Towards injectivity of the coherent model for connected *MELL* proof-nets

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Abstract

We present some advances in the study of injectivity of the coherent multiset-based model for the proof-nets of multiplicative and exponential linear logic whose switching graphs are all acyclic and connected.

Context

The general idea of denotational semantics is to give a mathematical counterpart to a rewriting system. Every term t of the syntax is mapped to some mathematical object $[\![t]\!]$, thus defining a function $[\![\cdot]\!]$, called *semantics* or *interpretation*. The fundamental property that a semantics must satisfy is invariance under the rewrite rules: if t reduces to t', then $[\![t]\!] = [\![t']\!]$. This invariance entails the inclusion of the *syntactic equivalence* relation on terms, which identifies terms that can be transformed one into the other by applying some rewrite rules, in the *semantic equivalence*, which equates terms with the same interpretation. We say that the interpretation is *injective* if this inclusion is an equality.

In the second part of the last century, with the discovery of the Curry-Howard correspondence, the study of these equivalence relations, historically at the heart of theoretical computer science, became relevant in proof theory: a proof can be seen as a program and execution corresponds to cut-elimination. In the framework of linear logic [Gir87], the question of injectivity was addressed in [TdF03] and turned out to be quite complex: contrary to what happens in the simply typed λ-calculus where any model (satisfying "typical ambiguity", see [Jol00]) is injective, there are semantics of linear logic (satisfying typical ambiguity) that are not injective. For instance, while in [Car15] it was proven that the relational model is injective for multiplicative and exponential linear logic (*MELL*), this is not the case for the coherent model, as counterexamples were exhibited in [TdF03].

The question of injectivity is also a way to determine whether or not two proofs are to be considered equal, which is the traditional proof-theoretical problem of the identity of proofs. With linear logic the notion of proof-net, a new formalism for proofs, is born. It allows to capture more faithfully the essence of a proof since it identifies proofs which, in Gentzen's sequent calculus, are morally the same: for instance, proofs that only differ in the order in which some rules are applied. To study the question of injectivity is then a way to "measure" the quality of the representation of proofs as proof-nets and to wonder if it is possible to make "more identifications" than proof-nets.

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Introduction

We resume the work on the injectivity of the coherent multiset-based semantics which started in [TdF03] with the first positive and negative results. While it was shown that injectivity fails when considering all proof-nets of *MELL*, it was also conjectured that injectivity holds for connected proof-nets, that is proof-nets such that all their switching graphs are acyclic and connected.[1] In the same paper, a sufficient condition to achieve this result was determined: the existence of an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions. Still in [TdF03], it was shown that this sufficient condition holds with the restriction that every contraction is terminal.

We recently managed to establish the result with the restriction that every contraction is atomic, meaning that the types of its premises are atomic formulas. More precisely, given an injective "relational" experiment, we are able to define a partial coherence relation in such a way that the labels of the premises of any atomic contraction are incoherent and, for every non-atomic contraction, the coherence relation between the labels of its premises is either incoherence or undefined.

1 Proof-nets and experiments

Our logical framework is a particular subsystem of cut-free *MELL* proof-nets *without weakenings*. Formulas are generated by the following grammar, where X denotes any atomic formula:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid ?A$$

In this setting, we provide the definitions of proof-structure, proof-net and connected proof-net.

Definition 1.1. A *proof-structure* is a labelled directed graph R, with labels of the nodes in $\{ax, \otimes, ?, \bullet\}$ and such that:

- Every arc of R is called a *premise* of its head and a *conclusion* of its tail;
- Every node of R labelled by ax is called an *axiom*, has no premises and exactly two conclusions, labelled by dual atomic formulas;
- Every node of R labelled by $\otimes$ is called a *tensor*, has exactly one conclusion, labelled by a formula $A \otimes B$, and has exactly two premises, one of which is called its *left premise*, is labelled by A and by the integer 1, whereas the other is called its *right premise*, is labelled by B and by the integer 2;
- Every node of R labelled by ? is called a *why not*, has exactly one conclusion, labelled by a formula $?A$, and has at least one premise. Such a node has all of its premises labelled by A and is called a *dereliction* if it has exactly one premise, a *contraction* otherwise;
- Every node of R labelled by $\bullet$ is called a *conclusion* and possesses exactly one premise and no conclusions.

Moreover, a proof-structure is equipped with a total order of its conclusions: if $c_1, \ldots, c_k$ are the conclusions of R, then $k \geq 1$ and the premise of c_i is labelled by the integer i for all $i \in \{1, \ldots, k\}$. The *type* of an arc a of R is the formula labelling a. A *switching* of R is a function φ mapping every contraction of R to one of its premises. A *switching graph* of R induced by φ is a proof-structure obtained from R by replacing a with an arc having the same tail as a and a fresh conclusion as head, for every premise a of a contraction node c of R such that $\varphi(c) \neq a$. Finally, we say that R is a *proof-net* when the underlying undirected graph of every switching graph of R is acyclic, a *connected* proof-net if such graphs are also connected.

