Aix-Marseille Université
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Master's thesis and internship report

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Injectivity in linear logic

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Marseille, 27 June 2022

The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs. Given two routes, it is not right to take either of these two or look for a third; it is necessary to investigate the area lying between the two routes.

David Hilbert

Abstract

We review some recent results concerning the question of injectivity in the multiplicative and exponential fragment of linear logic. We review the notion of obsessional experiment, the properties of the multiplicative case and we revisit a sufficient condition of local injectivity. Then, we study the Taylor expansion of λ-terms. We prove the commutation of Taylor support and head reduction. As a corollary, we also establish the property that, if a λ-term has a head normal form, then its head reduction terminates.

Keywords: linear logic, obsessional experiment, Taylor expansion, lambda calculus, proof net, denotational semantics.

Contents

Chapter 1

Introduction

What is a proof? This simple yet non trivial question received several answers in the history of thought and, to date, none of them is completely satisfactory.

For Gerhard Gentzen, a proof was essentially a tree built by applying rules to sequences of formulas called sequents. Figure 1.1 shows an example of such a proof in linear logic. If we accepted this as the definition of proof, then what we get by swapping the tensor rules would be considered as a different proof. This looks artificial and contrary to the intuition that proving $A \otimes B^\perp$ before or after $B \otimes C^\perp$ makes no difference. Such intuition is supported by the idea that we should not distinguish a proof which provides first a demonstration of the fact that A linearly implies B, then an argument to state that B linearly implies C, from the proof using the same arguments but in the opposite order. Hence, this is not a good definition of proof.

With the discovery of linear logic by Jean-Yves Girard in his article [Gir87], a notion of proof net was born. This new formalism allows us to capture more accurately the essence of a proof, since it identifies proofs which, in Gentzen's formalism, were morally the same. In particular, we can represent the sequent proofs mentioned before with a single proof net, illustrated in figure 1.2. Thus, proof nets contain no redundant information about the order of application of the rules.

We now wonder if it is possible to make "more identifications" than proof nets. With the idea of "measuring" the quality of the proof net formalism as a way to represent proofs, one can examine three equivalence relations on proof nets: the syntactic, semantic and observational equivalences. Basically, we can

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\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax \qquad \dfrac{\dfrac{}{\vdash B, B^\perp}\,ax \qquad \dfrac{}{\vdash C, C^\perp}\,ax}{\vdash B^\perp, B \otimes C^\perp, C}\,\otimes}{\vdash A \otimes B^\perp, B \otimes C^\perp, C, A^\perp}\,\otimes}{\vdash A \otimes B^\perp, B \otimes C^\perp, C \parr A^\perp}\,\parr
$$

Figure 1.1: A sequent calculus proof.

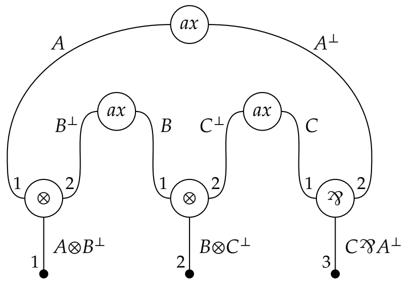

Figure 1.2: A proof net.

say that two proof nets are syntactically equivalent if we can transform one in the other by applying to them some specific syntactic manipulations (that is to say, those defining the cut elimination procedure). On the other hand, we have that two proof nets are semantically equivalent with respect to a model if that model interprets them as the same mathematical object. Lastly, two proof nets are observationally equivalent with respect to a particular type, usually called "ground" or "observable", if they normalize to the same value in each context of that type. Observational equivalence was born in the field of programming languages semantics: in order to compare two pieces of code (equivalently, by following the point of view of Curry-Howard correspondence, two fragments of proofs), it is natural to plug them into a program producing a ground value, such as a number and check whether they both produce the same output.

When comparing the syntactic and semantic equivalences, we address the question of injectivity: for any given model, we ask if it interprets syntactically non equivalent proofs with different objects. If, on the other hand, we compare the syntactic and observational equivalences, then we address the question of separability: we wonder if, given syntactically non equivalent proofs, there is a context where they behave differently. We mention here that the study of both injectivity and separability inspired this work, through an accurate analysis of the paper [Pag07] and part of the article [TdF03]. The reason we only deal with the question of injectivity in this work is that our original contributions mostly concern this matter.

To conclude, we mention that proof nets (more generally, proof structures) are also interesting as pure computational objects. In particular, the procedure of cut elimination is defined via local transformations. Thus, this formalism is suitable for examining the dynamics of normalization and for establishing the most fundamental properties of the system, such as strong normalization: the fact that any chain of reductions ends by producing a normal object.

We now outline the structure of the document. We start this chapter with a section in which we recall some basic notions and provide the notations which are used in the sequel. In section 1.2, some notions of graph theory are recalled and are used to give a precise definition of proof structures and proof nets. In section 1.3, we pinpoint significant paths and trees in proof structures and we study their properties. Then, in chapter 2, we revisit some crucial results of the article [TdF03] concerning the question of injectivity whereas, in chapter 3, we study the Taylor expansion of lambda terms and use it to prove some expected properties. In the conclusion, we look back at the accomplished results and we try to assess the current state of the art.

1.1 Notations

We specify in this section all notations we adopt for this document.

First of all, we use the symbol $\mathbb{N}$ to denote the set of non negative integers, that is $0, 1, 2 \ldots$ When we enumerate a list of elements $a_1, \ldots, a_k$, it is intended that k is a non negative integer and that the list is empty if $k = 0$. We adopt the same conventions for sets. In addition, if we are dealing with indices, we write $i_1, \ldots, i_h = 1, \ldots, k$ meaning that $i_1, \ldots, i_h \in \{1, \ldots, k\}$. Finally, whenever A is a finite subset of $\mathbb{N}$, we can consider the least upper bound of A with respect to the usual ordering of non negative integers. We denote this number $\sup A$. We remark, in particular, that $\sup A = 0$ if A is the empty set.

Let A be a set. We denote $\bigcup A$ the reunion of A, that is the set the elements of which are precisely the elements of the elements of A. To put it another way, the reunion of A is just the union of the elements of A. If B is a subset of A, the set difference of A and B, denoted $A \setminus B$, is the set whose elements are precisely the elements of A which are not elements of B.

A relation is a subset of a cartesian product of sets. If the cartesian product only involves n copies of the same set A, we talk about an n-ary relation on A. If R is a binary ($n = 2$) relation on A, we say that R is reflexive if it contains the identity relation on A, that is the set $\{(a, a) : a \in A\}$. On the other hand, we say that R is symmetric if, for any two elements $a, b \in A$, we have $(a, b) \in R$ if and only if $(b, a) \in R$. Lastly, the relation R is transitive when, for all $a, b, c \in A$, the conditions $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$. The reflexive closure of R, denoted $R^?$, is defined as the smallest reflexive relation on A which contains R or, equivalently, as the union of R with the identity relation on A. Now, if n is a non negative integer, then the n-th power of R, denoted R^n, is the relation on A defined inductively as follows:

- If $n = 0$, then R^n is the identity relation on A.
- If $n \geq 1$ then, for all $a, c \in A$, we have $(a, c) \in R^n$ when there exists $b \in A$ such that $(a, b) \in R^{n-1}$ and $(b, c) \in R$.

Lastly, the reflexive transitive closure of R, denoted R^*, is the smallest reflexive and transitive relation on A which contains R. Equivalently, we define:

$$R^* := \bigcup \{R^n : n \in \mathbb{N}\}$$

We call permutation over $1, \ldots, n$ a bijective map from the set $\{1, \ldots, n\}$ to itself. If A is a finite set, a finite multiset with support A is a map from A to the set of strictly positive integers. Now, for every $a \in A$, let μ_a be a finite multiset with support a. Then, let μ be a finite multiset with support $\{\mu_a : a \in A\}$. To be capable of expressing a notion of multiset union indexed over a finite multiset, we define the finite multiset $\bigcup\mu$, called reunion of μ. It is a finite multiset with support the reunion of A and such that, whenever $i \in \bigcup A$, we have:

$$(\bigcup\mu)(i) := \sum_{i \in a \in A} \mu(\mu_a)\mu_a(i)$$

Finally, if μ is a finite multiset with support A, an element x of A is also called an element of μ and we write $x \in \mu$. The multiplicity of x in μ is just $\mu(x)$. Now, to facilitate the intuition that finite multisets are a generalization of finite sets, we introduce the notation $\{n_1[x_1], \ldots, n_k[x_k]\}$ to talk about the finite multiset μ with support $\{x_1, \ldots, x_k\}$ and such that $\mu(x_i) = n_i$ for all indices $i = 1, \ldots, k$. The notation $n[x]$ can therefore be understood as n distinct occurrences of the element x. It is then natural to write just x instead of $1[x]$. From now on, for the sake of simplicity, we talk about multisets meaning finite multisets.

A finite word over a set A is just a finite sequence of elements of A and it is denoted $w_1 \ldots w_n$. The length of $w_1 \ldots w_n$ is the non negative integer n. When $n = 0$, we have the empty word, denoted ε. The set A is called alphabet and its elements are called letters. We can define the juxtaposition of two finite words $u = u_1 \ldots u_n$ and $v = v_1 \ldots v_m$ as the finite word $u_1 \ldots u_n v_1 \ldots v_m$, denoted uv. Juxtaposition is obviously an associative operation on finite words. Finally, we say that v is a factor of u if there exist two finite words x and y over A such that $u = xvy$.

Lastly, we recall the notion of *MELL* formula. Throughout the document, it is assumed that an infinite set of symbols called atomic formulas is fixed once for all and that we have a symmetric binary relation *NOT* on this set such that, for every atomic formula X, there exists a unique atomic formula Y satisfying $(X, Y) \in \mathit{NOT}$. Now let **L** be the alphabet made up of the atomic formulas, the symbols $\otimes$, $⅋$, $!$, $?$ called tensor, par, bang and why not respectively, the comma , and the parentheses (and). The set of *MELL* formulas (or formulas of *MELL*) is the set of finite words over **L** produced by the following inductive definition:

- If X is an atomic formula, then X is an *MELL* formula.
- If A and B are *MELL* formulas, then so are $(A \otimes B)$ and $(A ⅋ B)$.
- If A is an *MELL* formula, then so are $!A$ and $?A$.

Outermost parentheses are not written. It is important to notice that we do not deal with units in this work. We then define a function, called linear negation, which associates with every *MELL* formula A another *MELL* formula, denoted $A^\perp$. This function is defined inductively as follows:

- If X is an atomic formula, then $X^\perp$ is the only atomic formula Y such that $(X, Y) \in \mathit{NOT}$.
- If A and B are *MELL* formulas, then:

$$(A \otimes B)^\perp = A^\perp ⅋ B^\perp$$
$$(A ⅋ B)^\perp = A^\perp \otimes B^\perp$$

- If A is an *MELL* formula, then $(!A)^\perp = ?A^\perp$ and $(?A)^\perp = !A^\perp$.

