An Anti-Locally-Nameless Approach to Formalizing Quantifiers

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- Formalization of syntactic results by proof assistants.

Motivations

- Formalization of syntactic results by proof assistants.
- Avoiding α-equivalence.

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For some y not free in Γ nor in $\forall x A$ we have:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi\\ \Gamma \vdash A[y/x]\end{array}}{\Gamma \vdash \forall x A}\,(\forall I) \qquad \dfrac{\begin{array}{c}\vdots\,\rho\\ \Delta, A[t/x] \vdash B\end{array}}{\Delta, \forall x A \vdash B}\,(\forall L)}{\Gamma, \Delta \vdash B}\,(cut)$$

Case of a generalization versus an instantiation

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Cut elimination step:

$$\dfrac{\begin{matrix}\vdots\,\pi[t/y]\\ \Gamma \vdash A[t/x]\end{matrix} \qquad \begin{matrix}\vdots\,\rho\\ \Delta, A[t/x] \vdash B\end{matrix}}{\Gamma, \Delta \vdash B}\,(cut)$$

First example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{Px, Py \vdash Py}\,(ax)
        }{Px, \forall x Px \vdash Py}\,(\forall L)
      }{Px, \forall x Px \vdash \forall x Px}\,(\forall I)
    }{Px, \forall x Px \vdash Py \vee \forall x Px}\,(\vee IR)
  }{Px, \forall x Px \vdash \forall y(Py \vee \forall x Px)}\,(\forall I)
  \qquad
  \dfrac{
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  \dfrac{
    \dfrac{
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}{Px, \forall x Px \vdash Px \vee \forall x Px}\,(\vee IR)
\qquad
\dfrac{}{Px \vee \forall x Px \vdash Px \vee \forall x Px}\,(ax)
$$

$$
\dfrac{Px, \forall x Px \vdash Px \vee \forall x Px \qquad Px \vee \forall x Px \vdash Px \vee \forall x Px}{Px, \forall x Px \vdash Px \vee \forall x Px}\,(cut)
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Case of a commutative cut

For some y not free in Δ, A nor in $\forall x B$ we have:

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Second example

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{Py, Px \vdash Px}\ (ax)}{Py, \forall x Px \vdash Px}\ (\forall L)}{Py, \forall x Px \vdash \forall x Px}\ (\forall I) \qquad \dfrac{\dfrac{\dfrac{}{Py \vdash Py}\ (ax)}{\forall x Px \vdash Py}\ (\forall L)}{\forall x Px \vdash \forall x Px}\ (\forall I)}{Py, \forall x Px \vdash \forall x Px}\ (cut)
$$

Second example

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{Py, Px \vdash Px}\,(ax)}{Py, \forall x Px \vdash Px}\,(\forall L)}{Py, \forall x Px \vdash \color{orange}{\forall x Px}}\,(\forall I) \qquad \dfrac{\dfrac{}{Py \vdash Py}\,(ax)}{\color{orange}{\forall x Px} \vdash Py}\,(\forall L)}{\dfrac{Py, \forall x Px \vdash P\color{red}{y}}{Py, \forall x Px \vdash \forall x Px}\,\color{orange}{(\forall I)}}\,(cut)
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One possible solution

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  each occurrence of the ($\forall I$) rule has its own variable.

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  each occurrence of the $(\forall I)$ rule has its own variable.
- Every derivation is *equivalent* to a *nice* derivation.
- We can apply any cut elimination step to a *nice* derivation.
- But we have to resort to variable renaming several times...

