An Internal Language for Autonomous Categories

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February 22, 2022

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Introduction

Logics	Categories

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λ-calculi	Cartesian Closed Categories
Intuitionistic type theories	Toposes

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Linear term calculi	Autonomous categories

Autonomous = symmetric monoidal closed

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IMLL and the linear term calculus

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IMLL is the intuitionistic multiplicative fragment of linear logic.
Formulas are defined by the following grammar:

$$A ::= X \mid 1 \mid A \otimes A \mid A \multimap A$$

Linear lambda calculus

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let t be $x \otimes y$ in u	$FV(t) \cap FV(u) = \varnothing$	$FV(t) \cup F$
	$x, y \in FV(u)$	$F = FV(u) \setminus \{x, y\}$

Typing system: axiom, exchange and unit rules

$$\frac{}{x : A \vdash x : A}\ (axiom)$$

$$\frac{\Gamma, x : A, y : B, \Delta \vdash t : C}{\Gamma, y : B, x : A, \Delta \vdash t : C}\ (exchange)$$

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Typing system: connectives

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash u : B}{\Gamma, \Delta \vdash t \otimes u : A \otimes B} \; (\otimes - \mathit{intro})$$

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$$\frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ (\multimap - elim)$$

Evaluation equations

$$\text{let } \star \text{ be } \star \text{ in } u \rightarrow_{\beta} u$$

$$\text{let } t \text{ be } \star \text{ in } \star \rightarrow_{\eta} t$$

$$\text{let } t \otimes u \text{ be } x \otimes y \text{ in } v \rightarrow_{\beta} v[t/x, u/y]$$

$$\text{let } t \text{ be } x \otimes y \text{ in } x \otimes y \rightarrow_{\eta} t$$

$$(\lambda x.t)u \rightarrow_{\beta} t[u/x]$$

$$\lambda x.(tx) \rightarrow_{\eta} t$$

Terms with holes

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We define three kinds of contexts by the following grammars:

$$C_0[] ::= x[] \mid C_0[]t \mid tC_0[]$$
$$C_1[] ::= C_0[] \mid t \otimes C_1[] \mid C_1[] \otimes t$$
$$C_2[] ::= C_1[] \mid \lambda x.C_2[]$$

Commutative conversion rules

$$\text{let } C_0[\text{let } t \text{ be } p \text{ in } v] \text{ be } \star \text{ in } w \equiv \text{let } t \text{ be } p \text{ in let } C_0[v] \text{ be } \star \text{ in } w$$

$$\text{let } C_1[\text{let } t \text{ be } p \text{ in } v] \text{ be } x \otimes y \text{ in } w \equiv \text{let } t \text{ be } p \text{ in let } C_1[v] \text{ be } x \otimes y \text{ in } w$$

$$\text{let } t \text{ be } \star \text{ in } C_2[u] \equiv C_2[\text{let } t \text{ be } \star \text{ in } u] \text{ if } \star \text{ does not occur in } C_2[\,]$$

$$\text{let } t \text{ be } x \otimes y \text{ in } C_2[u] \equiv C_2[\text{let } t \text{ be } x \otimes y \text{ in } u] \text{ if } x, y \notin FV(C_2[\,])$$

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A typed term is a typing judgement $\Gamma \vdash t : A$ where t is a plain term modulo the symmetric transitive reflexive closure of all reduction relations and modulo commutative conversion rules.

Interpreting the deduction system

The interpretation $[\![X]\!]$ in an autonomous category $\mathcal{C}$ of an atomic type X is an object of $\mathcal{C}$.

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We define the interpretation of a context by:

$$[\![-]\!] = 1$$
$$[\![\Gamma, A]\!] = [\![\Gamma]\!] \otimes [\![A]\!]$$

Bookkeeping morphisms

- $\text{split}_{\Gamma,\Delta} \colon [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \otimes [\![\Delta]\!]$

$$\begin{aligned}
\text{split}_{-,\Delta} &= \lambda_{\Delta}^{-1} \\
\text{split}_{\Gamma,-} &= \rho_{\Gamma}^{-1} \\
\text{split}_{\Gamma,A} &= \text{id}_{\Gamma,A} \\
\text{split}_{\Gamma,(\Sigma,A)} &= \alpha_{\Gamma,\Sigma,A} \circ (\text{split}_{\Gamma,\Sigma} \otimes \text{id}_A)
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- $\mathsf{join}_{\Gamma,\Delta} \colon [\![\Gamma]\!] \otimes [\![\Delta]\!] \to [\![\Gamma,\Delta]\!]$
- $\mathsf{exch}_{\Gamma,A,B,\Delta} \colon [\![\Gamma,A,B,\Delta]\!] \to [\![\Gamma,B,A,\Delta]\!]$

