{array}
\quad
\begin{array}{c}
\text{d}
\ell_{T5,d}
\end{array}
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III. Lattice description of syzygies
Definition of syzygies

**Syzygies.**

- The space of **syzygies** of \( F = \{ T_1, \cdots, T_n \} \subseteq \text{RO}(G, <) \) is the kernel of

\[
\ker(T_1) \times \cdots \times \ker(T_n) \longrightarrow V, \quad (v_1, \cdots, v_n) \mapsto v_1 + \cdots + v_n.
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- The space of syzygies of $F$ is denoted by $\text{syz} (F)$.

**Example.**

![Diagram showing the reduction operators and their effects on elements c, d, a, b, e.](image-url)
Definition of syzygies

**Syzygies.**

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\[
\text{syz}(T_1, \cdots, T_5) = \mathbb{K}\{s_1, s_2\} \text{ where }
\]

\[
\begin{array}{c}
e \\
\downarrow T_1 \uparrow T_2 \\
\downarrow T_3 \\
b \\
\downarrow a \\
\downarrow T_4 \\
\downarrow T_5 \\
d
\end{array}
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**Example.**

\[
\text{syz}(T_1, \ldots, T_5) = \mathbb{K}\{s_1, s_2\} \quad \text{where} \quad s_1 = (- (e - c), (e - b) - (c - b), 0, 0, 0)
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**Notation.** We denote by $u_{i,g} = (0, \cdots, 0, g - T_i(g), 0, \cdots, 0)$. 

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\[
\text{syz}(T_1, \cdots, T_5) = \mathbb{K}\{s_1, s_2\} \quad \text{where}
\]
\[
s_1 = (-(e-c), (e-b)-(c-b), 0, 0, 0)
\]
\[
= u_{2,e}
\]
\[
s_2 = (e-c, 0, -(e-a), -(d-c), d-a)
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 &= u_{2,e} - u_{2,c} - u_{1,e} \\
s_2 &= (e - c, 0, -(e - a), -(d - c), d - a) \\
 &= u_{5,d} - u_{4,d} - u_{3,e} + u_{1,e}
\end{align*}
\]
Lattice structure on \( \text{RO}(G, <) \) and syzygies

Lattice structure on \( \text{RO}(G, <) \).

- The map \( \text{RO}(G, <) \rightarrow \text{Subspaces}(V), \ T \mapsto \ker(T) \) is a bijection.
Lattice structure on \( \text{RO}(G, <) \) and syzygies

**Lattice structure on \( \text{RO}(G, <) \).**

- The map \( \text{RO}(G, <) \rightarrow \text{Subspaces}(V), \; T \mapsto \ker(T) \) is a bijection.
- \( \text{RO}(G, <) \) admits a lattice structure where:
  - \( T_1 \preceq T_2 \) if \( \ker(T_2) \subseteq \ker(T_1) \),
  - \( T_1 \land T_2 := \ker^{-1}(\ker(T_1) + \ker(T_2)) \),
  - \( T_1 \lor T_2 := \ker^{-1}(\ker(T_1) \cap \ker(T_2)) \).
Lattice structure on \( \text{RO} \( G, \prec \) \) and syzygies

**Lattice structure on \( \text{RO} \( G, \prec \) \).**

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  - \( T_1 \preceq T_2 \) if \( \ker \( T_2 \) \subseteq \ker \( T_1 \) \),
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  - \( T_1 \vee T_2 := \ker^{-1} \( \ker \( T_1 \) \cap \ker \( T_2 \) \) \).

**Proposition i.** Let \( P = (T, T') \subset \text{RO} \( G, \prec \) \). We have a linear isomorphism

\[
\ker \left( T \vee T' \right) \sim \text{syz} \( P \).
\]
Lattice structure on $\text{RO} (G, <)$ and syzygies

Lattice structure on $\text{RO} (G, <)$.

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- $T_1 \preceq T_2$ if $\ker (T_2) \subseteq \ker (T_1)$,
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Proposition i. Let $P = (T, T') \subset \text{RO} (G, <)$. We have a linear isomorphism

$$\ker (T \vee T') \overset{\sim}{\longrightarrow} \text{syz} (P), \quad v \mapsto (−v, v).$$
Lattice structure on \( \text{RO}(G, <) \) and syzygies

**Lattice structure on \( \text{RO}(G, <) \).**

- The map \( \text{RO}(G, <) \to \text{Subspaces}(V), \ T \mapsto \ker(T) \) is a bijection.
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**Proposition i.** Let \( P = (T, T') \subset \text{RO}(G, <) \). We have a linear isomorphism

\[
\ker(T \vee T') \xrightarrow{\sim} \text{syz}(P).
\]

**Proposition ii.** Let \( F = \{T_1, \cdots, T_n\} \subset \text{RO}(G, <) \). For every integer \( 2 \leq i \leq n \), we have a short exact sequence

\[
0 \to \text{syz}(T_1, \cdots, T_{i-1}) \xrightarrow{\iota_i} \text{syz}(T_1, \cdots, T_i) \xrightarrow{\pi_i} \text{syz}(T_1 \wedge \cdots \wedge T_{i-1}, T_i) \to 0.
\]
Construction of a basis of $\text{syz}(F)$

- How to construct a basis of $\text{syz}(F)$?
Motivations

Reduction operators

Lattice description of syzygies

Construction of a basis of $\text{syz}(F)$

How to construct a basis of $\text{syz}(F)$?