A different approach

Two kinds of variables

Gentzen's approach, with e not occurring in Γ nor A:

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  - Constants at the level of terms and formulas.
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  - Constants at the level of terms and formulas.
  - Variables at the level of proofs.
- f-variables ("formula") in $\mathcal{V}$, denoted $x, y, z, \ldots$

$\mathcal{E}$-terms are given by:

$$t ::= x \mid e \mid g\, t \ldots t$$

Terms and substitution

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The substitution $t[u/x]$ of an f-variable x by an $\mathcal{E}$-term u in an $\mathcal{E}$-term t is defined by induction on t:

$$\begin{aligned} x[u/x] &= u \\ y[u/x] &= y && \text{(if } x \neq y\text{)} \\ e[u/x] &= e \\ (g t_1 \dots t_k)[u/x] &= g(t_1[u/x]) \dots (t_k[u/x]) \end{aligned}$$

Formulas and substitution

$\mathcal{E}$-formulas are given by:

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The substitution $A[u/x]$ of an f-variable x by an $\mathcal{E}$-term u in an $\mathcal{E}$-formula A is defined by induction on A:

$$\begin{aligned}
(Pt_1 \dots t_k)[u/x] &= P(t_1[u/x]) \dots (t_k[u/x]) \\
(B \star C)[u/x] &= (B[u/x]) \star (C[u/x]) \\
(\forall x B)[u/x] &= \forall x B \\
(\forall y B)[u/x] &= \forall y (B[u/x]) && (\text{if } x \neq y) \\
(\exists x B)[u/x] &= \exists x B \\
(\exists y B)[u/x] &= \exists y (B[u/x]) && (\text{if } x \neq y)
\end{aligned}$$

Instantiation rules

If t is an f-closed term:

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists I) \qquad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}\ (\forall L)$$

Instantiation rules

If t is an f-closed term:

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Notice that $Pt \to \exists x P x$ is provable if and only if t is f-closed.

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Notice that $Pt \to \exists x Px$ is provable if and only if t is f-closed.

Not a problem: provability of f-closed formulas (i.e. with no free occurrence of f-variable) in our system corresponds to provability of formulas in usual intuitionistic sequent calculus.

Generalization Rules

If e does not occur in Γ, A, C:

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The e-variable e is called the eigenvariable of these rules.

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The e-variable e is called the eigenvariable of these rules.

We wish for uniqueness of eigenvariables, but this is not possible due to proof transformations with duplications involved.

The anti-locally-nameless approach

We fix $\mathcal{E} = \mathbb{N}$ and we lift eigenvariables when crossing a generalization rule upwards.

$$\frac{\Gamma\uparrow \vdash A\uparrow[0/x]}{\Gamma \vdash \forall x A}\ (\forall I) \qquad \frac{\Gamma\uparrow, A\uparrow[0/x] \vdash B\uparrow}{\Gamma, \exists x A \vdash B}\ (\exists L)$$

A concrete example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P00 \vdash P00}\,(ax)}{\forall y P0y \vdash P00}\,(\forall L)}{\forall x \forall y Pxy \vdash P00}\,(\forall L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P01 \vdash P01}\,(ax)}{\forall y P0y \vdash P01}\,(\forall L)}{\forall x \forall y Pxy \vdash P01}\,(\forall L)}{\forall x \forall y Pxy \vdash \forall y Py0}\,(\forall I)}{\forall x \forall y \vdash P00 \wedge \forall y Py0}\,(\wedge I)}{\forall x \forall y Pxy \vdash \forall x (Pxx \wedge \forall y Pyx)}\,(\forall I)$$

Parallel substitution

Given a function $r\colon \mathcal{E} \to \mathcal{E}$-terms, we define the parallel substitution $t[r]$ by induction on t:

$$x[r] = x$$

$$e[r] = r(e)$$

$$(gt_1 \dots t_k)[r] = g(t_1[r]) \dots (t_k[r])$$

We also have $A[r]$ defined by induction on A:

$$(Pt_1 \dots t_k)[r] = P(t_1[r]) \dots (t_k[r])$$

$$(B \to C)[r] = (B[r]) \to (C[r])$$

$$(\forall x B)[r] = \forall x (B[r])$$

$$(\exists x B)[r] = \exists x (B[r])$$

Composition and commutation lemmas

We say that r is f-closed if $r(e)$ is f-closed for all $e \in \mathcal{E}$.

