An application of parallel cut elimination in unit-free multiplicative linear logic to the Taylor expansion of proof nets

Jules Chouquet\textsuperscript{1}, Lionel Vaux Auclair\textsuperscript{2}

\textsuperscript{(1)} IRIF, Université Paris-Diderot
\textsuperscript{(2)} I2M, Aix Marseille Université

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Motivations/framework

- Quantitative semantics of λ-calculus and linear logic
- Commutation between cut-elimination/normalization and Taylor expansion of proof nets
- Combinatorial study of parallel cut elimination

Main result

Parallel reduction over infinite linear combinations of differential nets is well defined
1 Quantitative semantics

2 Quantitative syntax

3 Taylor expansion

4 Linear logic proof nets

5 Our contribution
Denotational semantics of $\lambda$-calculus

\[ M, N ::= x \mid \lambda x M \mid MN \]
\[ (\lambda x M) N \rightarrow_\beta M[N/x] \]
\[ [[M]]_\mathcal{M} \leadsto \text{Something in a structure } \mathcal{M} \text{ invariant under } \rightarrow_\beta \]

For a program $P : A \rightarrow B$, $[[P]]$ can be a function or a relation between $[[A]]$ and $[[B]]$. 
Denotational semantics of $\lambda$-calculus

An example: relational semantics

Interpret $\lambda$-terms in the category $\mathcal{R}$ of sets and relations.

**Types** \[ \rightsquigarrow \] **Sets**

**Terms** \[ \rightsquigarrow \] **Relations**

$M \rightarrow_{\beta} N \Rightarrow [[M]]_{\mathcal{R}} = [[N]]_{\mathcal{R}} \rightsquigarrow$ Composition of relations

$soundness\, theorem$ \[ [[MN]] = [[M]] \circ [[N]] \]
Quantitative approach
Think about resources

- Quantitative meaning of $[[M]]$ in $(\lambda x(xx))M$ \(\text{wrt} \) $(\lambda xx)M$?
- Interpret probabilistic reduction?
- ...

Girard (Normal functors, 1988)

Uses of arguments \(\rightsquigarrow\) degree of a monomial in a power series.

Types: $[[A]] \subseteq S^{|A|}$ where $S$ is a semiring

Programs: power series
Quantitative semantics
Example: Probabilistic coherent spaces

$S$ \Rightarrow \mathbb{R}^+
Types \Rightarrow [[A]] = (|A|, P(A) \subseteq (\mathbb{R}^+)^{|A|})
Programs \Rightarrow \Rightarrow \Rightarrow [[P]] : [[A]] \rightarrow [[B]]$ an analytic map
Composition \Rightarrow Composition of matrices

Key result
Probabilistic coherent spaces are fully abstract for probabilistic PCF
(Ehrhard, Pagani, Tasson 2015)
Quantitative Semantics

Example 2: multirelations

$S$ \mapsto \text{Boolean semiring}

Types \mapsto \left[ [A \to B] \right] = M_{\text{fin}}(\mid A \mid) \times \mid B \mid

Programs \mapsto P : A \to B \Rightarrow \left[ [P] \right] \subseteq M_{\text{fin}}(\mid A \mid) \times \mid B \mid

Invariance \mapsto \text{Composition of multirelations.}

Key idea

let $M : A \to B$, $N : A$.

$([a_1, \ldots, a_k], b) \in [[M]]$ will match with $k$ uses of the argument $N$ in the application $(MN)$. 
1 Quantitative semantics

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Resource calculus
A quantitative syntax

\[ m, n ::= x | \lambda x m | \langle m \rangle[n_1, \ldots, n_k] \text{ (k-linear application)} \]

\[ \langle \lambda x m \rangle[n_1, \ldots, n_k] \rightarrow_\partial \sum_{\sigma \in S_n} m\left[ n_{\sigma(1)}/x_1, \ldots, n_{\sigma(k)}/x_k \right] \]

\[ \langle \lambda x \langle x \rangle [x] \rangle[z] \rightarrow_\partial 0 \quad \partial \leftarrow \langle \lambda x x \rangle[z, z] \]

