Joint work with Simon Vaux (T2K)...

Julie Chouquet (Irif)...

Choccola

Syon, November 9, 2017

• Finiteness of antireduction...

• Taylor expansion...

• Linear logic proof nets...
Outline

- Introduction

- Main result: How can we bound the loss of size in parallel reduction?

Applications:

1. Simulate cost-elimination into Taylor expansion
   - In particular, reduction and expansion commute.

2. Normalization by evaluation for box-connected MELL nets
   - With Giulia Guerrieri, Luc Pellissier,
   Lorenzo Tortora de Falco, Lloyd H. Vean
Introduction • Proof nets • Taylor expansion • reduction
Tim: We want to study the Taylor expansion, and keep « good properties » as in $\lambda$-calculus.

What is a good property?

$\Rightarrow$ If $M \rightarrow^p N$, $T(M) \Rightarrow T(N)$

and $\Rightarrow$ keeps the coefficients finite.

In particular, $T(\mathrm{nf}(M)) = \mathrm{nf}(T(M))$. 
Tim: We want to study the Taylor expansion, and keep «good properties» as in λ-calculus.

What is a good property?

\[ \Rightarrow \text{If } M \rightarrow_p N, \quad T(M) \Rightarrow T(N) \]

and \( \Rightarrow \) keeps the coefficients finite.

in particular, \( T(\text{nf}(M)) = \text{nf}(T(M)) \).

Ehrlund & Reignier 2008: Uniformity and the Taylor expansion

\[ \Rightarrow \text{The proof can't be adapted to proof nets.} \]
Vaux 2017: Taylor expansion, \( \beta \)-reduction, and normalization.

Another technique, for algebraic \( \lambda \)-calculus, using that:

- For all \( m \in \mathcal{I}(M) \), \( \text{depth}(m) \leq \text{depth}(M) \)

- If \( m \equiv n \), \( \#m \leq \Psi(\text{depth}(m), \#n) \) and \( \text{depth}(n) \leq \Psi(\text{depth}(m)) \)
Vaux 2017: Taylor expansion, β-reduction, and normalization.

Another technique, for algebraic λ-calculus, using that:

- For all \( m \in |T(M)| \), \( \text{depth}(m) \leq \text{depth}(M) \)

- If \( m \rightarrow n \), \( \#m \leq \Psi(\text{depth}(m), \#n) \)

and \( \text{depth}(n) \leq \Psi(\text{depth}(m)) \)

**Consequence:** If \( M \rightarrow_p N \), then, for all \( n \in |T(N)| \)

\[ \{ m \in |T(M)| ; m \rightarrow n \} \] is finite,

and \( T(M) \Rightarrow T(N) \) keeps finite coefficients.
Framework:

- Multiplicative, Exponential fragment of Linear Logic (MELL)
  - Proof nets (Girard, 80's)

\[ A \otimes A^* \]

Identity

\[ \vdash A, A^* \]

\[ \vdash \Gamma, A \quad \vdash A^*, \Delta \]

\[ \vdash \Gamma, \Delta \]

\[ A \text{ cut} A^* \]
D Multiplicatives

\[ \Gamma, A \vdash \Delta, B \]

\[ \Gamma, \Delta \vdash A \otimes B \]

\[ \Gamma, A, B \vdash \Gamma, A \otimes B \]

\[ \Gamma, A \otimes B \]

\[ A \otimes B \]
Exponentials

\[ \Gamma \vdash \Gamma, ?A \]

\[ \Gamma, ?A, ?A \]

\[ \Gamma \vdash \Gamma, ?A \]

\[ \Gamma \vdash ?A, \ldots, ?A^n, A \]

\[ \Gamma \vdash ?A, \ldots, ?A^n, \neg A \]

\[ \Gamma \vdash \Gamma, A \]

\[ \Gamma \vdash \Gamma, ?A \]

\[ \neg A \]

\[ \neg A \]

\[ \neg A \]

\[ \neg A \]

\[ \neg A, \ldots, \neg A^n, A \]

\[ \neg A, \ldots, \neg A^n, \neg A \]

\[ \neg A \]

\[ \neg A \]
Cut elimination: reduction rules

\[
\begin{align*}
A & \rightarrow A^1 \\
\text{cut} & \\
\end{align*}
\]

\[
\begin{align*}
A & \rightarrow A^1 \\
\text{cut} & \\
\end{align*}
\]