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Lemma (Composition).

$$t[r][s] = t[e \mapsto r(e)[s]]$$

$$A[r][s] = A[e \mapsto r(e)[s]]$$

Lemma (Commutation). If r is f-closed, then:

$$t[u/x][r] = t[r][u[r]/x]$$

$$A[u/x][r] = A[r][u[r]/x]$$

Formalizing lifting

Let $S\colon \mathbb{N} \to \mathbb{N}$-terms be defined by $S(n) = n + 1$. We define:

$$t{\uparrow} = t[S]$$

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Formalizing lifting

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And given $r\colon \mathbb{N} \to \mathbb{N}$-terms, we define $\Uparrow r\colon \mathbb{N} \to \mathbb{N}$-terms by:

$$\Uparrow r(n) = \begin{cases} 0 & \text{if } n = 0 \\ r(n-1){\uparrow} & \text{otherwise} \end{cases}$$

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$$\Uparrow r(n) = \begin{cases} 0 & \text{if } n = 0 \\ r(n-1)\uparrow & \text{otherwise} \end{cases}$$

By construction $\uparrow$ is f-closed and $\Uparrow r$ is f-closed if r is f-closed.

Lemma (Lifting).

$$t{\uparrow}[{\Uparrow}r] = t[r]{\uparrow}$$

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Proof. By the composition lemma, we have:

$$t{\uparrow}[{\Uparrow}r] = t[n \mapsto (n+1)[{\Uparrow}r]] = t[n \mapsto r(n){\uparrow}] = t[r]{\uparrow}$$

Analogous for formulas. □

Substitution in proofs, case of a generalization

Given a derivation π of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed r by induction on the size of π.

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Back to cut elimination

Given an $\mathbb{N}$-term v, we define $v\Downarrow\colon \mathbb{N} \to \mathbb{N}$-terms by:

$$v\Downarrow(n) = \begin{cases} v & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

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By construction $v\Downarrow$ is f-closed if v is f-closed.

Lemma (Deletion).

$$t{\uparrow}[v{\Downarrow}] = t$$
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Proof. By the composition lemma, we have:

$$t{\uparrow}[v{\Downarrow}] = t[n \mapsto (n+1)[v{\Downarrow}]] = t[n \mapsto n] = t$$

Analogous for formulas. □

Lemma (Substitution)**.** If π is a derivation of $\Gamma{\uparrow} \vdash A{\uparrow}[0/x]$ and t is an f-closed term, then $\pi[t{\Downarrow}]$ is a derivation of $\Gamma \vdash A[t/x]$.

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Case of a generalization versus an instantiation revisited

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$$

Case of a generalization versus an instantiation revisited

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\pi\\ \Gamma{\uparrow} \vdash A{\uparrow}[0/x]\end{array}}{\Gamma \vdash \forall x A}\,(\forall I) \qquad \cfrac{\begin{array}{c}\vdots\,\rho\\ \Delta, A[t/x] \vdash B\end{array}}{\Delta, \forall x A \vdash B}\,(\forall L)}{\Gamma, \Delta \vdash B}\,(cut)$$

Cut elimination step:

$$\cfrac{\begin{array}{c}\vdots\,\pi[t{\Downarrow}]\\ \Gamma \vdash A[t/x]\end{array} \qquad \begin{array}{c}\vdots\,\rho\\ \Delta, A[t/x] \vdash B\end{array}}{\Gamma, \Delta \vdash B}\,(cut)$$