1.2 Proof nets

Since there exist a lot of variants for the definition of proof net in the literature, it is appropriate to specify what we mean exactly by this notion. As we want to define proof nets as particular graphs, first of all we recall a possible definition of graph.

Definition 1.1. A *graph* is a tuple $G = (V, A, \ell_V, \ell_A)$ consisting in a finite set V whose elements are called *vertices*, an irreflexive binary relation A on V whose elements are called *arcs*, a function ℓ_V, called *vertex labeling*, associating to each vertex of G a multiset whose elements are called *labels* and a function ℓ_A, called *arc labeling*, associating with each arc of G a multiset the elements of which are also called *labels*. A vertex or arc of G is *unlabeled* if the multiset associated with it is empty. In addition, when $a = (x, y)$ is an arc of G, we say that x is the *tail* of a and y is the *head* of a. We also say that x and y are *adjacent* and that a vertex z is *incident* to a if $z = x$ or $z = y$. Furthermore, we call *undirected closure* of G any graph having V as set of vertices and the symmetric closure of A as set of arcs.

In what follows, we always want to consider graphs modulo isomorphism. We then recall this notion.

Convention. For simplicity, if $\Phi\colon U \to V$ is a map, we denote in the same way the function from $U \times U$ to $V \times V$ which associates to a pair (u, u') the ordered couple $(\Phi(u), \Phi(u'))$.

Definition 1.2. Let $G = (U, A, \ell_U, \ell_A)$ and $H = (V, B, \ell_V, \ell_B)$ be two graphs. An *isomorphism* between G and H is a bijection $\Phi\colon U \to V$ such that, if $a \in U \times U$, then $a \in A$ if and only if $\Phi(a) \in B$ and such that $\ell_U = \ell_V \circ \Phi$, $\ell_A = \ell_B \circ \Phi$.

We say that G and H are *isomorphic* if there exists an isomorphism between them and, in that case, we write $G = H$.

It is well known that one can manipulate graphs by making disjoint unions of graphs, by deleting or contracting arcs and by suppressing vertices. Each of these operations on graphs is precisely defined in [BM08].

Definition 1.3. Let u and v be non adjacent vertices of a graph G. If $u = v$, the graph we get from G by identifying u and v is G. If not, it is defined as follows:

- We first suppress u and v (necessarily with all arcs they are incident to).
- We then add a vertex w which is labeled with all labels of u and v.
- Finally, for all vertices x such that (x, u) or (x, v) is an arc of G, we add an arc (x, w) with all labels of both arcs, if they exist, otherwise all labels of the existing arc. In the same manner, for every vertex y such that (u, y) or (v, y) is an arc of G, we add an arc (w, y) having all labels of both arcs, if they exist, otherwise all labels of the existing arc.

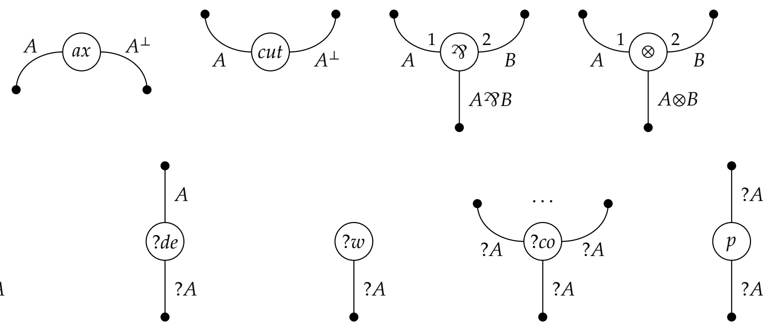

Figure 1.3: *MELL* links.

Remark 1.1. The previous definition only makes sense because the arc relation on G is irreflexive and we pick u and v as non adjacent vertices.

Definition 1.4. Let $a = (u, v)$ and $b = (x, y)$ be two distinct arcs with the same label of a graph G and suppose that u and x, v and y are pairs of non adjacent vertices. The graph obtained from G by identifying a and b is the graph we get by first deleting one of the arcs a or b and then identifying u with x and v with y.

Remark 1.2. Since the operation of identifying distinct arcs in a graph is local, if we apply it repeatedly, then we will end up with the same graph no matter the order in which we identify the arcs.

Convention. We represent arcs oriented from top to bottom, meaning that the head is always lower than the tail.

Definition 1.5. We call *MELL links* the graphs depicted in figure 1.3 and, more specifically, from left to right, top to bottom, we have the *axiom*, *cut*, *par*, *tensor*, *of course*, *dereliction*, *weakening*, *contraction* and *pax* links. Each possesses exactly one labeled vertex n which, with a slight abuse of notation, is usually referred to as a link, just as the graph it belongs to. All other vertices are unlabeled. We say that an arc is a *premise* if n is its head, a *conclusion* if n is its tail. The *arity* of a link is the number of its premises. Contraction links may have arity k for any integer $k \geq 2$. We call *type* of an arc the formula of *MELL* which labels that arc. In addition, the axiom and weakening links are called *initial links*, whereas the weakening and contraction links are called *structural links*. A *fragment* of *MELL* is a subset of the set of *MELL* links. In particular, the fragment consisting of the axiom, cut, par and tensor links is denoted *MLL*.

Now let R be the graph obtained by first taking a non trivial disjoint union of *MELL* links, possibly with repetitions of the same link and then identifying couples of arcs, labeled by the same *MELL* formula, such that one is a premise, the other is a conclusion, they both have an unlabeled vertex as a tail or a head and they do not share the other vertex. Now suppose that every arc of R is the conclusion of a unique link and the premise of at most one link. Assume that F

is the set of arcs of R which are not the premise of any link and define s as the cardinality of F. We call a *pseudo proof structure* the graph produced from R by ordering the elements of F, that is, by labeling them with $1, \ldots, s$ in such a way that different arcs possess distinct labels. The elements of F and their types are both called *conclusions* of R. A link of R is called *terminal* when its conclusion is a conclusion of R. We define the *conclusion* of R as the list $\Gamma = C_1, \ldots, C_s$ where C_i is the type of an arc $c_i \in F$ and c_i is labeled by i for all $i = 1, \ldots, s$. A pseudo proof structure is *cut free* if it was produced without using the cut link and we say that it belongs to a particular fragment of *MELL* if it was obtained by only using links of that fragment.

A *proof structure* is a pseudo proof structure R satisfying the two following conditions:

(1) *!-box condition.* With any of course link n is associated a unique subgraph B of R which is a pseudo proof structure such that one of its conclusions is the conclusion of n and any other conclusion is the conclusion of a pax link. We call B an *exponential box* or just a *box* and we visually represent it by using a rectangular frame. We call n a *front door* or *pal door* of B.

    With every pax link n is associated a unique exponential box B such that one among the conclusions of B is the conclusion of n. We say that n is a *pax door* of B.

(2) *Nesting condition.* Any two boxes are disjoint or included one in the other.

Let B be a box of a proof structure R. We say that an arc of B is contained in B if it is not a conclusion of B. In the same way, a link of B is contained in B if it is not a door of B. Finally, we say that a proof structure is contained in B when its arcs are all contained in B. We can now define the *depth* of B as the number of boxes in which B is strictly contained. We can also define the depth of a link or of an arc of R as the number of boxes in which it is contained. Lastly, the *box complexity* of R is 0, if R has no boxes, otherwise it is defined as $m + 1$, where m is the maximal depth of its boxes.

Remark 1.3. The labels 1 and 2 on the premises of par and tensor links allow us to distinguish the left and the right premise.

Remark 1.4. All pseudo proof structures of *MLL* are proof structures.

Remark 1.5. The depth of a box is well defined thanks to the immediate remark that boxes are proof structures.

Remark 1.6. The biggest proof structure contained in a box is simply the proof structure obtained from that box by deleting its conclusions.

Lemma 1.1. *Let B be a box of depth p. The doors of B have depth p, their conclusions have depth p and their premises have depth $p + 1$.*

Proof. We reason by induction on p. First, suppose $p = 0$. If there existed boxes containing a door or the conclusion of a door of B, then they would contain the premise of that door, because boxes are pseudo proof structures. Therefore, by the nesting condition, they would also contain B, contradicting the hypothesis that the depth of B is $p = 0$. We can then conclude that the doors of B and their conclusions have depth 0. Their premises have depth at least 1, since B is a box containing them. If a box contains the premise of a door of B, it is contained in B by the nesting condition and the hypothesis that B has depth 0. Then the box is B, because they have a common door and by definition the box associated to a door is unique. Hence, the premises of the doors of B have depth 1.

Now assume $p \geq 1$ and that the result holds for any box of depth $p - 1$. Let B_0 be the box of depth 0 containing B and let R_0 be the biggest proof structure contained in B_0. We know for sure that B is also a box of R_0, because boxes are proof structures and R_0 is maximal among those which B_0 contains. Since the depth of B in R_0 is $p - 1$, by applying the inductive hypothesis and taking into account the presence of the box B_0 we get the desired conclusion. □

Lemma 1.2. *Let n be a link of depth p in a proof structure R. If n is not the door of a box, then its premises and its conclusion have depth p.*

Proof. Again, we reason by induction on p. Suppose $p = 0$. If the conclusion of n were contained in a box, then n would, too. This goes against the hypothesis that n has depth $p = 0$, hence the conclusion of n has depth 0. If a premise of n were contained in a box B, then this premise would not be a conclusion of that box. Since the conclusion of n has depth 0, it must be a conclusion of B. Then n would be a door of B, contradicting our hypotheses. Therefore, the premises of n have depth 0. In the case $p \geq 1$, we repeat the argument seen in the proof of the previous result. □

We recall the notion of path. It is important to stress that, for this definition of path, the orientation of arcs matters.

Definition 1.6. Let G be a graph. A *path* in G is a sequence of arcs of G denoted $a_0 \dots a_k$ for which there are pairwise distinct vertices $x_0, \dots, x_k$ of G such that x_{i-1} is the head of a_{i-1} and the tail of a_i for all indices $i = 1, \dots, k$. Moreover, a path $a_0 \dots a_k$ is called a *cycle* when the head of a_k and the tail of a_0 are the same vertex. We say that G is *acyclic* if there exists no cycle in G. On the other hand, the graph G is *connected* if, for any two distinct vertices x and y of G, there is a path $a_0 \dots a_k$ in G from x to y, that is, such that the tail of a_0 is x and the head of a_k is y.

We finally provide the definition of proof net.