<table>
<thead>
<tr>
<th>\lambda\text{-calculus}</th>
<th>resource calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>( MN ) ( \rightsquigarrow ) ( \langle M \rangle[N_1, \ldots, N_k] )</td>
<td></td>
</tr>
<tr>
<td>( (\lambda x(x)x)z ) ( \rightsquigarrow ) ( \langle \lambda x \langle x \rangle [x] \rangle[z, z] )</td>
<td></td>
</tr>
<tr>
<td>( \downarrow_\beta )</td>
<td>( \downarrow_\partial )</td>
</tr>
<tr>
<td>( zz ) ( \rightsquigarrow ) ( \langle z \rangle[z] + \langle z \rangle[z] )</td>
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</table>
Multilinear Approximations

We say $m$ is an approximation of $M$, and write $m \triangleleft M$ if:

- $m = M = x$
- $m = \lambda x \cdot n$, $M = \lambda x \cdot N$ and $n \triangleleft N$.
- $m = \langle m' \rangle[n_1, \ldots, n_k]$ for some $k \in \mathbb{N}$, $M = M' \cdot N$, and $m \triangleleft M$, $n_i \triangleleft N$.

**Definition**

We extend $\rightarrow_\partial$ to a parallel reduction $\Rightarrow_\partial$.

**Property: simulation of $\rightarrow_\beta$ with approximants**

If $m \triangleleft M$ and $M \rightarrow M'$, $m \rightarrow 0$ or $\exists m'$ s.t. $m \Rightarrow_\partial m'$ and $m' \triangleleft M'$.

**Example**

- $N \rightarrow N' \Rightarrow MN \rightarrow MN'$
- $m \triangleleft M$, $n_i \triangleleft N \Rightarrow \langle m \rangle[n_1, \ldots, n_k] \triangleleft MN$.
- $n_i \Rightarrow_\partial n_i' \Rightarrow \langle m \rangle[n_1, \ldots, n_k] \Rightarrow_\partial \langle m \rangle[n_1', \ldots, n_k'] \triangleleft MN'$ (ind. hyp.)
1. Quantitative semantics

2. Quantitative syntax

3. Taylor expansion

4. Linear logic proof nets

5. Our contribution
Taylor expansion
A bridge between syntax and semantics

Semantic approach: Interpret a term/function as an infinite series of approximants.

Syntactic Taylor expansion:

\[
T(MN) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \langle T(M) \rangle [T(N), \ldots, T(N)]_k
\]

\[
T(\lambda x M) = \lambda x T(M), \quad T(x) = x.
\]

Remark

\(T(M)\) is a weighted sum of all resource nets \(m\) s.t. \(m \prec M\)
Simulation

Wanted result

Extend $\Rightarrow_\partial$ to infinite sums of terms ($\Rightarrow_\partial$), in order to have $M \rightarrow_\beta N \Rightarrow T(M) \Rightarrow_\partial T(N)$, and define $NF(T(M))$.

Problem

Can $\Rightarrow_\partial$ be always well-defined?

Counterexample

$$\sum_{k \in \mathbb{N}} \langle \lambda x x \rangle [\langle \lambda x x \rangle \ldots [y]] \ldots ] \Rightarrow_\partial \infty \cdot y$$

If $\mathcal{S}$ is not a complete semiring, the reduction is not defined on all series of terms.
Some convergence results

\[ \Rightarrow_\partial \] and normalization are well defined and commute with Taylor expansion:

- Classical \( \Lambda \): Ehrhard Regnier 2007.
- Non deterministic \( \Lambda \) with finite sums: Pagani, Tasson, Vaux 2016.
Idea of the proof in (Vaux, CSL 2017)

\[ M \xrightarrow{\beta} M' \]

\[ T(M) \Rightarrow_{\partial} T(M') \]

Sketch

\[ \{ \text{appdepth}(m); m \in T(M) \} \text{ bounded by } M \]
\[ \Downarrow \]
\[ \{ \#m; m \in T(M), m \Rightarrow_{\partial} m' \} \text{ bounded by } \#m' \]
\[ \Downarrow \]
\[ \{ m_i \in T(M); m_i \Rightarrow_{\partial} m' \} \text{ is finite.} \]
\[ \Downarrow \]

Hence, \( m' \) has a finite coefficient in \( T(M') \).
1. Quantitative semantics