First example revisited

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P2, P0 \vdash P0}\,(ax)
}{P2, \forall x Px \vdash P0}\,(\forall L)
}{P1, \forall x Px \vdash \forall x Px}\,(\forall I)
}{P1, \forall x Px \vdash P0 \vee \forall x Px}\,(\vee IR)
}{P0, \forall x Px \vdash \forall y(Py \vee \forall x Px)}\,(\forall I)
\qquad
\dfrac{
\dfrac{}{P0 \vee \forall x Px \vdash P0 \vee \forall x Px}\,(ax)
}{\forall y(Py \vee \forall x Px) \vdash P0 \vee \forall x Px}\,(\forall L)
}{P0, \forall x Px \vdash P0 \vee \forall x Px}\,(cut)
$$

First example revisited

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{P1, P0 \vdash P0}\,(ax)
    }{P1, \forall x Px \vdash P0}\,(\forall L)
  }{P0, \forall x Px \vdash \forall x Px}\,(\forall I)
}{P0, \forall x Px \vdash P0 \lor \forall x Px}\,(\lor IR)
\qquad
\dfrac{}{P0 \lor \forall x Px \vdash P0 \lor \forall x Px}\,(ax)
$$

$$
\dfrac{P0, \forall x Px \vdash P0 \lor \forall x Px \qquad P0 \lor \forall x Px \vdash P0 \lor \forall x Px}{P0, \forall x Px \vdash P0 \lor \forall x Px}\,(cut)
$$

Case of a commutative cut revisited

$$\dfrac{\dfrac{\vdots\,\pi}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\vdots\,\rho}{\Delta\uparrow, A\uparrow \vdash B\uparrow[0/x]}}{\Delta, A \vdash \forall x B}\,(\forall I)}{\Gamma, \Delta \vdash \forall x B}\,(cut)$$

$$\dfrac{\dfrac{\vdots\,\pi}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\vdots\,\rho}{\Delta\uparrow, A\uparrow \vdash B\uparrow[0/x]}}{\Delta, A \vdash \forall x B}\,(\forall I)}{\Gamma, \Delta \vdash \forall x B}\,(cut)$$

Cut elimination step:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi[\uparrow]}{\Gamma\uparrow \vdash A\uparrow} \qquad \dfrac{\vdots\,\rho}{\Delta\uparrow, A\uparrow \vdash B\uparrow[0/x]}}{\Gamma\uparrow, \Delta\uparrow \vdash B\uparrow[0/x]}\,(cut)}{\Gamma, \Delta \vdash \forall x B}\,(\forall I)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P1, P0 \vdash P0}\,(ax)}{P1, \forall x Px \vdash P0}\,(\forall L)}{P0, \forall x Px \vdash \forall x Px}\,(\forall I) \qquad \dfrac{\dfrac{\dfrac{}{P0 \vdash P0}\,(ax)}{\forall x Px \vdash P0}\,(\forall L)}{\forall x Px \vdash \forall x Px}\,(\forall I)}{P0, \forall x Px \vdash \forall x Px}\,(cut)$$

Second example revisited

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P2, P0 \vdash P0}\;(ax)
}{P2, \forall x Px \vdash P0}\;(\forall L)
}{P1, \forall x Px \vdash \forall x Px}\;(\forall I)
\qquad
\dfrac{
\dfrac{}{P0 \vdash P0}\;(ax)
}{\forall x Px \vdash P0}\;(\forall L)
}{
\dfrac{P1, \forall x Px \vdash P0}{P0, \forall x Px \vdash \forall x Px}\;(\forall I)
}\;(cut)
$$

Expressiveness analysis

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

From LJ to ALN, case of a generalization

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ.

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n, 0/y] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n, 0/y]\\ \hline\hline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n, 0/y] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n, 0/y] \\ \hdashline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n, 0/y] \\ \hdashline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_1{\uparrow}/x_1, \ldots, t_n{\uparrow}/x_n] \vdash B[y/x][t_1{\uparrow}/x_1, \ldots, t_n{\uparrow}/x_n, 0/y]\\ \hdashline\end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][0/y]\\ \hdashline\end{array}$$