Definition 1.7. A *switching* of a proof structure R is a set S whose elements are exactly one premise of each par and contraction link of R with depth zero. We can then define the *correctness graph* of R with respect to the switching S as the graph $S(R)$ produced from R by deleting every premise of a par or contraction link with depth zero which is not an element of S and by contracting every arc contained in a box of R.

A proof structure R is an *AC proof net* (respectively, an *ACC proof net*) when, for every box B of R with depth zero, the maximal proof structure contained in

B is an *AC* proof net (respectively, *ACC* proof net) and R satisfies the so called *acyclicity criterion* (respectively, *acyclicity and connectedness criterion*): whenever S is a switching of R, the unlabeled undirected closure of $S(R)$ is a forest, that is an acyclic graph (respectively, a tree, that is an acyclic and connected graph).

Remark 1.7. The previous definition reveals that there are actually two notions of proof net. One can prove that the notion of *AC* proof net corresponds to the standard notion of sequent calculus proof, provided that one adds to the usual rules the so called mix rule, allowing to infer $\vdash \Gamma, \Delta$ from any two sequents $\vdash \Gamma$ and $\vdash \Delta$. On the other hand, an *ACC* proof net can be transformed in a sequent calculus proof without the need for supplementary rules, but only if it belongs to a fragment of *MELL* not containing the weakening link.

Convention. For now on, whenever we refer to proof nets, we mean either *AC* proof nets or *ACC* proof nets. In addition, when we mention proof nets several times in the same context, we consistently refer every time to the same kind of proof net.

1.3 Paths and trees in proof structures

Convention. In this section, unless otherwise specified, we suppose that R is a cut free proof structure.

Remark 1.8. Let a be an arc of R. If $a_0 \dots a_k$ and $b_0 \dots b_h$ are paths of R such that $a_0 = b_0 = a$, then one is the prefix of the other. In fact, assuming $h \leq k$ without loss of generality and bearing in mind that we always consider directed paths, we must have $a_i = b_i$ for all indices $i = 0, \dots, h$ by definition of proof structure (one easily checks this by induction on i). In particular, there is just one path of length k whose first arc is a. This motivates the following definition.

Definition 1.8. We say that the path $a_0 \dots a_k$ of R is:

- Issued by an arc a of R if $a_0 = a$.
- Issued by a link of R if it is issued by the conclusion of that link.
- A *descending path* if a_k is a conclusion of R.

Remark 1.9. Let $a_0 \dots a_k$ be a path of R. Then, by the two previous lemmas, the depth of a_{i-1} is greater than or equal to the depth of a_i for all $i = 1, \dots, k$. Also, the equality holds if and only if the arc a_i is not the conclusion of an of course or pax link.

We now give a useful characterization for the notion of depth of a link or of an arc, which has the quality of not involving boxes.

Proposition 1.1. *The depth of a link or of an arc in R is p if and only if the descending path issued by that link or arc crosses the premises of exactly p of course or pax links of R.*

Proof. Since the depth of a link is always equal to the depth of its conclusion, it is enough to consider the case of an arc a. Let $a_0 \dots a_k$ be the descending path issued by a. We start by proving the reverse implication of the statement. Our hypothesis is that $a_0 \dots a_k$ crosses the premises of precisely p of course or pax links. Suppose that these links are $n_1, \dots, n_p$ and that n_{i-1} is crossed before n_i for $i = 2, \dots, p$. Let $B_1, \dots, B_p$ be the unique associated boxes. For $i = 2, \dots, p$, since B_i is a proof structure, it must contain the premise of n_{i-1}. By the nesting condition, the box B_{i-1} is included in B_i. Moreover, this is a strict inclusion: by remark 1.9, the depth of n_{i-1} is strictly greater than that of n_i, hence n_{i-1} and n_i cannot be two doors of the same box by lemma 1.1. Now assume that B is a box containing a. Since a_k is not contained in a box, we can consider the first index $i \in \{1, \dots, k\}$ such that a_i is not contained in B. Then a_{i-1} is contained in B and this entails that a_i is a conclusion of B. Thus, the arc a_{i-1} is the premise of an of course or pax link n, which is in particular a door of B. However, we must have $n = n_j$ for some $j \in \{1, \dots, p\}$ and therefore $B = B_j$, because the box associated to a door is unique. We have then proven that there is no other box containing a aside from $B_1, \dots, B_p$. Hence, the depth of a is p.

The direct implication is now immediate: obviously, for some non negative integer q, the path $a_0 \dots a_k$ crosses the premises of precisely q of course or pax links of R. Then, by the reverse implication, the depth of a is q. The hypotheses impose $p = q$ and we are done. ◻

We have the following characterization of exponential boxes in the absence of weakening links.

Proposition 1.2. *Let R be an ACC proof net, let $\bar{R}$ be an undirected closure of R with the same labels as R on vertices and let m and n be two of course or pax links of R with depth p. Then m and n are doors of the same box if and only if there is a path $a_0 \dots a_k$ of $\bar{R}$ such that the tail of a_0 is m, the head of a_k is n and every link crossed by this path has depth strictly greater than p.*

Proof. By definition of *ACC* proof net, if m and n are doors of a box B, then the maximal proof structure R' contained in that box is an *ACC* proof net. Thus, its undirected closure $\bar{R}'$ with the same labels as R' on vertices is connected. Then there exists a path $a_0 \dots a_k$ of $\bar{R}'$ such that the tail of a_0 is m and the head of a_k is n. In addition, since the depth of B is p by lemma 1.1, any link crossed by the path $a_0 \dots a_k$ has depth strictly greater than p. Lastly, this path is in $\bar{R}$, because $\bar{R}'$ is a subgraph of $\bar{R}$.

We then prove the reverse implication. Let B be the box associated with m. From now on, if (u, v) is an arc of R, we say that (v, u) is contained in B if (u, v) is. Then we claim that a_i is contained in B for every index $i = 0, \dots, k$. In fact, if this were not the case, we could consider an index $h \in \{0, \dots, k\}$ such that a_i is contained in B for all $i = 0, \dots, h-1$ and a_h is not contained in B. Since the tail of a_0 is m, we must have $h \geq 1$. We can then observe that a_{h-1} is the premise of a door of B with conclusion a_h. This door is a link with depth p by lemma 1.1, hence we have a contradiction with our hypotheses. We can conclude that a_k is contained in B. By the hypothesis that the head of a_k is n, it is also contained in the box B' associated with n. Since B and B' satisfy the nesting condition, they are necessarily included one in the other. Finally, since they both have depth p by lemma 1.1, we must have $B = B'$. □

We now need another general definition about graphs.

Definition 1.9. Let $a = (u, w)$ be an arc of a graph G. The graph obtained from G by splitting a is the graph we get by deleting a, adding an unlabeled vertex v and the arcs (u, v) and (v, w), each with the same labels as a.

The analogous of remark 1.2 holds.

Definition 1.10. Let n be a terminal link of R and let R' be a graph. We say that R' is obtained from R by *removal* of n if it is produced by splitting the premises of n, then suppressing n and ordering the arcs which are not premises of a link in such a way that:

- If a and b are conclusions of R and not conclusions of n, then a precedes b in R' whenever this happens in R.
- If a is a conclusion of R and b is a premise of n, then a precedes b in R' if and only if a precedes the conclusion of n in R.
- If n is a par or tensor link with left premise a and right premise b, then a precedes b in R'.

Remark 1.10. If R is a pseudo proof structure or a proof structure, then so is R'.

Definition 1.11. The *open graph* of a pseudo proof structure R is defined as the graph $\mathbf{O}(R)$ we get by splitting the conclusions of the axiom links of R and by suppressing the labeled vertices (necessarily with the two incident arcs) of any axiom link of R.

Remark 1.11. In some cases, we want to "forget" that a specific proof structure R verifies the conditions on boxes. In other words, we do not want to consider the proof structure R, but rather the underlying pseudo proof structure, which we denote $\mathbf{P}(R)$. Intuitively, we can visualize $\mathbf{P}(R)$ as the graph we produce by erasing the rectangular frames of all boxes of R. The major difference is that in $\mathbf{P}(R)$ we do not know anymore which of course and pax links are the doors of the same box.

Notation. For simplicity, we denote $\mathbf{OP}(R)$ the open graph of $\mathbf{P}(R)$.

Remark 1.12. There are evident bijections associating with any path of R a path of $\mathbf{P}(R)$ and one of $\mathbf{OP}(R)$. Besides, these bijections preserve descending paths. Hence, thanks to proposition 1.1, the depth of a link or of an arc is well defined in each of these graphs and coincides with the depth in R.

Definition 1.12. The *distance* in R of an arc a from an initial link is the smallest non negative integer k for which there exists a path $a_0 \ldots a_k$ of R such that a_0 is a conclusion of an initial link and $a_k = a$.

Remark 1.13. A path as in the previous definition always exists, by definition of proof structure. In addition, notice that the distance of a from an initial link is zero if and only if a is a conclusion of an initial link.

Definition 1.13. If n is a link of R with conclusion a, the tree of a in R, denoted T_a^R or just T_a if there is no ambiguity, is defined as follows, by induction on the distance d of a from an initial link:

- If $d = 0$, then the tree of a in R is just a.
- If $d \geq 1$ then, for some integer $k \geq 1$, the arc a is the conclusion of a link n with premises $a_1, \ldots, a_k$. The tree of a in R is the graph produced by first taking the disjoint union of the graphs $T_{a_1}, \ldots, T_{a_k}$ and of the link n, then identifying the premise a_i of n with the corresponding arc of T_{a_i} for each index $i = 1, \ldots, k$.

Remark 1.14. Let a and b be arcs of R. Then one can easily prove, by induction on the distance of a from an initial link, that the following properties hold:

(i) The graph T_a is a tree, that is, connected and acyclic.

(ii) The graph T_a is a subgraph of $\mathbf{OP}(R)$.

(iii) If b is an arc of T_a, then T_b is a subgraph of T_a.

(iv) If b is an arc of T_a, there exists a path $a_0 \ldots a_k$ of R with $a_0 = b$ and $a_k = a$.

(v) A path $a_0 \ldots a_k$ of R with $a_k = a$ is a path of T_a.

In addition to these:

(vi) The graphs T_a and T_b do not share any arc if and only if a is not an arc of T_b and b is not an arc of T_a (by properties (iv), (v) and remark 1.8).

(vii) If a and b are two distinct conclusions of R, then the graphs T_a and T_b are disjoint in $\mathbf{OP}(R)$ (by properties (iv) and (vi)).

(viii) If $a_1, \ldots, a_k$ are the conclusions of R, then $\mathbf{OP}(R)$ is the disjoint union of the graphs $T_{a_1}, \ldots, T_{a_k}$ (by properties (ii), (v) and (vii)).