[1] The proof structure associated with any *MELL* sequent calculus proof without weakenings is a connected proof-net.

We now turn our attention from syntax to semantics, by recalling the notion of experiment. We assume that an interpretation of atomic formulas by coherent spaces, that is a function associating a coherent space $\mathcal{X}$ with any atomic formula X, is given. It is well known that one can extend such an interpretation to all formulas.

Notation 1.1. If α is a conclusion of an axiom n of a proof-structure, we denote $\alpha^\perp$ the other conclusion of n.

Definition 1.2. Let R be a proof-structure. An *experiment* of R is a map e which associates with every arc of type A of R an element of the web of the coherent space associated with A and satisfies the following conditions:

- If α is a conclusion of an axiom of R, then $e(\alpha) = e(\alpha^\perp)$;
- If a is the conclusion of a tensor of R with left premise b and right premise c, then $e(a) = (e(b), e(c))$;
- If a is the conclusion of a why not of R with premises $b_1, \dots, b_k$, then $e(a) = \{e(b_1), \dots, e(b_k)\}$.

We say that e is *injective* if $e(\alpha_1) \neq e(\alpha_2)$ for all distinct arcs α_1, α_2 of R of the same atomic type.

Remark 1.1. Recall that, if $\mathcal{B}$ is a coherent space, an element of the web of $?\mathcal{B}$ is a finite multiset of elements of the web of $\mathcal{B}$ which are pairwise incoherent. Therefore, the definition of experiment implicitly requires that, if $b_1, \dots, b_k$ are the premises of a contraction of R, then $e(b_i) \asymp e(b_j)$ for all $i, j \in \{1, \dots, k\}$. Hence, the existence of an injective experiment is not trivial. In addition, if e is injective, then $e(a) \neq e(a')$ for any two distinct arcs a, a' of the same type and the previous condition becomes $e(b_i) \smile e(b_j)$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$.

In the paper [TdF03] it is proven that, if there exists an injective experiment for every connected proof-net, then the coherent multiset-based semantics is injective for connected *MELL* proof-nets.

Conjecture. *If R is a connected proof-net, then there exists an injective experiment of R.*

Remark 1.2. If R is a proof-net that is not connected then, in general, there exists no injective experiment. This suggested the counterexamples in [TdF03] not only to the existence of an injective experiment, but also to the injectivity of the coherent multiset-based semantics for all *MELL* proof-nets.

The intuitive idea behind our work towards a proof of the previous conjecture is the following. If we ignore the coherence constraints, then a "relational" injective experiment always exists, because Remark 1.1 does not apply. Then we can start with no coherence relations at all and add them little by little, making sure that we are really making progress, meaning that we are not inducing coherence on two premises of any contraction. This motivates the definition of pre-experiment as a partial function, which will be our main tool in the next section.

2 The case of atomic contractions

Notation 2.1. If R is a proof-structure, we call P_R the set of unordered pairs $\{a, a'\}$ such that a, a' are distinct arcs of R of the same type and P_R^{at} the set of those elements $\{\alpha, \alpha'\} \in P_R$ such that the type of α, α' is atomic.

Definition 2.1. If R is a proof-structure, a *pre-experiment* of R is a partial function $e\colon P_R^{at} \to \{\frown, \smile\}$, where $\frown$ and $\smile$ are just two formal symbols, such that, for all $\{\alpha, \alpha'\} \in P_R^{at}$, if $e(\{\alpha, \alpha'\})$ is defined, then $e(\{\alpha^\perp, \alpha'^\perp\})$ is also defined and $e(\{\alpha, \alpha'\}) \neq e(\{\alpha^\perp, \alpha'^\perp\})$. In addition, the pre-experiment e uniquely extends to a partial function $\bar{e}\colon P_R \to \{\frown, \smile\}$, which is defined by induction on the type A of the arcs a, a' of a pair $\{a, a'\} \in P_R$:

- If A is an atomic type, then $\bar{e}(\{a, a'\}) = e(\{a, a'\})$;
- If $A = B \otimes C$ for some types B and C, then a, a' must be conclusions of two tensor nodes of R having left premises b, b' and right premises c, c' respectively. We define:

$$\bar{e}(\{a, a'\}) = \begin{cases} \frown & \text{if } \bar{e}(\{b, b'\}) = \bar{e}(\{c, c'\}) = \frown \\ \smile & \text{if } \bar{e}(\{b, b'\}) = \smile \text{ or } \bar{e}(\{c, c'\}) = \smile \end{cases}$$

  If neither of the conditions on the right holds, then the partial function $\bar{e}$ is undefined on the pair $\{a, a'\}$;