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\begin{align*}
A & \rightarrow A^1 \\
\text{cut} & \\
\end{align*}
\]

\[
\begin{align*}
A & \rightarrow A^1 \\
\text{cut} & \\
\end{align*}
\]
Cut elimination: reduction rules

\[ A \rightarrow A \]

\[ A \bullet B \bullet A' \rightarrow A \]

\[ A \bullet B \bullet B' \rightarrow A' \]

\[ A \bullet B \rightarrow A \]

\[ A \bullet B \rightarrow B \]

\[ A \bullet B \rightarrow A' \]

\[ A \bullet B \rightarrow B' \]

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\[ A \bullet B \rightarrow B' \]
Taylor expansion: \( \text{DLL}_0 \)

It is defined as linear combinations of resource nets:

- MELL without the promotion boxes
- Plus costructural rules:

  - coderestriction:

  - co-weakening:

  - co-contraction:
We only need to consider two reduction rules.
We only need to consider two reduction rules:
If \( P = R \)

then, \( \tau(P) = \)

\[
\sum_{n \in \mathbb{N}} \tau(R)_n \cdots \tau(R)_{n_n} \cdot \frac{1}{n!}
\]

\( n_1, \ldots, n_n \in \text{DLL}_0 \)
Reducing in $T(P)$: we need to define a parallel reduction $\Rightarrow$.

- If $p \rightarrow q+$ then $p \Rightarrow q$
Problem: extend to infinite sums.

Let \( p_n = \left[ \ldots \begin{array}{c} \text{cut} \\ \text{cut} \\ \text{cut} \\ \text{cut} \end{array} \right] \)

- Where \( t \) is a tree of DLL.

\[
\sum_{n \in \mathbb{N}} p_n \rightarrow t \cdot \infty
\]
Problem: extend to infinite sums.

- Let $p_n = \sum_{n \in \mathbb{N}} p_n \Rightarrow t$

- Where $t$ is a tree of DLL.

$\sum_{n \in \mathbb{N}} p_n \Rightarrow t \cdot \infty$

More generally, $\uparrow p = \{ q \in DLL; q \Rightarrow p \}$ is infinite for all $p$. 

\[ \checkmark \]
· · Antireduction · Size · Paths · ·
We need to obtain the following result:

For all $q \in \mathcal{D}L_0$, $\uparrow q \cup |\mathcal{T}(P)|$ is finite.

and all $P \in \mathcal{M}ELL$
We need to obtain the following result:

\[ \forall q \in \text{DLL}, \quad \uparrow q \cup |\uparrow \exists (P)| \text{ infinite} \]

and all \( p \in \text{MELL} \)

we will show:

\[ \max \left\{ \# p ; p \in |\uparrow \exists (P)| , \ p \Rightarrow q \right\} \leq \Phi(p, \# q) \]
We need to obtain the following result:

For all \( q \in \text{DLL} \), \( \Uparrow q \cup |\mathcal{T}(P)| \) is finite.

and all \( P \in \text{MELL} \)

we will show:

\[
\max \left\{ \#p \mid p \in |\mathcal{T}(P)|, \ p \Rightarrow q \right\} \leq \varphi(P, \#q)
\]

\( \Rightarrow \) \( |I| \overset{A}{=} \text{Size / number of wires} \)
How?

We set the following quasi invariant:

\[
cc(p) = \max \{ cc(\gamma); \gamma \text{ is a switching path of } p \}
\]

where

\[
cc(\gamma) = \text{number of cuts crossed by } \gamma.
\]
Paths

\[\text{ax}\]

\[\text{cut}\]

\[!\]

?
Exemples:

\[ P = \]

\[ \text{cc}(P) = 1 \]
Exemples :

\[ p = \]

\[ cc(p) = 1 \]

\[ q = \]

\[ cc(q) = 2 \]
For all $n \in \mathbb{N}$, $\text{cc}(p_n) = 2$.
Lemma: Let $p \in \text{PcMELL}$, $p \in |T(P)|$, $cc(p) \leq 2^{\#P}$

Idea of the proof: $P = \{R\}$

Since $\pi_i \in |T(R)|$, $\text{Hi}: cc(\pi_i) \leq 2^{\#R}$

Induction on exponential depth
Reduction:

In order to establish that \( \#p \leq \eta(c(c(p), \#q)) \) whenever \( p \Rightarrow q \), we decompose the reduction:

1. **Multiplicative phase**: \( p \Rightarrow_{\otimes \mid \otimes} q^{-} \)
   
   \( \rightarrow \) Only \( \otimes / \otimes \) and \( ? / ! \)

2. **Axiomatic phase**: \( q^{-} \Rightarrow_{\text{ax/cut}} q \)
   
   \( \rightarrow \) Only \( \text{ax/cut} \).
Multiplicative phase:

Let \( p \rightarrow_{\otimes} q \), where:

\[ p = \]

\[ q = \]
Multiplicative phase:

- Let $p \Rightarrow^* q$, where:

$\quad p = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad q = \quad \quad\quad \quad \quad \quad \quad \quad \quad \quad \quad$

and $q = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad$

Lemma: If $p \Rightarrow^* q$, then $\#P \leq 2 \# q$. 

Axiomatic phase:

Let $p \Rightarrow_{ax} q$, and $cc(p) = n$.

First, observe that between

\[ \text{ax} \quad \text{cut} \quad \ldots \quad \text{cut} \quad \text{ax} \quad \text{cut} \quad \ldots \quad \text{cut} \quad k \quad \text{cut} \]

and

the loss of size is $\frac{1}{2} 2k + 1$
Here, whenever $\text{cc}(p) = n$, 
\[ p = \text{ax} \quad \text{cut} \quad \ldots \quad \text{cut} \quad \text{cut} \quad \leq n \quad \text{cut} \quad \leq n \]
and \[ q = \quad \text{cut} \quad \text{cut} \quad \text{cut} \quad \text{cut} \]
Here, whenever \( cc(p) = n \),

\[
p = \underbrace{\text{ax} \text{ cut} \ldots \text{ cut}}_{\leq n} \quad \text{and} \quad q = \underbrace{1 \ldots 1}_{\leq n}
\]

Then, we have:

**Lemma:** If \( p \models_{\text{ax}} q \) and \( cc(p) = n \), then:

\[
\#p \leq 2(n+1)\#q
\]
First conclusion: If \( p \Rightarrow_{\sigma/k} q^- \Rightarrow_{\omega} q \), then

\[-\# p \leq 2 \# q^-,\]

\[-\# q^- \leq 2 (cc(q^-) + 1) \# q.\]
First conclusion: If $p \Rightarrow_{\alpha/\beta} q \Rightarrow_{\alpha} q$, then

- $\#p \leq 2\#q$
- $\#q^{-} \leq 2(\text{cc}(q^{-}) + 1)\#q$.

Good, but not enough: we are looking for $\# p \leq \text{cc}(p), \# q$.

We need to examine the relation between $\text{cc}(p)$ and $\text{cc}(q^{-})$.
.. Variations. Slipknot ..
Question: If \( p \rightarrow q \), what about \( cc(q) \)?

It increases. Consider as example this configuration («slipknot»):

\[
\begin{align*}
\text{cc} &= 1 \\
\end{align*}
\]

\[\rightarrow\]

\[
\begin{align*}
\text{cc} &= 2 \\
\end{align*}
\]
**Question:** If $p \Rightarrow q$, what about $cc(q)$?

It increases. Consider as example this configuration («slipknot»):

$$cc = 1$$

$$\rightarrow$$

$$cc = 2$$

It increases a lot in this other example:

$$cc = 4$$

$$\rightarrow$$

$$cc = 12$$
But, from the knowledge of $cc(p) = n$, we can bound the number of slipknots.

\[
\begin{align*}
\text{height} &\leq n \\
\text{width} &\leq n
\end{align*}
\]
Moreover, if a slipknot is at width $k$, it has at most $n-k$ "sub-slipknots":

\[
\text{width } k
\]

\[
\{ n-k \text{ slipknots} \}
\]
Moreover, if a slipknot is at width $k$, it has at most $n-k$ "sub-slipknots":

\[
\text{width } k \nn-k \text{ slipknots.}
\]

Since, during the reduction, each slipknot generates 2 cuts, we can conclude (with an induction on the width)