We have $\text{syz}(T_1, T_2) \subseteq \text{syz}(T_1, T_2, T_3) \subseteq \cdots \subseteq \text{syz}(T_1, \cdots, T_n)$. 
How to construct a basis of $\text{syz}(F)$?

- We have $\text{syz}(T_1, T_2) \subseteq \text{syz}(T_1, T_2, T_3) \subseteq \cdots \subseteq \text{syz}(T_1, \cdots, T_n)$.
- Main step: construct a supplement of $\text{syz}(T_1, \cdots, T_{i-1})$ in $\text{syz}(T_1, \cdots, T_i)$. 

Example.

```
    c
   ↘ ↘
  T_2 ↓  \\
 ▶ ▶
  T_1 ↙ ↙  \\

    b a
d
   ↘ ↘
  T_4 ↙  \\
   ↘ ↘
  T_5 ↙  \\
```
Construction of a basis of $\text{syz}(F)$

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    - This supplement is constructed using the isomorphism
      $$\text{syz}(T_1 \cdots, T_i)/\text{syz}(T_1, \cdots, T_{i-1}) \cong \ker((T_1 \land \cdots \land T_{i-1}) \lor T_i).$$
Construction of a basis of $\text{syz}(F)$

- How to construct a basis of $\text{syz}(F)$?
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      $$\text{syz}(T_1, \cdots, T_i) / \text{syz}(T_1, \cdots, T_{i-1}) \cong \ker ((T_1 \land \cdots \land T_{i-1}) \lor T_i).$$

Example.
Construction of a basis of \( \text{syz}(F) \)

- How to construct a basis of \( \text{syz}(F) \)?
  - We have \( \text{syz}(T_1, T_2) \subseteq \text{syz}(T_1, T_2, T_3) \subseteq \cdots \subseteq \text{syz}(T_1, \cdots, T_n) \).
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      \]

Example.

**Step 1.** We have \( \ker(T_1 \lor T_2) = \mathbb{K}\{e - c\} \).
Construction of a basis of $\text{syz}(F)$

- **How to construct a basis of $\text{syz}(F)$?**
  - We have $\text{syz}(T_1, T_2) \subseteq \text{syz}(T_1, T_2, T_3) \subseteq \cdots \subseteq \text{syz}(T_1, \cdots, T_n)$.
  - Main step: construct a supplement of $\text{syz}(T_1, \cdots, T_{i-1})$ in $\text{syz}(T_1, \cdots, T_i)$.
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**Example.**

**Step 1.** We have $\ker(T_1 \lor T_2) = \mathbb{K}\{e - c\}$.
- $e - c = e - T_1(e)$. 

\[ 
\begin{array}{ccc}
\text{e} & \downarrow T_2 & \text{b} \\
\text{c} & \downarrow T_4 & \text{d} \\
\text{a} & \downarrow T_5 & \text{d}
\end{array} 
\]
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Université Paris-Est Marne-la-Vallée
Szygues among reduction operators
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**Example.**

![Diagram](attachment:diagram.png)

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Construction of a basis of \( \text{syz}(F) \)

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**Example.**

![Diagram](image)

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**Step 2.** We have $\ker((T_1 \land T_2) \lor T_3) = \{0\}$. 

![Diagram showing relationships between reduction operators](image-url)
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- **Step 3.** $\ker ((T_1 \wedge T_2 \wedge T_3) \vee T_4) = \{0\}$.
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**Example.**

\[
\begin{array}{c}
\text{Step 4.} \text{ We have} \\
\ker((T_1 \wedge T_2 \wedge T_3 \wedge T_4) \lor T_5) = \mathbb{K}\{d - a\}. \\
\text{ } \\
\text{ } \\
\text{We get the second basis element:} \\
s_2 = u_{5,d} - u_{4,d} - u_{3,e} + u_{1,e}.
\end{array}
\]
A lattice criterion for rejecting useless reductions

**The criterion.** Let $F = \{T_1, \cdots, T_n\} \subset \text{RO}(G, <)$.
A lattice criterion for rejecting useless reductions

The criterion. Let $F = \{T_1, \cdots, T_n\} \subset RO(G, <)$.

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- The useless reductions are labelled by leading terms of syzygies.
  - These labels are $\ell_{T_i, g}$, where $g \notin \text{im}((T_1 \land \ldots \land T_{i-1}) \lor T_i)$.

**Example.**

![Diagram](image)
A lattice criterion for rejecting useless reductions

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![Diagram](image.png)
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We have:

- $\ker(T_1 \lor T_2) = \mathbb{K}\{e - c\}$.
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Example.

We have:

▷ $\ker (T_1 \lor T_2) = \mathbb{K}\{ e - c \}$.

▷ $T_1 \lor T_2(e) = c$.

▷ The labelled reduction $\ell_{T_2, e}$ is useless.
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The criterion. Let \( F = \{ T_1, \cdots, T_n \} \subset \text{RO}(G, <) \).

- The useless reductions are labelled by leading terms of syzygies.
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We have:

- \( \ker (T_1 \vee T_2) = \mathbb{K}\{e - c\} \).
- \( T_1 \vee T_2(e) = c \).
- The labelled reduction \( \ell_{T_2, e} \) is useless.

- \( \ker ((T_1 \wedge T_2 \wedge T_3 \wedge T_4) \vee T_5) = \mathbb{K}\{d - a\} \).
- The labelled reduction \( \ell_{T_5, d} \) is useless.
Thank you for listening!