- If $A = ?B$ for some type B, then a, a' must be conclusions of why not nodes of R with premises $b_1, \dots, b_k$ and $b'_1, \dots, b'_{k'}$ respectively, for some strictly positive integers k and k'. We define:

$$\bar{e}(\{a, a'\}) = \begin{cases} \frown & \text{if } \exists\, i \in \{1, \dots, k\}, i' \in \{1, \dots, k'\} \colon \bar{e}(\{b_i, b'_{i'}\}) = \frown \\ \smile & \text{if } \forall\, i \in \{1, \dots, k\}, i' \in \{1, \dots, k'\} \colon \bar{e}(\{b_i, b'_{i'}\}) = \smile \end{cases}$$

  Again, if neither of the conditions on the right holds, the partial function $\bar{e}$ is undefined on $\{a, a'\}$.

Notation 2.2. If e is a pre-experiment of a proof-structure R and $\{a, a'\} \in P_R$, we can unambiguously denote $e(a, a')$ the element $\bar{e}(\{a, a'\})$.

The following definition introduces two key properties of pre-experiments: admissibility and atomicity. The former is the requirement that any two premises of any contraction are *not* coherent, whereas the latter states that two arcs of the same atomic type are incoherent if and only if they are premises of the same contraction.

Definition 2.2. A pre-experiment e of a proof-structure R is *admissible* if, for every two distinct premises a, a' of the same contraction of R, we do *not* have $e(a, a') = \frown$, meaning that $e(a, a')$ is either $\smile$ or undefined. We say that e is *atomic* if, for every $\{\alpha, \alpha'\} \in P_R^{at}$, we have $e(\alpha, \alpha') = \smile$ if and only if α, α' are premises of the same contraction of R.

Remark 2.1. If e is an admissible pre-experiment of a proof-structure R and e is defined on each element of P_R^{at}, then e is (essentially) an injective experiment of R.

Remark 2.2 (Connectedness is crucial). Let R be a proof-structure and let α, α' be distinct arcs of R of atomic type and premises of the same contraction of R. If $\alpha^\perp, \alpha'^\perp$ are premises of the same contraction of R, then R is not a connected proof-net and no pre-experiment of R is atomic.

Lemma 2.1. *Let R be a connected proof-net. Then there exists an atomic pre-experiment of R.*

Proof. For every $\{\alpha, \alpha'\} \in P_R^{at}$, if α, α' are premises of the same contraction of R, then neither $\alpha^\perp$ nor $\alpha'^\perp$ is, because R is a connected proof-net. We can thus define:

$$e(\alpha, \alpha') = \begin{cases} \frown & \text{if } \alpha^\perp, \alpha'^\perp \text{ are premises of the same contraction of } R \\ \smile & \text{if } \alpha, \alpha' \text{ are premises of the same contraction of } R \end{cases}$$

As usual, if neither of the two conditions on the right holds, the partial function e is undefined on $\{\alpha, \alpha'\}$. □

Remark 2.3. An immediate consequence of Lemma 2.1 is that, if R is a connected proof-net and every premise of a contraction of R is a conclusion of an axiom of R, then there exists an injective experiment of R.

We now want to prove the existence of an atomic and *admissible* pre-experiment (Proposition 2.1), which is a stronger result than the one expressed in Remark 2.3 and is relevant for the general case with no restrictions on the position of contractions. First, we need some basic notions about proof-structures.

Definition 2.3 (Tree above an arc, address of an arc)**.** Let R be a proof-structure. If a is an arc of R, we define the *distance of a from an axiom* as the smallest non-negative integer h for which there exists a descent path[2] $a_0 \dots a_h$ of R such that a_0 is a conclusion of an axiom of R and $a_h = a$. The *tree above* a, denoted T_a, is defined by induction on the distance d of a from an axiom:

- If $d = 0$, then the tree above a is the arc a in which the label of the head is replaced by $\bullet$.
- Otherwise, the arc a is the conclusion of a tensor or why not node n of R with premises $b_1, \dots, b_\ell$. The tree above a is produced by first identifying the head of b_i in T_{b_i} and the tail of a for all indices $i \in \{1, \dots, \ell\}$, then replacing the labels of the tail and of the head of a with the label of n and $\bullet$ respectively.

[2] A descent path is a sequence of arcs $a_0 \dots a_h$ such that the head of a_{i-1} is the tail of a_i for all $i \in \{1, \dots, h\}$.