**Lemma**: If $p \xrightarrow{\rho/\pi} q$, then $cc(q) \leq 2 \cdot cc(p)$!
Second conclusion:  \[ p \Rightarrow_{\mathcal{Q}}^1 q^- \Rightarrow_{\mathcal{Q}}^2 q \]

- \( \#p \leq 2 \#q^- \)

- \( \#q^- \leq 2 (\text{cc}(q^-) + 1) \#q \)

\( \leq 2 \left( 2 \text{cc}(p)! + 1 \right) \#q \) \hspace{1cm} \text{since } \text{cc}(q^-) \leq 2 \text{cc}(p)!
Second conclusion: \[ p \Rightarrow_{\pi} q \Rightarrow_{\omega} q \text{, then:} \]

- \[ \# p \leq 2 \# q \]
- \[ \# q \leq 2(\text{cc}(q) + 1) \# q \]

\[ \leq 2 \left(2 \text{cc}(p)! + 1\right) \# q \quad \text{since} \quad \text{cc}(q) \leq 2 \text{cc}(p)! \]

Finally:

Proposition: \[ p \Rightarrow q \text{, then:} \]

\[ \# p \leq (8 \text{cc}(p)! + 4) \# q \]
But again, this is not enough; our proposition covers only the connected structures.

We can do better, and plug to our proof the specific case of 0-ary connectives:

tensor: \(\Box\)

contraction: ？

parr: \(\Box\)

c-o-contraction: ！
We need to consider another reduction:

- Evanescent phase:

$$P \xrightarrow{w/\emptyset} q$$

$$P = \begin{array}{c}
\begin{array}{c}
! \\
(? \overset{\text{cut}}{\mapsto}) \\
! \\
\end{array}
\end{array}$$

and

$$q = \begin{array}{c}
\begin{array}{c}
! \\
\overset{?}{\rightarrow} \\
? \\
\end{array}
\end{array}$$
We allow structures with weakenings, only if there exists a function $f : \text{weakenings} \to \text{trees}$ such that $f(?) \neq ?$.

Jumps: We can present the structures as follows:

We can also have:
We extend the definition of paths, along the jumps:
With similar (and more technical) arguments than before, we can bound the number of weakenings in antireducts.

In particular, if \( p \Rightarrow_{w/a} q \), where \( p = \)
With similar (and more technical) arguments than before, we can bound the number of weakenings in antireducts.

In particular, if $p \Rightarrow_{w/\ell} q$, where $p =$

\[\begin{array}{cc}
! & ? \\
\text{cut} & \text{cut}
\end{array}\]

\[\text{\Rightarrow}_{\text{w/\ell}} \quad q\]

$\#p \leq \text{veryexponential}(\text{cc}(p), \#q)$. 
Third conclusion. Theorem...
Theorem: let $P \in ME_{LL}$, $q \in D_{LLO}$.

$|T(P)| \cup ^{q}$ is finite.
Theorem: let $P \in \text{MELL}$, $q \in \text{DLL}$.  

$|\mathcal{T}(P)| \cap \uparrow q$ is finite.

\begin{align*}
\text{Proof:} \\
\quad \begin{cases}
\quad \text{For all } p \in |\mathcal{T}(P)|, \quad cc(p) \leq 2^{\#p} \\
\quad \text{For all } p \in \uparrow q, \quad \#p \leq \varphi(cc(p), \#q) \\
\quad \text{Then, for all } p \in |\mathcal{T}(P)| \cap \uparrow q, \quad \#p \leq \varphi(2^{\#p}, \#q) .
\end{cases}
\end{align*}
First application. Simulation.
An example of simulation: the contraction case.

Consider the following MELL nets:

\[
\begin{array}{c}
\text{P} \\
\text{cut} \\
\text{Q}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\text{P} \\
\text{cut} \\
\text{Q}'
\end{array}
\]
An example of simulation: the contraction case.