Chapter 2

Injectivity and obsessionality

We deal with the question of injectivity for the multiplicative and exponential fragment of linear logic, which was tackled by Lorenzo Tortora de Falco in his article [TdF03]. This will be our primary reference in this chapter. We precisely state our problem in mathematical terms and then we revisit the main results, providing extra details, examples and missing proofs. As in our reference, we restrict our study to the coherent multiset based semantics of linear logic. This is justified by a proposition, suggested by Thomas Ehrhard and established in the paper [TdF00], which allows to extend all positive results about injectivity to the case of relational semantics.

We focus on the key notion of obsessional experiment, thanks to which we get a sufficient condition of local injectivity: if R is a (standard) *ACC* proof net and there exists an experiment of R with some particular properties, then R is "alone" in its semantic equivalence class, that is, there are no other (standard) *ACC* proof nets with the same semantics as R.

This tool is used in [TdF03] to provide a positive or negative answer to the question of injectivity in specific fragments of linear logic. Notably, the author proves that the answer is positive in the "weakly polarized" fragment of linear logic, which contains the simply typed lambda calculus. On the other hand, he builds a counterexample to the injectivity of coherent semantics in the general case.

The chapter is structured as follows. The first section is aimed at providing the ingredients we need to precisely state the question of injectivity. We recall, in particular, the definitions of coherent space and experiment and we give an original definition which formalizes occurrences of subformula. In section 2.2, we provide a positive answer to the question of injectivity in the multiplicative fragment of linear logic. In section 2.3, we introduce obsessional experiments, we fix some ambiguities in the statement of lemma 2.2.1 of [TdF03] and finally provide an original proof of this result. In section 2.4, we then review another crucial result of [TdF03] and we give some details on intermediary results. We finally review, in section 2.5, a sufficient condition of local injectivity and then we comment on the result achieved.

2.1 Preliminaries

In order to formally state the question of injectivity, we have to precisely define the two equivalence relations we want to compare. First and foremost, we give some definitions.

Definition 2.1. We say that a cut free pseudo proof structure R is *standard* if all conclusions of axiom links of R have atomic types and if every conclusion of a structural link is not a premise of a pax or contraction link.

Remark 2.1. One easily proves that, by performing η-expansions of axioms, by erasing structural links and by commuting them with pax links, every cut free pseudo proof structure can be translated into a unique standard pseudo proof structure. In particular, we can turn any proof structure into a unique standard proof structure.

Remark 2.2. Because of the position of the structural links in a standard pseudo proof structure, all premises of pax or contraction links must be conclusions of pax or dereliction links.

The previous remark justifies the following definition.

Definition 2.2. Suppose that a is the conclusion of a pax or dereliction link m in a standard pseudo proof structure R. The dereliction link above a (or above m) is the dereliction link with conclusion c_0 for which there are a non negative integer h and h arcs $c_1, \ldots, c_h$ such that $c_0 \ldots c_h$ is a path of R, $c_h = a$ and c_i is the conclusion of a pax link for all indices $i = 1, \ldots, h-1$. If a is the conclusion of a contraction link m with premises $a_1, \ldots, a_k$, we call dereliction links above m the dereliction links above $a_1, \ldots, a_k$.

In some cases, it is important to stress the difference between subformulas and occurrences of subformula: if on one hand A is a subformula of $A \otimes A$, we have two distinct occurrences of A in $A \otimes A$, in short the one on the left and the one on the right. This difference is actually crucial in the sequel, which is why we provide an original definition to formalize our intuition.

Definition 2.3. Let A be an *MELL* formula. The set of *occurrences of subformula* of A is a subset $osf(A)$ of the cartesian product of the set of *MELL* formulas and the set of finite words over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$, determined by means of the following inductive definition:

- If A is an atomic formula, then $osf(A) := \{(A, \varepsilon)\}$.
- If $A = B \invamp C$ or $A = B \otimes C$, then $osf(A)$ is the set whose elements are the ordered pair (A, ε), every pair $(F, \mathsf{L}w)$ such that $(F, w) \in osf(B)$ and every couple $(F, \mathsf{R}w)$ such that $(F, w) \in osf(C)$.
- If $A = !B$ or $A = ?B$, then $osf(A)$ is the set whose elements are the ordered pair (A, ε) and every couple $(F, \mathsf{C}w)$ such that $(F, w) \in osf(B)$.

We now recall the notions of coherent space and experiment.

Definition 2.4. A *coherent space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \supset\!\!\subset)$ where $|\mathcal{A}|$ is a set called *web* and $\supset\!\!\subset$ is a binary reflexive and symmetric relation on the web called *coherence*. Let $x, y \in |\mathcal{A}|$. We adopt the following terminologies and notations:

- We say that x and y are *coherent* and we write $x \supset\!\!\subset y[\mathcal{A}]$ if $(x, y) \in \supset\!\!\subset$.
- We say that x and y are *strictly coherent* and we write $x \frown y[\mathcal{A}]$ if they are coherent and $x \neq y$.
- We say that x and y are *incoherent* and we write $x \asymp y[\mathcal{A}]$ if they are not strictly coherent.
- We say that x and y are *strictly incoherent* and we write $x \smile y[\mathcal{A}]$ if x and y are not coherent.

In addition, we call *clique* of $\mathcal{A}$ any finite multiset of elements of $|\mathcal{A}|$ which are pairwise coherent. Finally, suppose that it is given an interpretation of atomic formulas of *MELL* by coherent spaces, that is a function associating a coherent space $\mathcal{A}$ with every atomic formula A of *MELL*. Then we can extend this map on the set of all *MELL* formulas, by the following inductive definition:

- If $\mathcal{A}$ is a coherent space, then we define $\mathcal{A}^\perp$ as the coherent space whose web is $|\mathcal{A}|$ and such that, for all elements $x, y \in |\mathcal{A}|$, we have $x \supset\!\!\subset y[\mathcal{A}^\perp]$ if and only if $x \asymp y[\mathcal{A}]$.
- If $\mathcal{A}$ and $\mathcal{B}$ are coherent spaces, then $\mathcal{A} \otimes \mathcal{B}$ is the coherent space having the cartesian product of $|\mathcal{A}|$ and $|\mathcal{B}|$ as web and such that, when we have $x, x' \in |\mathcal{A}|$ and $y, y' \in |\mathcal{B}|$, the condition $(x, y) \supset\!\!\subset (x', y')[\mathcal{A} \otimes \mathcal{B}]$ holds if and only if $x \supset\!\!\subset x'[\mathcal{A}]$ and $y \supset\!\!\subset y'[\mathcal{B}]$.
- If $\mathcal{A}$ and $\mathcal{B}$ are coherent spaces, then $\mathcal{A} \parr \mathcal{B}$ is just $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$.
- If $\mathcal{A}$ is a coherent space, then $!\mathcal{A}$ is the coherent space whose web is the set of cliques of $\mathcal{A}$ and such that, whenever x and y are cliques of $\mathcal{A}$, we have $x \supset\!\!\subset y[!\mathcal{A}]$ if and only if $\bigcup\{x, y\}$ is a clique of $\mathcal{A}$.
- If $\mathcal{A}$ is a coherent space, then $?\mathcal{A}$ is just $(!\mathcal{A}^\perp)^\perp$.

Remark 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two coherent spaces. An immediate consequence of the previous definition is that, when we have $x, x' \in |\mathcal{A}|$ and $y, y' \in |\mathcal{B}|$, the condition $(x, y) \supset\!\!\subset (x', y')[\mathcal{A} \parr \mathcal{B}]$ holds if and only if either $(x, y) = (x', y')$, or $x \frown x'[\mathcal{A}]$, or $y \frown y'[\mathcal{B}]$.

Convention. From now on, unless expressly stated otherwise, we stick to the following notations:

- We denote the arcs of a proof structure or of a pseudo proof structure by lowercase latin letters $a, b, c, \ldots$
- We denote the types of these arcs by using the corresponding uppercase latin letters $A, B, C, \ldots$

- We denote the coherent spaces associated with *MELL* formulas by using the corresponding calligraphic latin letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

Also, if n is a positive integer and $\Gamma = A_1, \ldots, A_n$ is a list of *MELL* formulas, we define by induction on n the formula $\parr\Gamma$ produced by linking all formulas of Γ with par connectors:

- If $n = 1$, then $\parr\Gamma := A_1$.
- If $n \geq 1$ and $\Delta := A_1, \ldots, A_{n-1}$, then $\parr\Gamma := (\parr\Delta) \parr A_n$.

In addition, we slightly abuse notation by also denoting $\parr\Gamma$ the coherent space interpreting this formula. On the other hand, if Γ is the empty list, the notation $\parr\Gamma$ only refers to the unique coherent space with an empty web.

Definition 2.5. Let R be a proof structure and let p be the box complexity of R. An *experiment* of R is a map which associates with every arc a of type A of R a multiset of elements of $|\mathcal{A}|$, defined by induction on p as follows:

- If $p = 0$, every arc a of R must satisfy the following conditions:
  - ◇ If a is a conclusion of an axiom link of R and the other conclusion of this link is b, then $e(a) = e(b)$ and this set is a singleton.
  - ◇ If a is a premise of a cut link of R and we call b the other premise of this link, then $e(a) = e(b)$ and this set is a singleton.
  - ◇ If a is the conclusion of a par or tensor link of R with left premise a_1 and right premise a_2, then we have $e(a) = \{(x_1, x_2)\}$ with $x_1 \in e(a_1)$ and $x_2 \in e(a_2)$.
  - ◇ If a is the conclusion of a dereliction link and we call a_1 the premise of this link, then $e(a) = \{\{x_1\}\}$ with $x_1 \in e(a_1)$.
  - ◇ If a is the conclusion of a weakening link, then $e(a) = \{\emptyset\}$.
  - ◇ If a is the conclusion of a contraction link of arity $k \geq 2$ and we call $a_1, \ldots, a_k$ the premises of this link then, provided that $x_i \in e(a_i)$ for all indices $i = 1, \ldots, k$, we have $e(a) = \{\bigcup\{x_1, \ldots, x_k\}\}$.
- If $p \geq 1$, all arcs of R with depth 0 must meet the previous requirements. In addition, every box B of R with depth 0 must satisfy what follows. Let c and c' be the conclusion and the premise respectively of the front door of B. Also assume that $a_1, \ldots, a_m$ and $a'_1, \ldots, a'_m$ are the conclusions and the premises respectively of the pax doors of B. Then let R' be the biggest proof structure contained in B. There exist a unique non negative integer h and a unique multiset $\{e_1, \ldots, e_h\}$ of experiments of R' which meet the following conditions:
  - ◇ For every arc a of R', we have $e(a) = \bigcup\{e_1(a), \ldots, e_h(a)\}$.
  - ◇ We have $e(c) = \{\{x_1, \ldots, x_h\}\}$ with $x_j \in e_j(c')$ for every $j = 1, \ldots, h$.