Now let $a_0 \dots a_h$ be the unique descent path of R such that $a_0 = a$ and a_h is the premise of a conclusion of R. The non-negative integer h is called the *distance of a from a conclusion*. Finally, suppose that k is the number of conclusions of R. The *address* of a, denoted $\mathsf{adr}(a)$, is a finite word over the alphabet $\{1, \dots, k, \mathsf{L}, \mathsf{C}, \mathsf{R}\}$ and is defined by induction on the distance d of a from a conclusion:

- If $d = 0$, then a is the premise of a conclusion of R and the address of a is just the integer labelling a in R;
- Otherwise, the arc a must be a premise of a node n with conclusion b.
  - If n is a tensor, we define $\mathsf{adr}(a) = \mathsf{adr}(b)\mathsf{L}$ if a is the left premise of n, otherwise $\mathsf{adr}(a) = \mathsf{adr}(b)\mathsf{R}$;
  - Otherwise n is a why not and we define $\mathsf{adr}(a) = \mathsf{adr}(b)\mathsf{C}$.

An address is *atomic* if it is the address of an arc of atomic type of R.

One easily sees that the notion of address is strictly stronger than the notion of type: two arcs with the same address have the same type, but the converse does not hold in general. The idea is that addresses distinguish the distinct occurrences of A in $A \otimes A$.

Remark 2.4. If R is a proof-structure, then any two premises of a contraction of R have the same address by definition. Conversely, if two distinct arcs b_1, b_2 of R have the same address, then there are two distinct premises a_1, a_2 of a contraction of R such that $b_i \in T_{a_i}$ for each $i \in \{1, 2\}$.

Remark 2.5. If R is a proof-structure and a is an arc of R such that no contraction of R occurs in T_a, then any two distinct arcs of T_a have different addresses.

Notation 2.3. Let R be a proof-structure, let a be an arc of R such that no contraction of R occurs in T_a and let b be an arc of T_a with address w. Then b is denoted $a[w]$, unambiguously thanks to Remark 2.5.

We can finally state the following key result.

Lemma 2.2. *Let R be a connected proof-net and let e be an atomic pre-experiment of R. If a_1, a_2 are different arcs of R of the same type with addresses w_1, w_2 respectively and such that $e(a_1, a_2) = \frown$, then:*

(i) No contraction of R occurs in T_{a_i} for each $i \in \{1, 2\}$;

(ii) For every v such that $w_1 v$ and $w_2 v$ are atomic addresses, we have that $a_1[w_1 v]^\perp$ and $a_2[w_2 v]^\perp$ are premises of the same contraction of R.

Proof. By induction on the distance of a_1 from an axiom. □

Proposition 2.1. *If R is a connected proof-net, there exists an atomic and admissible pre-experiment of R.*

Proof. By Lemma 2.1, there exists an atomic pre-experiment of R. By using Lemma 2.2, we will prove that any atomic pre-experiment of R is admissible. We will do so by showing that, if there is an atomic pre-experiment e of R which is not admissible, then R is not connected. By Definition 2.2, there exist two distinct premises a, a' of a contraction c of R such that $e(a, a') = \frown$. Therefore, the two assertions of Lemma 2.2 apply. Now let w be the address of a, a' and let φ be any switching of R with $\varphi(c) = a'$ and $\varphi(c_v) \neq a[wv]^\perp$ for every v such that wv is an atomic address, where c_v is the contraction of R having $a[wv]^\perp$ and $a'[wv]^\perp$ among its premises. Let R^φ be a switching graph of R induced by φ. We claim that R^φ is not a connected graph. More precisely, there is no path $\theta = a_0 \dots a_k$ of R^φ such that $a_0 = a$ and $a_k = a'$. Indeed, let us suppose, for the sake of contradiction, that such a θ exists. Since $a \in T_a$ and $a' \notin T_a$, there exists $i \in \{1, \dots, k\}$ such that a_i is the first arc of θ which is not in T_a. Since $\varphi(c) = a'$, it must be the case that a_{i-1} and a_i are the conclusions of an axiom n of R. Let X be the type of a_i. We know that, if v is such that wv is the address of a_{i-1}, then a_i is a premise of c_v and $\varphi(c_v) \neq a_i$. Hence $i = k$, meaning that a' is a conclusion of n. In particular, since a, a' are premises of the same contraction, they have the same type, which has to be X. Therefore, we have $k = 1$. So a, a' are both conclusions of n, which gives the contradiction $X = X^\perp$. We can conclude that R^φ is not connected, so R is not a connected proof-net. □

References

[Car15] Daniel de Carvalho. "The relational model is injective for multiplicative exponential linear logic". In: *Preprint arXiv:1502.02404* (2015).

[Gir87] Jean-Yves Girard. "Linear logic". In: *Theoretical computer science* 50.1 (1987), pp. 1–101.

[Jol00] Thierry Joly. "Codages, séparabilité et représentation de fonctions en lambda-calcul simplement typé et dans d'autres systèmes de types". PhD thesis. Paris 7, 2000.

[TdF03] Lorenzo Tortora de Falco. "Obsessional experiments for linear logic proof-nets". In: *Mathematical Structures in Computer Science* 13.6 (2003), pp. 799–855.