From LJ to ALN, case of a generalization

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[y/x]\end{matrix}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{matrix}\vdots \\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][0/y] \\ \hdashline \end{matrix}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[y/x]\end{matrix}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{matrix}\vdots \\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][y/x][0/y] \\ \hline \end{matrix}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ.
Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][y/x][0/y] \\ \hdashline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][0/x]\\ \hdashline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][0/x] \\ \hdashline \end{array}$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_1/x_1, \ldots, t_n/x_n]\!\uparrow \vdash B[t_1/x_1, \ldots, t_n/x_n]\!\uparrow\![0/x] \\ \hline\hline \end{array}$$

From LJ to ALN, case of a generalization

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[y/x]\end{matrix}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma[t_1/x_1, \ldots, t_n/x_n]\uparrow \vdash B[t_1/x_1, \ldots, t_n/x_n]\uparrow[0/x]\end{matrix}}{\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash \forall x(B[t_1/x_1, \ldots, t_n/x_n])}\ (\forall I)$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma[t_1/x_1, \ldots, t_n/x_n]\!\uparrow\, \vdash B[t_1/x_1, \ldots, t_n/x_n]\!\uparrow\![0/x]\end{array}}{\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash \forall x (B[t_1/x_1, \ldots, t_n/x_n])}\ (\forall I)$$

Lemma. If the free variables of Γ and A are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\mathrm{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\mathrm{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that y is not free in Γ nor in $\forall x B$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[y/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma[t_1/x_1, \ldots, t_n/x_n]\uparrow \vdash B[t_1/x_1, \ldots, t_n/x_n]\uparrow[0/x]\end{array}}{\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash (\forall x B)[t_1/x_1, \ldots, t_n/x_n]}\ (\forall I)$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_i/x_i, 0/y_k] \vdash B[t/x][t_i/x_i, 0/y_k] \\ \hline\hline \end{array}$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_i/x_i, 0/y_k] \vdash B[t/x][t_i/x_i, 0/y_k] \\ \hdashline \end{array}$$

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots\\ \Gamma[t_i/x_i, 0/y_k] \vdash B[t_i/x_i, 0/y_k][t[t_i/x_i, 0/y_k]/x]\\ \hdashline \end{array}$$

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots \\ \Gamma[t_i/x_i, 0/y_k] \vdash B[t_i/x_i, 0/y_k][t[t_i/x_i, 0/y_k]/x] \\ \hdashline \end{array}$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[t/x]\end{matrix}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{matrix}\vdots \\ \Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x] \\ \hdashline \end{matrix}$$

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[t/x]\end{matrix}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]\end{matrix}}{\Gamma[t_i/x_i] \vdash \exists x(B[t_i/x_i])}\ (\exists I)$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash B[t/x]\end{matrix}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]\end{matrix}}{\Gamma[t_i/x_i] \vdash \exists x(B[t_i/x_i])}\ (\exists I)$$

From LJ to ALN, case of an instantiation

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]\end{array}}{\Gamma[t_i/x_i] \vdash {\color{red}(\forall x B)[t_i/x_i]}}\ (\exists I)$$

Let t be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]\end{array}}{\Gamma[t_i/x_i] \vdash (\forall x B)[t_i/x_i]}\ (\exists I)$$

The other cases are trivial. □

Lemma. If the e-variables of Γ and A are among $e_1, \dots, e_n$ and $\Gamma \vdash_{\mathrm{ALN}} A$, then $\Gamma[x_1/e_1, \dots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \dots, x_n/e_n]$ for any variables $x_1, \dots, x_n$ not bound in Γ nor in A.