◇ For all $i = 1, \ldots, m$, if $x^i_j \in e_j(a'_i)$ for all indices $j = 1, \ldots, h$, we have:

$$e(a_i) = \bigcup\{x^i_1, \ldots, x^i_h\}$$

In addition, if n is a positive integer, if $\Gamma = A_1, \ldots, A_n$ is the conclusion of R, if $a_1, \ldots, a_n$ are the conclusions of R, if e is an experiment of R and $x_i \in e(a_i)$ for all indices $i = 1, \ldots, n$, the element[1] $(x_1, \ldots, x_n)$ of $|\mathbin{⅋}\Gamma|$ is called the *conclusion* or *result* of e.

An experiment can be understood and represented as a labeling of the arcs of a proof structure. However, in contrast with the definition of [Gir87] which only concerns the multiplicative case, an experiment here associates with each arc a finite number of labels (possibly zero or more than one).

Remark 2.4. The definition of experiment implicitly requires that the following coherence conditions are also satisfied:

(i) If a is the conclusion of a pax or contraction link of R with depth 0 and if $?B$ is its type, then $e(a)$ is an element of $|?\mathcal{B}|$, that is a clique of $\mathcal{B}^\perp$.

(ii) If a is the conclusion of an of course link of R with depth 0 and if $!B$ is its type, then $e(a)$ is an element of $!\mathcal{B}$, that is a clique of $\mathcal{B}$.

Definition 2.6. Let R be a proof structure with conclusion Γ. The *interpretation* of R is the subset $[\![R]\!]$ of $|\mathbin{⅋}\Gamma|$ whose elements are the results of the experiments of R.

Remark 2.5. The interpretation of a proof structure R depends on the coherent spaces which are associated with the occurrences of atomic subformula of the conclusions of R.

Definition 2.7. Let R and R' be proof structures with the same conclusions, let R_0 and R'_0 be the unique standard proof structures corresponding to R and R' respectively. We say that R and R' are *syntactically equivalent* or *$\beta\eta$-equivalent* if $R_0 = R'_0$, *semantically equivalent* when $[\![R]\!] = [\![R']\!]$ for all choices of the coherent spaces of the occurrences of atomic subformula which are associated with the conclusions of R.

Remark 2.6. One can prove that $[\![R]\!] = [\![R_0]\!]$. Then two proof structures with the same conclusions are semantically equivalent if and only if the standard proof structures associated with them are. This property allows to restrict our study to standard proof structures without loss of generality.

Convention. For the rest of this chapter, we refer to standard proof structures just as proof structures and to standard proof nets just as proof nets.

We can now precisely state what we mean by injectivity.

Definition 2.8. Let F be a fragment of *MELL*. We say that the coherent multiset based semantics is:

[1] We are slightly abusing notation by omitting some parentheses.

- *Locally injective* for F in a proof net R of F if, for all proof nets R' of F with the same conclusions as R and such that $[\![R]\!] = [\![R']\!]$, we have $R = R'$.
- *Injective* for F if it is locally injective for F in all proof nets of F.

We finally introduce some definitions about experiments.

Definition 2.9. Let F be a fragment of *MELL* and let R be a proof net of F. An experiment of R with result γ contains all information about R with respect to F if, for all proof nets R' of F with the same conclusions as R and $\gamma \in [\![R']\!]$, we have $R = R'$.

Remark 2.7. If such an experiment exists, then the semantics is locally injective for F in R.

Definition 2.10. Let F be a fragment of *MELL* and let R be a proof structure of F. We say that an experiment e of R is *injective* (respectively, *simple*) if we have $e(a) \neq e(a')$ (respectively, $e(a) = e(a')$) for all distinct arcs a and a' of R with the same atomic type.

2.2 The multiplicative fragment

We prove that the question of injectivity has a positive answer if we restrict to proof nets of *MLL*. The proof of this result suggests the path to follow in order to address the problem in the general case.

The first result expresses the fact that a proof structure of *MLL* is uniquely determined once its conclusions and axiom links are known. Furthermore, an experiment is uniquely determined by its result. These properties can easily be proven by induction on the number of links of R. To be completely precise, we need the following preliminary definition.

Definition 2.11. Let R and R' be proof structures. An isomorphism Φ between $\mathbf{OP}(R)$ and $\mathbf{OP}(R')$ identifies an experiment e of R and an experiment e' of R' if we have $e = e' \circ \Phi$. If such an isomorphism exists, we write $e = e'$.

Here is the aforementioned result.

Lemma 2.1. *If R and R' are proof structures of MLL with the same conclusions, then we have $\mathbf{O}(R) = \mathbf{O}(R')$. Moreover, if e and e' are experiments of R and R' respectively with the same result, then $e = e'$.*

Theorem 2.1. *If e is an injective experiment of a proof net R of MLL, then e contains all information about R with respect to MLL.*

Proof. Let γ be the result of e and let R' be a proof net of *MLL* having the same conclusions as R and $\gamma \in [\![R']\!]$. Then there is an experiment e' of R' with result γ. By the previous lemma, there is an isomorphism Φ between $\mathbf{O}(R)$ and $\mathbf{O}(R')$ such that $e = e' \circ \Phi$.

Now notice that, by injectivity of e, for every arc a of $\mathbf{O}(R)$ with atomic type X there is a unique arc a' of $\mathbf{O}(R)$ of type $X^{\perp}$ such that $e(a) = e(a')$. This entails that a and a' are the conclusions of an axiom link of R. Notice that a' is also the unique arc b of $\mathbf{O}(R)$ such that $e'(\Phi(a)) = e'(\Phi(b))$, hence $\Phi(a)$ and $\Phi(a')$ are the conclusions of an axiom link of R'. Now Φ induces an isomorphism between R and R', that is $R = R'$. □

Remark 2.8. If R is a proof structure of *MLL* with exactly h axiom links then, for any choice of the coherent spaces we associate with the occurrences of atomic subformula of the conclusions of R, if the cardinality of each of their webs is at least h, there is an injective experiment of R (this is easily verified by induction on the number of links of R).

Corollary 2.1. *The coherent multiset based semantics is injective for MLL.*

Proof. Immediate consequence of remarks 2.7, 2.8 and theorem 2.1. □

2.3 Obsessional results

It turns out that, for most of the proofs we see in this chapter, we do not use the correctness of proof nets. In other words, these results hold more generally for proof structures, which justifies the following convention.

Convention. From now on, unless otherwise stated, it is intended that R and R' are proof structures.

We introduce the main tool of our analysis: obsessional experiments.

Definition 2.12. Let n be a positive integer and let e be an experiment of R. We say that e is *n-obsessional* if the following conditions hold:

- For every edge a of R with atomic type X, if $x, y \in e(a)$, then $x = y$.
- For each edge c of R with type $!A$, the multiset $e(c)$ is not empty and each of its elements has cardinality n.

In addition, if $n = 1$, then we prefer to call e a *1-experiment*.

We now want to generalize what we did in the previous section to the more interesting case of proof structures of *MELL*. One of the primary ingredients is the following result.

Proposition 2.1. *Let R and R' be proof structures with the same conclusions. If e and e' are experiments of R and R' respectively with the same result and e is n-obsessional, then e' is n-obsessional.*

In other words, we can "read the obsessional feature of an experiment in its result". We do not go over all the auxiliary results used in [TdF03] to establish the previous proposition, but we do revisit one of them, for which a proof was not provided. Also, this result is not used exclusively to prove proposition 2.1, which leads us to believe that it may turn out to be useful in future research on the topic. Since its proof is not particularly simple and even the statement was not completely precise in the source, we first provide some original definitions and remarks. To begin with, remark 1.8 and item (iv) of remark 1.14 justify the following definition.

Definition 2.13. Let a be an arc of R and let c be an arc of T_a. We call *distance* of c from a the non negative integer k for which there is a path $a_0 \dots a_k$ of R with $a_0 = c$ and $a_k = a$.

Definition 2.14. Let a be an arc of R and let c be an arc of T_a. The *address* of c is a finite word $adr_a^R(c)$ over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$, denoted $adr_a(c)$ when there is no ambiguity and defined as follows, by induction on the distance d of c from a:

- If $d = 0$, we define $adr_a(c) := \varepsilon$.
- If $d \geq 1$, let b be the conclusion of the link n of which c is a premise.
  - If n is a par or tensor link, we define $adr_a(c) := adr_a(b)\mathsf{L}$ if c is the left premise of n, otherwise $adr_a(c) := adr_a(b)\mathsf{R}$.
  - If n is an of course or dereliction link, we define $adr_a(c) := adr_a(b)\mathsf{C}$.
  - If n is a pax or contraction link, we define $adr_a(c) := adr_a(b)$.

Remark 2.9. We compute the address of c in b, where b is the conclusion of the link n of which c is a premise:

- If n is a par or tensor link, we have $adr_b(c) = \mathsf{L}$ if c is the left premise of n, otherwise $adr_b(c) = \mathsf{R}$.
- If n is an of course or dereliction link, then $adr_b(c) = \mathsf{C}$.
- If n is a pax or contraction link, then $adr_b(c) = \varepsilon$.

In particular, we have $adr_a(c) = adr_a(b)adr_b(c)$. This identity is no longer trivial if we want to consider any arc b crossed by the path from c to a, which justifies the two following results.

Lemma 2.2. *Let $a_0 \dots a_k$ be a path of R. Then we have $adr_{a_k}(a_0) = adr_{a_k}(a_i)adr_{a_i}(a_0)$ for all $i = 0, \dots, k$.*

Proof. We reason by induction on k. The statement is obvious if $k = 0$ or $i = 0$. Suppose $i \geq 1$ (in particular, the inequality $k \geq 1$ holds). By remark 2.9 and by inductive hypothesis, we have:

$$\begin{aligned} adr_{a_k}(a_0) &= adr_{a_k}(a_1)adr_{a_1}(a_0) \\ &= adr_{a_k}(a_i)adr_{a_i}(a_1)adr_{a_1}(a_0) = adr_{a_k}(a_i)adr_{a_i}(a_0) \end{aligned}$$ □

Lemma 2.3. *Let a be an arc of R, let b be an arc of T_a and let c be an arc of T_b. Then we have $adr_a(c) = adr_a(b)adr_b(c)$.*

Proof. The result easily follows from the previous lemma and from item (iv) of remark 1.14. □

The following result reveals a very natural relationship between addresses of arcs and occurrences of subformulas.