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Lemma (Substitution). If $\Gamma, x : A \vdash t : B$ and $\Delta \vdash u : A$, then:

$$\Gamma, \Delta \vdash t[u/x] : B$$

Moreover, we have the following condition:

$$[\![t[u/x]]\!] = [\![t]\!] \circ (\mathrm{id}_\Gamma \otimes [\![u]\!]) \circ \mathrm{split}_{\Gamma,\Delta}$$

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Theorem. If $\Gamma \vdash t : A$, $\Gamma \vdash s : A$ and $t \to_{\beta\eta} s$, then $[\![t]\!] = [\![s]\!]$.

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Theorem. If $\Gamma \vdash t : A$, $\Gamma \vdash s : A$ and $t \to_{\beta\eta} s$, then $[\![t]\!] = [\![s]\!]$.

It follows from this result that the interpretation of judgements is well defined on typed terms.

Categorical logic correspondence

The category of an autonomous theory

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$$(x : A, t : B) \sim (y : A, t[y/x] : B)$$

- $(x : B, t : C) \circ (y : A, u : B)$ is the morphism $(y : A, t[u/x] : C)$.
- The identity morphism on type A is $(x : A, x : A)$.

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- For all morphisms $f\colon A \to B$, $g\colon B \to C$, $h\colon A \otimes B \to C$ in $\mathcal{C}$, whenever t is typable with A, u is typable with B, v is typable with $A \multimap B$, r is typable with $A \otimes 1$, s is typable with $1 \otimes A$ and w is typable with $(A \otimes B) \otimes C$, we have the supplementary reductions:

$$[\mathrm{id}_A]t \to t$$

$$[g][f]t \to [g \circ f]t \qquad [f]t \otimes [g]u \to [f \otimes g](t \otimes u)$$

$$[\rho_A^{-1}]t \to t \otimes \star \qquad [\Lambda_{A,B,C}(h)]t \to \lambda x.[h](t \otimes x)$$

$$[\lambda_A^{-1}]t \to \star \otimes t \qquad [\mathrm{ev}_{A,B}](v \otimes u) \to vu$$

$$[\rho_A]r \to \text{let } r \text{ be } x \otimes y \text{ in let } y \text{ be } \star \text{ in } x$$

$$[\lambda_A]s \to \text{let } s \text{ be } y \otimes x \text{ in let } y \text{ be } \star \text{ in } x$$

$$[\alpha_{A,B,C}]w \to \text{let } w \text{ be } e \otimes z \text{ in let } e \text{ be } x \otimes y \text{ in } x \otimes (y \otimes z)$$

We give an interpretation in $\mathcal{C}$ of all judgements of $\mathbb{L}(\mathcal{C})$ by extending the definition to the following case:

$$[\![[f]t]\!]_{\Gamma \vdash B} = f \circ [\![t]\!]_{\Gamma \vdash A}$$

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One easily checks that the substitution lemma and the theorem expressing invariance of the semantics hold true.

Extending the categorical interpretation

We give an interpretation in $\mathcal{C}$ of all judgements of $\mathbb{L}(\mathcal{C})$ by extending the definition to the following case:

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One easily checks that the substitution lemma and the theorem expressing invariance of the semantics hold true. Thus, the interpretation is still well defined on typed terms.

$$\mathcal{C} \simeq \mathbb{C}(\mathbb{L}(\mathcal{C}))$$

Proof of the equivalence

$$F\colon \mathcal{C} \longrightarrow \mathbb{C}(\mathbb{L}(\mathcal{C})) \qquad\qquad G\colon \mathbb{C}(\mathbb{L}(\mathcal{C})) \longrightarrow \mathcal{C}$$