Consider the following MeLL nets:

We will show that $\tau(Q) \Rightarrow \tau(Q')$
First, notice that \( \tau(Q) = \)

\[
\sum_{n \in \mathbb{N}} \tau(P)_{p_1} \cdots \tau(P)_{p_n} \cdot \frac{1}{n!}
\]
First, notice that $\mathcal{T}(Q) = \sum_{n \in \mathbb{N}} \mathcal{T}(P)_p \cdot \ldots \cdot \mathcal{T}(P)_{p_n} \cdot \frac{1}{n!}$

and $\mathcal{T}(Q') = \sum_{p, p' \in \mathcal{T}(P)} \mathcal{T}(P)_p \cdot \mathcal{T}(P)_{p'}$
For n-ary connectives, the reduction contraction/cocontraction reduces into the 0-structure if $m \neq n$. 
For n-ary connectives, the reduction contraction/cocontraction reduces into the 0-structure if \( m \neq n \).

Then, \( \sum_{n \in \mathbb{N}} \tau(P_1) \cdots \tau(P_n) \frac{1}{n!} \)

reduces into:

\[
\sum_{P, P' \in \tau(P)} \frac{1}{2}
\]

\[
\sum_{P, P' \in \tau(P)} \frac{1}{2}
\]
So we have:

\[ \tau(k) \Rightarrow \sum_{P, P' \in \tau(P)} \tau(P) \tau(P') \frac{1}{2} + \frac{1}{2} \]
So we have:

\[ T(Q) = \sum_{P, P' \in \tau(P)} T(P)_p \cdot T(P)_p \cdot \frac{1}{2} + \frac{1}{2} \]

And, up to the commutativity of the contraction, this is equal to:

\[ T(Q') = \sum_{P, P' \in \tau(P)} T(P)_p \cdot T(P)_p \]
We can extend this study to all the exponential reductions:

Theorem: let $P, Q \in \text{MELL}$ or $P \rightarrow Q$

$\tau(P) \Rightarrow \tau(Q)$

Moreover, for all $P \in \text{MELL}$, $\eta(P) = \tau(\eta(P))$. 
Second application
Normalization by evaluation
We propose a method for obtaining the normal form of a proof net without proceeding to the cut elimination, but evaluating some points of its Taylor expansion.
Let $P \in \text{MELL}$. We can bound the term such that

$$P \Rightarrow \cdots \Rightarrow \text{nf}(P)$$
Let $P \in \text{MELL}$. We can bound the term such that

\[ P \rightarrow \cdots \rightarrow \text{nf}(P) \]

So, we can extend the finiteness result presented earlier, and state:

For all $q \in \text{nf}(P)$, \( q \cup \text{fix}(P) \) is finite.
Now, let's consider the following result (Guerrieri, Pellissier, Tortora, 2016):

- Let $R \in \text{MELL}$ in normal form.
- Let $\pi[2] \in T(R)$ consisting in the $2$-expansion of $R$.

(\(\Rightarrow\) The boxes are duplicated hereditarily twice)

Then, we have a procedure to recover $R$ from $\pi[2]$. 
So, let \( R = \ell f (P) \).

- We look into \( T(P) \) and isolate a finite subset \( X \),
  containing \( \lvert T(P) \rvert \cap \uparrow r [2] \).
So, let \( R = n_f(P) \).

- We look into \( T(P) \) and isolate a finite subset \( X \), containing \( |T(P)| \cap \cup n_2 \).

- We evaluate (linear reduction) the resource nets of \( X \), until we recognize \( r(2) \).
So, let $R = nf(P)$.

- We look into $c(P)$ and isolate a finite subset $X$, containing $\mathcal{L}(P) \cap \uparrow^{*} r[E]$

- We evaluate (linear reduction) the resource nets of $X$, until we recognize $r[E]$.

- We use the previous result to build $R$. 

Conclusions
• By an analysis of the size, and the paths of nets under parallel reduction, we presented a geometrical way to establish algebricic properties:
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- There exists a good notion of relation for infinite linear combinations of nets ("\( \Rightarrow \)) . It is well-defined hence it keeps coefficients finite.
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- There exists a good notion of relation for infinite linear combinations of nets ("\( \Rightarrow \)"). It is well-defined hence it keeps coefficients finite.

- In particular, if \( T(P) \) always converges.
• By an analysis of the size and the paths of nets under parallel reduction, we presented a geometrical way to establish algebraic properties:

  - There exists a good notion of relation for infinite linear combinations of nets ("⇒"). It is well-defined hence it keeps coefficients finite.

  - In particular, if \(T(P)\) always converges.

• Two applications of this result:

  - Cut elimination (resp. normalization) and Taylor expansion commute.

  - Normalization by evaluation.
... Thank you ...