Lemma 2.4. *Let a be an arc of R. If c is an arc of T_a with address w, then (C, w) is an occurrence of subformula of A.*

Proof. We reason by induction on the distance d of c from a. If $d = 0$, the result is trivial. Suppose $d \geq 1$ and let b be the premise of the link n with conclusion a such that c is an arc of T_b. If we define $v := adr_b(c)$, we know that (C, v) is an occurrence of subformula of B by inductive hypothesis. By lemma 2.3, we have $w = adr_a(b)v$. Finally, by remark 2.9, we have the following possibilities:

- If $adr_a(b) = \mathsf{L}$, then n is a par or tensor link and b is its left premise. Thus, we have $A = B \parr D$ or $A = B \otimes D$ for some *MELL* formula D and we can conclude, by definition 2.3, that (C, w) is an occurrence of subformula of A. The case $adr_a(b) = \mathsf{R}$ is completely analogous.
- If $adr_a(b) = \mathsf{C}$, then n is an of course or dereliction link. In particular, we must have $A = !B$ or $A = ?B$. In both cases, by definition 2.3 we obtain the desired conclusion.
- If $adr_a(b) = \varepsilon$, then n is a pax or contraction link. But then we have $A = B$ and we are done. □

We now revisit a definition given in [TdF03].

Definition 2.15. Let A be an *MELL* formula and $x \in |\mathcal{A}|$. The *multiset projection* of x on an occurrence of subformula (F, w) of A is the multiset $|x|_{F,w}$ we define as follows, by induction on the length of w:

- If $w = \varepsilon$, then $|x|_{F,w} := \{x\}$.
- If $w = \mathsf{L}u$ (respectively, $w = \mathsf{R}u$) for some finite word u over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$, then either $A = B \parr C$ or $A = B \otimes C$ and (F, u) is an occurrence of subformula of B (respectively, C). By definition of par or tensor of two coherent spaces, there are $y \in |\mathcal{B}|$, $z \in |\mathcal{C}|$ such that $x = (y, z)$. Hence, we can define $|x|_{F,w} := |y|_{F,u}$ (respectively, $|x|_{F,w} := |z|_{F,u}$).
- If $w = \mathsf{C}u$ for some finite word u over $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$, we have either $A = !B$ or $A = ?B$ and (F, u) is an occurrence of subformula of B. Therefore, we can define $|x|_{F,w} := \bigcup\{|y|_{F,u} : y \in x\}$.

Remark 2.10. If A is an *MELL* formula, if (B, u) is an occurrence of subformula of A and (C, v) is an occurrence of subformula of B then one easily verifies, by induction on the length of u, that the following properties hold:

(i) The ordered pair (C, uv) is an occurrence of subformula of A.

(ii) For all $x \in |\mathcal{A}|$ and $z \in |\mathcal{C}|$, we have $z \in |x|_{C,uv}$ if and only if $z \in |y|_{C,v}$ for some $y \in |x|_{B,u}$.

As a straightforward consequence of property (ii), we have:

(iii) For all $x, x' \in |\mathcal{A}|$, the inclusion $|x|_{B,u} \subseteq |x'|_{B,u}$ implies $|x|_{C,uv} \subseteq |x'|_{C,uv}$.

We can finally state and prove the result we mentioned at the beginning of this section.

Proposition 2.2. *Let a be an arc of R, let B be an MELL formula which is not a why not formula and let w be a finite word over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$. If e is an experiment of R then, for every $x \in |\mathcal{B}|$, the two following statements are equivalent:*

$\langle 1 \rangle$ *There exists an arc b of type B and address w in T_a such that $x \in e(b)$.*

$\langle 2 \rangle$ *The ordered couple (B, w) is an occurrence of subformula of A and, for a certain $y \in e(a)$, we have $x \in |y|_{B,w}$.*

Proof. We first prove that $\langle 1 \rangle$ implies $\langle 2 \rangle$. By lemma 2.4, we know that (B, w) is an occurrence of subformula of A. In order to prove that $x \in |y|_{B,w}$ for a certain $y \in e(a)$, we reason by induction on the distance k of b from a. If $k = 0$, then we have $a = b$ and thus, by picking $y := x$, we are done. Now assume $k \geq 1$ and let $a_0 \dots a_k$ be the path of R with $a_0 = b$ and $a_k = a$. Since $a \neq b$, we have $w = w_0 u$ for some letter w_0 and for some finite word u over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$. Let h be the biggest index $i \in \{1, \dots, k\}$ with $A_{i-1} \neq A_i$. Observe that such an index exists because $A_0 \neq A_1$ by the hypothesis that B is not a why not formula. Now, since $A_{i-1} = A_i$ for each $i = h+1, \dots, k$, the arc a_i is the conclusion of a pax or contraction link. Similarly, since $A_{h-1} \neq A_h$, we know that a_h is the conclusion of a par, tensor, of course or dereliction link. By lemma 2.2 and remark 2.9, we then have $adr_a(a_{h-1}) = w_0$ and $adr_{a_{h-1}}(b) = u$. Notice that the distance of b from a_{h-1} is $h - 1 < k$. We can then apply the inductive hypothesis, which provides an element $z \in e(a_{h-1})$ such that $x \in |z|_{B,u}$. We can now consider the following possibilities:

- If a_h is the conclusion of a par or tensor link, then $h = k$. Suppose $w_0 = \mathsf{L}$. Then a_{k-1} is the left premise of that link. If a' is the right premise, there is $z' \in e(a')$ such that $y := (z, z') \in e(a)$, by definition of experiment. Hence we have $|y|_{B,w} = |z|_{B,u}$, by definition 2.15. The case $w_0 = \mathsf{R}$ is completely analogous.

- If a_h is the conclusion of an of course link, then $h = k$ and $w_0 = \mathsf{C}$. There is an element $y \in e(a)$ such that $z \in y$, by definition of experiment. Then, by definition 2.15, we get $x \in |y|_{B,w}$.

- If a_h is the conclusion of a dereliction link, we have $w_0 = \mathsf{C}$. By definition of experiment, there exist two elements $z' \in e(a_h)$ and $y \in e(a)$ such that $z \in z' \subseteq y$. Therefore, by definition 2.15, we can conclude that $x \in |y|_{B,w}$.

We now prove that $\langle 2 \rangle$ implies $\langle 1 \rangle$. We reason by induction on the length of w. If $w = \varepsilon$, then $A = B$ and $x = y$, so we can pick $b := a$ and we are done. Now assume that $w = w_0 u$ for a certain letter w_0 and for some finite word u over the alphabet $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}$. We now distinguish the following cases:

- Suppose that a is the conclusion of a par or tensor link with left premise a' and right premise a''. By definition of experiment, we know that there exist $y' \in e(a')$ and $y'' \in e(a'')$ such that $y = (y', y'')$. If $w_0 = \mathsf{L}$, then (B, u) is an occurrence of subformula of A' by definition 2.3 and $|y|_{B,w} = |y'|_{B,u}$ by definition 2.15. By inductive hypothesis, there exists an arc b of type B and address u in $T_{a'}$ such that $x \in e(b)$. Moreover, by remark 2.9, we have $adr_a(a') = w_0$ and so, by lemma 2.3, the address of b in T_a is w. We use the same argument if $w_0 = \mathsf{R}$.

- If a is the conclusion of an of course link with premise a', we have $w_0 = \mathsf{C}$ and, by definition 2.3, the couple (B, u) is an occurrence of subformula of A'. Moreover, by definition 2.15, there exists $z \in y$ such that $x \in |z|_{B,u}$. By definition of experiment we have $z \in e(a')$, too. By inductive hypothesis, there is an arc b of type B and address u in $T_{a'}$ such that $x \in e(b)$. Now we can use remark 2.9 and lemma 2.3 exactly as we did above to obtain that the address of b in T_a is w.

- If a is the conclusion of a pax or dereliction link and if a' is the premise of the dereliction link above a, then we can repeat the same argument as in the previous case. We just point out that proving $z \in e(a')$ is not as trivial as before. However, this is not really an obstacle, as it can be easily done by induction on the distance of a' from a.

- We finally consider the case in which a is the conclusion of a contraction link with premises $a_1, \ldots, a_k$. Once again, we necessarily have $w_0 = \mathsf{C}$. If the type of a is $?A'$, then the couple (B, u) is an occurrence of subformula of A' by definition 2.3 and $x \in |z|_{B,u}$ for some $z \in y$ by definition 2.15. We now apply the definition of experiment to get an index $i \in \{1, \ldots, k\}$ and an element $y_i \in e(a_i)$ such that $z \in y_i$. Notice that a_i is the conclusion of a pax or dereliction link. Thus, if a' is the conclusion of the dereliction link above a_i, we have $z \in e(a')$ as we saw before. This is enough to conclude, as usual. □

Remark 2.11. The hypothesis that B is not a why not formula is not necessary in the proof that ⟨2⟩ implies ⟨1⟩.

Corollary 2.2. *Let a be a conclusion of R, let e be an experiment of R, let (B, w) be an occurrence of subformula of A and let $b_1, \ldots, b_h$ be all arcs of type B and address w in T_a. If B is not a why not formula and $e(a) = \{y\}$, we get $|y|_{B,w} = e(b_1) \cup \cdots \cup e(b_h)$.*

Proof. Straightforward consequence of the previous proposition. □

2.4 Open pseudo proof structures

We prove that the interpretation of R determines R “up to the axiom links and the boxes”. We revisit the result in [TdF03] expressing this property, providing some supplementary details. We start with the following definition.

Definition 2.16. If k is the maximal arity of the contraction links of R, then the *contraction size* of R is defined as the non negative integer:

$$h(R) := \begin{cases} \max\{1, k\} & \text{if } R \text{ has at least a box} \\ 0 & \text{otherwise} \end{cases}$$

We borrow from [TdF03] a very useful proposition, which expresses one of the fundamental properties of n-obsessional experiments.

Proposition 2.3. *Let e be an n-obsessional experiment of R. If a is an arc of R having depth p, then $e(a)$ is a multiset containing exactly n^p occurrences of the same element.*

The following definition is now justified.

Definition 2.17. Let e be an n-obsessional experiment of R and let a be an arc of R. The *repeated element* of $e(a)$, denoted $\langle e(a)\rangle$, is any element of $e(a)$.

Remark 2.12. If e is an n-obsessional experiment of R and a is the conclusion of an of course link of R with premise a' then, by definition of experiment, we get $\langle e(a)\rangle = \{n[\langle e(a')\rangle]\}$.

We then recall the following result, proven in [TdF03]. It is the analogue for *MELL* of lemma 2.1.