From ALN to LJ, case of a generalization

Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\mathrm{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in Γ nor in A.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\mathrm{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in Γ nor in A.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma\!\uparrow \vdash B\!\uparrow\![0/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\mathrm{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in Γ nor in A.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\uparrow \vdash B\uparrow[0/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

Let y be a fresh variable.

$$\begin{array}{c}\vdots\\ \Gamma\uparrow[x_1/e_1\uparrow, \ldots, x_n/e_n\uparrow, y/0] \vdash B\uparrow[0/x][x_1/e_1\uparrow, \ldots, x_n/e_n\uparrow, y/0]\\ \hdashline \end{array}$$

Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\mathrm{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in Γ nor in A.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

$$\frac{\begin{array}{c}\vdots\\ \Gamma\uparrow \vdash B\uparrow[0/x]\end{array}}{\Gamma \vdash \forall x B}\ (\forall I)$$

Let y be a fresh variable.

$$\begin{array}{c}\vdots\\ \Gamma\uparrow[x_1/e_1\uparrow, \ldots, x_n/e_n\uparrow, y/0] \vdash B\uparrow[0/x][x_1/e_1\uparrow, \ldots, x_n/e_n\uparrow, y/0]\\ \hdashline \end{array}$$

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From ALN to LJ, case of a generalization

Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in Γ nor in A.

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$$\dfrac{\begin{matrix}\vdots \\ \Gamma\uparrow \vdash B\uparrow[0/x]\end{matrix}}{\Gamma \vdash \forall x B}\ (\forall I)$$

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$$\dfrac{\vdots \atop \Gamma\!\uparrow\, \vdash B\!\uparrow\![0/x]}{\Gamma \vdash \forall x B}\ (\forall I)$$

Let y be a fresh variable.

$$\dfrac{\vdots \atop \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[x_1/e_1, \ldots, x_n/e_n][y/x]}{\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash \forall x(B[x_1/e_1, \ldots, x_n/e_n])}\ (\forall I)$$

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From ALN to LJ, case of an instantiation

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

$$\cfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots \\ \Gamma[x_i/e_i, y/e'_k] \vdash B[t/x][x_i/e_i, y/e'_k] \\ \hdashline \end{array}$$

From ALN to LJ, case of an instantiation

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash B[t/x]}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\dfrac{\vdots}{\Gamma[x_i/e_i, y/e'_k] \vdash B[t/x][x_i/e_i, y/e'_k]}$$

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

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$$\begin{array}{c}\vdots \\ \Gamma[x_i/e_i, y/e'_k] \vdash B[x_i/e_i, y/e'_k][t[x_i/e_i, y/e'_k]/x] \\ \hdashline \end{array}$$

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$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash B[t/x]\end{array}}{\Gamma \vdash \exists x B}\ (\exists I)$$

$$\begin{array}{c}\vdots \\ \Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x] \\ \hdashline \end{array}$$

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$$\dfrac{\begin{array}{c}\vdots \\ \Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x]\end{array}}{\Gamma[x_i/e_i] \vdash \exists x (B[x_i/e_i])}\ (\exists I)$$

From ALN to LJ, case of an instantiation

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

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From ALN to LJ, case of an instantiation

Let t be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

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$$\dfrac{\vdots \\ \Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x]}{\Gamma[x_i/e_i] \vdash (\exists x B)[x_i/e_i]}\ (\exists I)$$

Proposition (Embedding LJ). If the free variables of Γ and A are among $x_1, \ldots, x_n$, then:

$$\Gamma \vdash_{\mathrm{LJ}} A \iff \Gamma[1/x_1, \ldots, n/x_n] \vdash_{\mathrm{ALN}} A[1/x_1, \ldots, n/x_n]$$

Conclusions

Proposition (Embedding LJ). If the free variables of Γ and A are among $x_1, \ldots, x_n$, then:

$$\Gamma \vdash_{\mathrm{LJ}} A \iff \Gamma[1/x_1, \ldots, n/x_n] \vdash_{\mathrm{ALN}} A[1/x_1, \ldots, n/x_n]$$

Proposition (Embedding ALN). If Γ and A are f-closed, their e-variables are among $e_1, \ldots, e_n$ and if $x_1, \ldots, x_n$ are variables not bound in Γ nor in A, then:

$$\Gamma \vdash_{\mathrm{ALN}} A \iff \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\mathrm{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$$

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