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$$\begin{aligned} F\colon \mathcal{C} &\longrightarrow \mathbb{C}(\mathbb{L}(\mathcal{C})) & G\colon \mathbb{C}(\mathbb{L}(\mathcal{C})) &\longrightarrow \mathcal{C} \\ A &\longmapsto A & A &\longmapsto A \\ (f\colon A \to B) &\longmapsto (x : A, [f]x : B) & (x : A, t : B) &\longmapsto [\![t]\!] \end{aligned}$$

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Now, if $f\colon A \to B$ is a morphism in $\mathcal{C}$ and $(x : A, t : B)$ is a morphism in $\mathbb{C}(\mathbb{L}(\mathcal{C}))$, then we have:

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$$GFf = G(x : A, [f]x : B) = [\![[f]x]\!]_{A \vdash B} = f \circ [\![x]\!]_{A \vdash A} = f$$

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$$\begin{array}{rcl} F\colon \mathcal{C} & \longrightarrow & \mathbb{C}(\mathbb{L}(\mathcal{C})) \\ A & \longmapsto & A \\ (f\colon A \to B) & \longmapsto & (x : A, [f]x : B) \end{array} \qquad \begin{array}{rcl} G\colon \mathbb{C}(\mathbb{L}(\mathcal{C})) & \longrightarrow & \mathcal{C} \\ A & \longmapsto & A \\ (x : A, t : B) & \longmapsto & [\![t]\!] \end{array}$$

One easily proves that F is a functor by using the supplementary relations for identities and for composition. The substitution lemma implies that G is a functor as well.

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$$FG(x : A, t : B) = F[\![t]\!] = (x : A, [\![t]\!]x : B)$$

For any integer $n \geqslant 0$ and for any terms $t_1, \ldots, t_n$ we define:

$$t_1 \otimes \cdots \otimes t_n := \begin{cases} \star & \text{if } n = 0 \\ (t_1 \otimes \cdots \otimes t_{n-1}) \otimes t_n & \text{otherwise} \end{cases}$$

Auxiliary results

For any integer $n \geqslant 0$ and for any terms $t_1, \dots, t_n$ we define:

$$t_1 \otimes \cdots \otimes t_n := \begin{cases} \star & \text{if } n = 0 \\ (t_1 \otimes \cdots \otimes t_{n-1}) \otimes t_n & \text{otherwise} \end{cases}$$

Lemma. If t_i is typable with A_i then, for all $k = 0, \dots, n$, we have:

$$[\text{split}_{(A_1,\dots,A_k),(A_{k+1},\dots,A_n)}](t_1 \otimes \cdots \otimes t_n) = (t_1 \otimes \cdots \otimes t_k) \otimes (t_{k+1} \otimes \cdots \otimes t_n)$$

$$[\text{join}_{(A_1,\dots,A_k),(A_{k+1},\dots,A_n)}]((t_1 \otimes \cdots \otimes t_k) \otimes (t_{k+1} \otimes \cdots \otimes t_n)) = t_1 \otimes \cdots \otimes t_n$$

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$$[\text{join}_{(A_1,\dots,A_k),(A_{k+1},\dots,A_n)}]((t_1 \otimes \cdots \otimes t_k) \otimes (t_{k+1} \otimes \cdots \otimes t_n)) = t_1 \otimes \cdots \otimes t_n$$

Lemma. If $x_1 : A_1, \dots, x_n : A_n \vdash t : B$, then in $\mathbb{L}(\mathcal{C})$ we have:

$$t = [\![t]\!](x_1 \otimes \cdots \otimes x_n)$$

Applications

The multisets of positive and negative type variables occurring in a type A are defined by mutual induction:

$$
\begin{aligned}
(X)^+ &= \{X\} & (X)^- &= \varnothing \\
(1)^+ &= \varnothing & (1)^- &= \varnothing \\
(A \otimes B)^+ &= (A)^+ \cup (B)^+ & (A \otimes B)^- &= (A)^- \cup (B)^- \\
(A \multimap B)^+ &= (A)^- \cup (B)^+ & (A \multimap B)^- &= (A)^+ \cup (B)^-
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A type A is binary if $(A)^+ = (A)^-$ and in this multiset each atomic formula appears exactly once.

Coherence theorem

The multisets of positive and negative type variables occurring in a type A are defined by mutual induction:

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Theorem (Coherence). If two arrows in the free autonomous category have the same binary type, then they are equal.

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