Proposition 2.4. *Let R and R' be proof structures with the same conclusions and let $n > \max\{h(R), h(R')\}$. Suppose that e and e' are n-obsessional experiments of R and R' respectively with the same result. Then $e = e'$ and in particular $\mathbf{OP}(R) = \mathbf{OP}(R')$.*

We prove in detail the following lemma.

Lemma 2.5. *Let e be an n-obsessional experiment of R and let a and a' be arcs of R of type A. If $\langle e(\alpha)\rangle = \langle e(\alpha')\rangle$ for all arcs α of T_a and α' of $T_{a'}$ with the same atomic type, then $\langle e(a)\rangle \asymp \langle e(a')\rangle[\mathcal{A}]$.*

Proof. First, define p as the sum of the number of arcs of T_a and the number of arcs of $T_{a'}$. We reason by induction on p. If $p = 2$, then a and a' are conclusions of initial links, hence we have $\langle e(a)\rangle = \langle e(a')\rangle$ by hypothesis or by definition of experiment. If $p \geq 3$ and the statement is true for every integer $q < p$, let m be the link of R with conclusion a and let m' be the link of R with conclusion a'. If the premises of m are $a_1, \ldots, a_k$ and those of m' are $a'_1, \ldots, a'_h$, we can suppose $k \geq 1$ without loss of generality.

Observe that A cannot be an atomic formula. If A is not a why not formula, then m and m' are the same kind of link and we get $k = h$. We can distinguish the following possibilities:

- In the case of a par link, we have $k = 2$ and $A = A_1 \parr A_2$. By applying the inductive hypothesis, we get $\langle e(a_i)\rangle \asymp \langle e(a'_i)\rangle[\mathcal{A}_i]$ for $i = 1, 2$. Hence, we have $(\langle e(a_1)\rangle, \langle e(a_2)\rangle) \frown (\langle e(a'_1)\rangle, \langle e(a'_2)\rangle)[\mathcal{A}_1^\perp \otimes \mathcal{A}_2^\perp]$ and this implies the desired result by definition of coherent space.

- In the case of a tensor link, we have $k = 2$ and $A = A_1 \otimes A_2$. As before, we get $\langle e(a_i)\rangle \frown \langle e(a'_i)\rangle[\mathcal{A}_i^\perp]$ for $i = 1, 2$. If we have strict coherence for $i = 1$ or $i = 2$, we have $(\langle e(a_1)\rangle, \langle e(a_2)\rangle) \frown (\langle e(a'_1)\rangle, \langle e(a'_2)\rangle)[\mathcal{A}_1^\perp \parr \mathcal{A}_2^\perp]$, else we get $\langle e(a_i)\rangle = \langle e(a'_i)\rangle$ for $i = 1, 2$. In both cases, we are done.

- In the case of an of course link, we have $k = 1$ and $A = !A_1$. By inductive hypothesis, we get $\langle e(a_1)\rangle \asymp \langle e(a'_1)\rangle[\mathcal{A}_1]$. If $\langle e(a_1)\rangle = \langle e(a'_1)\rangle$, then we are done, otherwise we have $\langle e(a_1)\rangle \smile \langle e(a'_1)\rangle[\mathcal{A}_1]$. In this case, the multiset $\{n[\langle e(a_1)\rangle], n[\langle e(a'_1)\rangle]\}$ is not a clique of $\mathcal{A}_1$. However, this is the same as the condition $\{n[\langle e(a_1)\rangle]\} \smile \{n[\langle e(a'_1)\rangle]\}[!\mathcal{A}_1]$, which is exactly what we wanted to achieve.

If $A = ?B$ for a certain *MELL* formula B, assume that m' is a weakening link and let $b_1, \ldots, b_k$ be the premises of the k dereliction links above m. We know that, for some non negative integers $p_1, \ldots, p_k$, the repeated element of $e(a)$ is the multiset $\{n^{p_1}[\langle e(b_1)\rangle], \ldots, n^{p_k}[\langle e(b_k)\rangle]\}$. By induction hypothesis, we have $\langle e(b_i)\rangle \asymp \langle e(b_j)\rangle[\mathcal{B}]$ for all indices $i, j = 1, \ldots, k$. Thus, the previous multiset is a clique of $\mathcal{B}$, or equivalently $\langle e(a)\rangle \asymp \emptyset[\mathcal{A}]$, which is the desired result.

We then consider the possibility that m and m' are dereliction links. In this case, we have $k = 1$ and $B = A_1$. Once again, we have $\langle e(a_1)\rangle \frown \langle e(a'_1)\rangle[\mathcal{A}_1^\perp]$ by inductive hypothesis, so $\{\langle e(a_1)\rangle\} \frown \{\langle e(a'_1)\rangle\}[!\mathcal{A}_1^\perp]$ and we are done.

We finally assume, without loss of generality, that m is a pax or contraction link and observe that, for all $i, j = 1, \ldots, k$, we have $\langle e(a_i)\rangle \asymp \langle e(a_j)\rangle[\mathcal{A}]$. Also, by inductive hypothesis, we have $\langle e(a_i)\rangle \asymp \langle e(a')\rangle[\mathcal{A}]$. Therefore, the multiset $\langle e(a_1)\rangle \cup \cdots \cup \langle e(a_k)\rangle \cup \langle e(a')\rangle$ is a clique of $\mathcal{A}^\perp$, which immediately gives the desired conclusion. □

We can now establish the following result.

Proposition 2.5. *For every integer $n \geq 1$, there is a simple n-obsessional experiment e of R.*

Proof. We reason by induction on the number p of links of R. If R only contains one link, then it is an initial link. If it is a weakening link with conclusion a, we define $e(a) := \{\emptyset\}$ and we are done. Otherwise, the proof structure R is just an axiom link with conclusions a and a'. In this case, we just pick $\mathcal{A}$ as a coherent space with at least one element x and we define $e(a) := e(a') := \{x\}$.

Now suppose $p \geq 1$ and assume that every proof structure with $p - 1$ links possesses a simple n-obsessional experiment. Since R is cut free, there exists a terminal link m of R. Let R' be a proof structure obtained by removal of m. By induction hypothesis, there is a simple n-obsessional experiment e' of R'. This induces a simple n-obsessional experiment e of R. We justify this claim only in the case in which m is a contraction link, because all other cases are obvious. If $a_1, \ldots, a_k$ are the premises of m and a is its conclusion, then $e(a_i)$ only contains one element y_i for all $i = 1, \ldots, k$, because the depth of terminal links of R is 0. We can then define $e(b) := e'(b)$ for each arc b of R' and $e(a) := \{y_1 \cup \cdots \cup y_k\}$. If e is an experiment of R, it is simple and n-obsessional, because e' is. Hence, we only need to make sure that e really is an experiment of R. It is sufficient to check that $y_1 \cup \cdots \cup y_k$ is an element of $|A|$. Since m is a contraction link, there is an *MELL* formula C such that $A = ?C$. By lemma 2.5, we have $y_i \asymp y_j[\mathcal{A}]$ for all indices $i, j = 1, \ldots, k$. Therefore, the multiset $y_1 \cup \cdots \cup y_k$ is a clique of $C^\perp$, that is an element of $|A|$. □

We finally state and prove the result we mentioned at the beginning of this section.

Theorem 2.2. *If R and R' are proof structures with the same conclusions and we have $[\![R]\!] = [\![R']\!]$, then $\mathbf{OP}(R) = \mathbf{OP}(R')$.*

Proof. Let $n > \max\{h(R), h(R')\}$. By proposition 2.5, we can consider a simple n-obsessional experiment e of R. Since $[\![R]\!] = [\![R']\!]$, there is an experiment e' of R' with the same result as e. By proposition 2.1, we know that the experiment e' is n-obsessional. By proposition 2.4, we conclude that $\mathbf{OP}(R) = \mathbf{OP}(R')$. □

2.5 Local injectivity

We finally review the result of local injectivity in [TdF03] and, in doing so, we highlight analogies and differences with the multiplicative case. The following abuse of notation is justified by proposition 2.3.

Convention. If e_1 is a 1-experiment of R and a is an arc of R, then the unique element of $e_1(a)$ is denoted $e_1(a)$ as well.

Definition 2.18. If e_1 is a 1-experiment of R, an n-obsessional experiment e_n of R is *induced* by e_1 when $\langle e_n(a)\rangle = e_1(a)$ for all arcs a of R with atomic type.

Remark 2.13. If e_1 is injective, then e_n is injective, too.

Remark 2.14. One might be tempted to think that $\langle e_n(a)\rangle = e_1(a)$ for all arcs a of R, but this is not the case in general. It is easy to fabricate a counterexample by using the fact that, by definition, if c is the conclusion of an of course link of R, then $\langle e_n(c)\rangle$ contains n elements, whereas $e_1(a)$ only contains one element.

We borrow from [TdF03] the following results.

Lemma 2.6. *Let e be an n-obsessional experiment of R and let m be a link of R with conclusion a. If $n > h(R)$ and the type of a is $?B$, then the following statements hold:*

(i) We have that m is a weakening link if and only if $\langle e(a)\rangle = \emptyset$.

(ii) We have that m is a dereliction link if and only if $\langle e(a)\rangle$ is a singleton.

(iii) We have that m is a pax link if and only if the cardinality of $\langle e(a)\rangle$ is n^p for some positive integer p.

(iv) We have that m is a contraction link with arity k if and only if the cardinality of $\langle e(a)\rangle$ is $n^{p_1} + \cdots + n^{p_k}$ for some non negative integers $p_1, \ldots, p_k$.

Lemma 2.7. *Let a and a' be two different arcs of R with type A, let e_1 be an injective 1-experiment of R and let e_n be an n-obsessional experiment induced by e_1. If we have $n > h(R) \geq 1$, then the following statements hold:*

(i) *If there exists an arc* α *of* R *with atomic type such that* α *is an arc of* T_a *or* $T_{a'}$*, we have* $\langle e_n(a)\rangle \neq \langle e_n(a')\rangle$.

(ii) *Otherwise, either we have* $T_a = T_{a'}$ *and then* $\langle e_n(a)\rangle = \langle e_n(a')\rangle$*, or* $T_a \neq T_{a'}$ *and then* $\langle e_n(a)\rangle \smile \langle e_n(a')\rangle$.

Lemma 2.8. *Let* a *and* a' *be different arcs of* R *with type* A *and let* e_1 *be an injective* 1*-experiment of* R*. Then the following statements hold:*

(i) *If* a *is an arc of* $T_{a'}$ *or* a' *is an arc of* T_a*, then we have* $e_1(a) = e_1(a')$ *if and only if* a *and* a' *are not conclusions of contraction links of* R.

(ii) *If* T_a *and* $T_{a'}$ *do not share any arc and there exists an arc* α *of* R *with atomic type such that* α *is an arc of* T_a *or* $T_{a'}$*, then* $e_1(a) \neq e_1(a')$.

