Injectivity of the coherent model for a fragment of connected *MELL* proof-nets

Antonio Bucciarelli[1], Raffaele Di Donna[2], and Lorenzo Tortora de Falco[3]

[1] Université Paris Cité, Paris, France
buccia@irif.fr
[2] Université Paris Cité, Paris, France and Università Roma Tre, Rome, Italy
didonna@irif.fr
[3] Università Roma Tre, Rome, Italy
tortora@uniroma3.it

1 Context

Linear logic, introduced by Jean-Yves Girard in [5] in 1987, is a refinement of both classical and intuitionistic logic in which formulas are treated as resources: the structural rules of contraction and weakening are restricted to formulas that are marked with modal operators and the semantic equivalence induced by the models of linear logic is non-trivial.

In this context, we ask ourselves the question of finding a canonical object representing the proofs in the same class of semantic equivalence. Formalization through proof-nets removes the redundant information of sequent calculus that concerns the order of application of the rules and allows us to define the cut-elimination procedure through local manipulations of graphs rather than global transformations of proof trees. Consequently, it is an appropriate formalism to study the dynamics of normalisation and to prove fundamental properties of the system such as strong normalisation: the fact that, given a proof-net R, every chain of reductions starting from R ends in a cut-free proof-net (see [5] and [8]).

Following the point of view of Curry-Howard's correspondence between proofs and programs and relying on the existing literature about the relationship between the dynamics of PCF and the models providing a mathematical representation of this language ([2], [6], [7] and [9] to name a few), we study the connections between proof-nets and their semantic interpretations.

Therefore, we focus on three fundamental notions: the syntactic, semantic and observational equivalences. The first one is intrinsic, whereas the others depend on the model and on the notion of observation we choose, respectively. Generally speaking, these three notions of equivalence are increasingly coarse: all models identify syntactically equivalent proofs and all reasonable notions of observational equivalence include the semantic equivalence of all models. A model is *injective* when the induced semantic equivalence coincides with syntactic equivalence, *fully abstract* when it coincides with observational equivalence. In general, full abstraction fails when some points of the model are not interpretations of proofs. A classic example is Scott's continuous model, that is not fully abstract for PCF since the "parallel or" function is not PCF-definable, as proven in [9]. On the other hand, when every point of a model is the interpretation of a proof, we say that *full completeness* holds. This last property, which was originally studied in [1], is often exploited as a sufficient condition for full abstraction, for instance in [6].

In linear logic, the question of injectivity was addressed for the first time by Tortora de Falco in [10], where he produced two counter-examples to the injectivity of the multiset-based coherent model for multiplicative exponential linear logic without units (*MELL*). On the other hand, the injectivity of the relational model for the full multiplicative exponential fragment of linear logic was recently proven by de Carvalho in [4] by employing the powerful Taylor expansion technique, which allows us to represent a proof-net as the infinite series of its linear approximations.

To ask the question of injectivity is also a way to address the problem of proof identity: one asks whether two proofs are to be considered equal. In other words, one aims to specify what is a proof. As already mentioned, proof-nets identify distinct sequent calculus proofs that are morally the same, because they only differ in the order in which some rules are applied. We could then say that proof-nets capture more faithfully the essence of a proof. With the idea of "measuring" the quality of the representation of proofs as proof-nets, we study the question of injectivity for the coherent model in order to understand if one could make "more identifications" than proof-nets.

2 Advances

We resume the work on the injectivity of multiset-based coherent semantics which started in [10]. It was conjectured that the result of injectivity can be extended to all connected proof-nets and it was given a sufficient condition to reach this conclusion: the existence of an injective experiment for all connected proof-nets only consisting of axioms, tensors, derelictions and contractions. It was also proven that this condition is satisfied if one assumes that every contraction is terminal.

2.1 Atomic pre-experiment and (C)-pairs

We can define a partial labelling of pairs of arcs, called atomic pre-experiment, in such a way that any two premises of an atomic contraction are incoherent and any two premises of a non-atomic contraction are incoherent or undefined. The definition relies on the fact that we're dealing with *connected* proof-nets: two arcs of the same type are incoherent if and only if, for every switching graph, the unique path linking the nodes of which these arcs are the conclusions contains neither of them. The atomic pre-experiment is an injective experiment when all contractions are atomic.

Under the same assumption, we generalize the notion of (C)-pair introduced in [10]. A pair of conclusions of atomic why not nodes with the same address is a (C)-pair if there is a switching graph in which at least one of them belongs to the unique path connecting the conclusions of the proof-net over which they are located. Once again, this definition requires that the proof-net is *connected*. When a pair of conclusions (a, a') of the proof-net has a unique (C)-pair, the atomic pre-experiment automatically guarantees the incoherence of (a, a').

2.2 Injectivity for connected $(?\parr)LL_{pol}$ proof-nets

We now consider the fragment $(?\parr)LL_{pol}$ of linear logic that is defined by the following grammar:

$$N, M ::= X \mid ?X \mid ?P \parr N \mid N \parr ?P \qquad P, Q ::= X^{\perp} \mid !X^{\perp} \mid !N \otimes P \mid P \otimes !N$$

In this very specific framework, it turns out that the question of injectivity has a positive answer because, even when a pair of conclusions does not have a unique (C)-pair, we can always find one on which we can harmlessly assign incoherence.

Theorem. *Multiset-based coherent semantics is injective for connected* $(?\parr)LL_{pol}$ *proof-nets.*

Given that connected $(?\parr)LL_{pol}$ proof-nets embed the simply typed λI-calculus, which is the simply typed λ-calculus without weakenings (see [3]), we also have a proof of the following result.

Corollary. *Multiset-based coherent semantics is injective for the simply typed* λI*-calculus.*

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