2. Quantitative syntax

3. Taylor expansion

4. Linear logic proof nets

5. Our contribution
MELL
Multiplicative nodes and reductions

\[ A, B ::= X \mid A \perp \mid A \otimes B \mid A \bowtie B \mid !A \mid ?A \]

\[
\begin{array}{c}
\quad a \\
\quad a_\perp \\
\downarrow \\
\quad a \quad a_\perp \\
\quad \rightarrow \\
\quad a
\end{array}
\]

\[
\begin{array}{c}
\quad a \otimes b \\
\quad a_\perp \bowtie b_\perp \\
\downarrow \\
\quad a_\perp \bowtie b_\perp \\
\quad \rightarrow \\
\quad a \quad b \quad b_\perp \\
\quad a_\perp
\end{array}
\]
Duplication of an exponential box

An example of non linear reduction
<table>
<thead>
<tr>
<th>Λ</th>
<th>MELL</th>
</tr>
</thead>
<tbody>
<tr>
<td>x, λxM</td>
<td>Multiplicatives</td>
</tr>
<tr>
<td>MN</td>
<td>Interaction of an exponential box</td>
</tr>
<tr>
<td>Several uses of the argument</td>
<td>Duplication of a box</td>
</tr>
<tr>
<td>Resource λ-calculus</td>
<td>Resource nets</td>
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</table>

**Taylor expansion**

As in λ-calculus, we consider sums of approximants. An approximant of a box consists in the duplication of its content.
Resource reduction
Linear fragment of differential interaction nets

Resource nets
Exponential box is removed, and will be simulated by the linear constructs \( ? \) and \( ! \):

\[
\sum_{\sigma \in \mathcal{S}_n} \sigma
\]
Taylor expansion of a box

\[ \sum_{n \in \mathbb{N}} \frac{1}{n!} p_1 \in T(P) | \quad \cdots \quad n \quad \cdots \quad p_n \in T(P) | \]

becomes

\[ !P \]

\[ \cdots \]

J.chouquet (Irif), L.Vaux (I2M) About Taylor expansion of LL-proof nets CSL 2018 22 / 29
Wanted result
Define \( \Rightarrow \) over infinite sums of nets, in order to simulate exponential cut elimination and normalization of an MELL net, into its Taylor expansion.

Can we define a parallel reduction over infinite sums of resource nets?

Counterexample
\[
\sum_{n \in \mathbb{N}} \underbrace{\cdots}_{n} \Rightarrow \infty \cdot | 
\]
1. Quantitative semantics

2. Quantitative syntax

3. Taylor expansion

4. Linear logic proof nets

5. Our contribution
In $\Lambda$, the convergence result is proved thanks to the following property:

If $m \in T(M)$, applicative depth $(m)$ is bounded by $M$

**Key idea of our result**

Applicative depth in $\Lambda$ corresponds in proof nets to the number of cuts crossed by a switching path.
Idea of the proof

**Theorem**

Let $P \xrightarrow{cut.el.} Q$, $q \in |T(Q)|$, then \( \{ p \in |T(P)| ; p \Rightarrow q \} \) is finite.

**Proof.**

1. If $p \in |T(P)|$, the paths of $p$ do not cross more than $\#P$ cuts
2. If $p \Rightarrow q$, $\#p \leq f(\#q$, number of cuts on a path of $p$)
3. Then $\{ \#p; p \in |T(P)|, p \Rightarrow q \}$ is bounded by $p$ and $q$.
4. then $\{ p \in |T(P)| ; p \Rightarrow q \}$ is finite
Main result

Corollary
For all MELL net $P$, if $\mathcal{T}(P) \Rightarrow \psi = \sum_{i \in I} a_i \cdot p_i$, then each resource net $p$ has a finite coefficient in $\psi$

Reduction
If $P \rightarrow Q$, then the reduction $\mathcal{T}(P) \Rightarrow \mathcal{T}(Q)$ is well defined

Normalization
$\text{NF}(\mathcal{T}(M)) = \sum_{m \in |\mathcal{T}(M)|} \mathcal{T}(M)_m \cdot \text{nf}(m) = \mathcal{T}(\text{NF}(M))$
Conclusion

- Quantitative semantics is a powerful approach to study important properties of various calculi
- Taylor expansion is a strong bridge between the calculus and its quantitative interpretation
- Linear logic is an efficient framework for quantitative semantics

- We study the calculus of linear logic proof nets
- We can mimic quantitative identities in the Taylor expansion setting (resource calculi)
- We show that this is sound \textit{wrt} the underlying algebraic structure (convergence result)
Thank you