(iii) *If* T_a *and* $T_{a'}$ *do not share any arc and there exists* no *arc* α *of* R *with atomic type such that* α *is an arc of* T_a *or* $T_{a'}$*, then* $e_1(a) \asymp e_1(a')$.

Remark 2.15. Observe that item (iii) of the previous result is a straightforward consequence of lemma 2.5.

Remark 2.16. By item (vi) of remark 1.14, the three statements of lemma 2.8 are mutually exclusive.

Remark 2.17. Obviously, the size of the type labeling the conclusion of a link is greater than or equal to the maximal size of the types which label the premises of that link. Moreover, equality holds if and only if we are considering a pax or contraction link. In particular, if $a_0 \dots a_k$ is a path of R, then for all $i = 1, \dots, k$ the size of A_{i-1} is smaller than or equal to the size of A_i and we have equality if and only if A_i is the conclusion of a pax or contraction link.

We provide details for the following result.

Lemma 2.9. *Let* a *and* a' *be two different arcs of* R *with type* A*, let* e_1 *be an injective* 1*-experiment of* R *and let* e_n *be an* n*-obsessional experiment induced by* e_1*. If we have* $n > h(R) \geq 1$*, then* $e_1(a) \smile e_1(a')$ *implies* $\langle e_n(a)\rangle \smile \langle e_n(a')\rangle$.

Proof. If p is the sum of the number of arcs of T_a and the number of arcs of $T_{a'}$, we can reason by induction on p. Suppose $p = 2$. Then a and a' are conclusions of initial links. Either they are both conclusions of weakening links and then it is not the case that $e_1(a) \smile e_1(a')$ because, by definition of experiment, we have $e_1(a) = \varnothing = e_1(a')$, or they are both arcs with atomic type. In the latter case, we have $e_1(a) = \langle e_n(a)\rangle$ and $e_1(a') = \langle e_n(a')\rangle$ by definition 2.18, hence we are done. If $p \geq 3$ and the statement holds for any integer $q < p$, then A is not an atomic formula. We can define m as the link of R with conclusion a and m' as the link of R with conclusion a'. If the premises of m are $a_1, \dots, a_k$ and those of m' are $a'_1, \dots, a'_h$, we can assume $k \geq 1$ without loss of generality.

We first consider the case in which A is not a why not formula. Since m and m' need to be the same kind of link, we have $k = h$ and we can distinguish the following possibilities:

- In the case of a par link, we have $k = 2$ and $A = A_1 \parr A_2$. Our hypothesis $e_1(a) \smile e_1(a')[\mathcal{A}]$ entails $e_1(a_i) \asymp e_1(a'_i)[\mathcal{A}_i]$ for $i = 1, 2$ and $e_1(a) \neq e_1(a')$. We can suppose $e_1(a_1) \neq e_1(a'_1)$ without loss of generality. Hence we have $e_1(a_1) \smile e_1(a_1)[\mathcal{A}_1]$. Therefore, by applying the inductive hypothesis, we obtain the condition $\langle e_n(a_1)\rangle \smile \langle e_n(a'_1)\rangle[\mathcal{A}_1]$. In particular, we must have $\langle e_n(a)\rangle \neq \langle e_n(a')\rangle$. Now, if $e_1(a_2) \neq e_1(a'_2)$, then by the same argument we get the condition $\langle e_n(a_2)\rangle \smile \langle e_n(a'_2)\rangle[\mathcal{A}_2]$ and we are done. Suppose that $e_1(a_2) = e_1(a'_2)$ and notice that, by remark 2.17, we cannot be in case (i) of lemma 2.8. Consequently, by remark 2.16, we are in case (iii). This allows us to apply item (ii) of lemma 2.7, which implies $\langle e_n(a_2)\rangle \asymp \langle e_n(a'_2)\rangle[\mathcal{A}_2]$. With this, we get the desired conclusion.

- In the case of a tensor link, we get $k = 2$ and $A = A_1 \otimes A_2$. This time, our hypothesis $e_1(a) \smile e_1(a')[\mathcal{A}]$ implies $e_1(a_i) \smile e_1(a'_i)[\mathcal{A}_i]$ for $i = 1$ or $i = 2$. Then, by inductive hypothesis, we have $\langle e_n(a_i)\rangle \smile \langle e_n(a'_i)\rangle[\mathcal{A}_i]$ for either $i = 1$ or $i = 2$ and we are done.

- In the case of an of course link, we get $k = 1$ and $A = !A_1$. Notice that, by remark 2.12, we get the four conditions $e_1(a) = \{e_1(a_1)\}$, $e_1(a') = \{e_1(a'_1)\}$, $\langle e_n(a)\rangle = \{n[\langle e_n(a_1)\rangle]\}$ and lastly $\langle e_n(a')\rangle = \{n[\langle e_n(a'_1)\rangle]\}$. Now, by using our hypothesis, we immediately get $e_1(a_1) \smile e_1(a'_1)[\mathcal{A}_1]$ and this implies $\langle e_n(a_1)\rangle \smile \langle e_n(a'_1)\rangle[\mathcal{A}_1]$ by inductive hypothesis. With this, we are done.

If $A = ?B$ for a certain *MELL* formula B and m' is a weakening link, then we have $\langle e_n(a')\rangle = \emptyset$ and $\langle e_n(a)\rangle \asymp \emptyset[\mathcal{A}]$ by definition of experiment. In addition, since m is not a weakening link by the hypothesis $p \geq 3$, we get $\langle e_n(a)\rangle \neq \emptyset$ by lemma 2.6 and so we are done.

Lastly, if m and m' are dereliction, pax or contraction links, let $b_1, \ldots, b_k$ be the premises of the dereliction links above m and let $b'_1, \ldots, b'_h$ be the premises of the dereliction links above m'. Then we have, for some non negative integers $p_1, \ldots, p_k, q_1, \ldots, q_h$, the following conditions:

$$
\begin{aligned}
e_1(a) &= \{e_1(b_1), \ldots, e_1(b_k)\} \\
e_1(a') &= \{e_1(b'_1), \ldots, e_1(b'_h)\} \\
\langle e_n(a)\rangle &= \{n^{p_1}[\langle e_n(b_1)\rangle], \ldots, n^{p_k}[\langle e_n(b_k)\rangle]\} \\
\langle e_n(a')\rangle &= \{n^{q_1}[\langle e_n(b'_1)\rangle], \ldots, n^{q_h}[\langle e_n(b'_k)\rangle]\}
\end{aligned}
$$

Our hypothesis $e_1(a) \smile e_1(a')[\mathcal{A}]$ now implies the condition $e_1(b_i) \smile e_1(b'_j)[\mathcal{B}]$ for some indices $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, h\}$. By inductive hypothesis, we must have $\langle e_n(b_i)\rangle \smile \langle e_n(b'_j)\rangle[\mathcal{B}]$, which entails the desired conclusion. □

One can consequently prove the following proposition from [TdF03].

Proposition 2.6. *If $n > h(R)$ and there exists an injective 1-experiment e_1 of R, then there exists an injective n-obsessional experiment e_n of R induced by e_1.*

If we consider *ACC* proof nets rather than just proof structures, then we get the following result.

Proposition 2.7. *If R and R' are ACC proof nets with the same conclusions and such that $\mathbf{P}(R) = \mathbf{P}(R')$, then $R = R'$.*

Proof. Immediate consequence of proposition 1.2: our hypothesis $\mathbf{P}(R) = \mathbf{P}(R')$ implies $\bar{R} = \bar{R}'$, which guarantees that the boxes of R and R' are the same. □

At last, we can express a sufficient condition of local injectivity.

Theorem 2.3. *Let R be a proof structure for which there is an injective 1-experiment. If R' is a proof structure having the same conclusions as R and such that $[\![R]\!] = [\![R']\!]$, then $\mathbf{P}(R) = \mathbf{P}(R')$.*

Proof. Let $n > \max\{h(R), h(R')\}$. We know, by proposition 2.6, that there is an injective n-obsessional experiment e of R. Let γ be the result of e. Observe that $\gamma \in [\![R]\!] = [\![R']\!]$, hence there is an experiment e' of R' with result γ. In addition, by proposition 2.1, we know that e' is n-obsessional. Then, by proposition 2.4, there is an isomorphism Φ between $\mathbf{OP}(R)$ and $\mathbf{OP}(R')$ such that $e = e' \circ \Phi$.

We now repeat the argument we saw in the proof of theorem 2.1: since e is injective, for every arc a of $\mathbf{OP}(R)$ with atomic type X there is a unique arc a' of $\mathbf{OP}(R)$ of type $X^\perp$ such that $e(a) = e(a')$. Then we necessarily have that a and a' are conclusions of the same axiom link of R. Moreover, the arc a' is the only arc b of $\mathbf{OP}(R)$ such that $e'(\Phi(a)) = e'(\Phi(b))$, hence $\Phi(a)$ and $\Phi(a')$ are conclusions of the same axiom link of R'. We can conclude that Φ induces an isomorphism between $\mathbf{P}(R)$ and $\mathbf{P}(R')$, that is $\mathbf{P}(R) = \mathbf{P}(R')$. □

Corollary 2.3. *Let R be an ACC proof net for which an injective 1-experiment exists. The coherent multiset based semantics is locally injective for ACC proof nets in R.*

Proof. Consider an *ACC* proof net R' with the same conclusions as R and such that $[\![R]\!] = [\![R']\!]$. By theorem 2.3, we get $\mathbf{P}(R) = \mathbf{P}(R')$. Then, by proposition 2.7, we can conclude that $R = R'$. □

Remark 2.18. It is important to see that the injective n-obsessional experiment e in the proof of theorem 2.3 heavily depends on the proof structure R', because the integer n does. It is natural to ask if it would be possible, in a certain sense, to make e independent of R'. More precisely, we would like to know if we can choose the experiment e in such a way that it contains all information about R with respect to *MELL*. More generally, one can ask if it is the case that, for any proof net R possessing an injective 1-experiment, there exists an experiment of R containing all information about R with respect to *MELL*. For sure, we have a negative answer if we look at the general case of *AC* proof nets. To prove this, we just exhibit a counterexample: let R be the *AC* proof net in figure 2.1. Notice that, if e is an experiment of R, then it must be an n-obsessional experiment for some positive integer n by our choice of R and item (i) of remark 2.4. Thus, the experiment e labels the arcs of R as shown in the previous figure, with $x \in |\mathcal{A}|$ and $y \in |\mathcal{B}|$. In particular, if we pick $n = 1$ and $x \neq y$, then we have an injective 1-experiment of R. Now we observe that the experiment e does not contain all information about R with respect to *MELL* because there exist an *AC* proof net

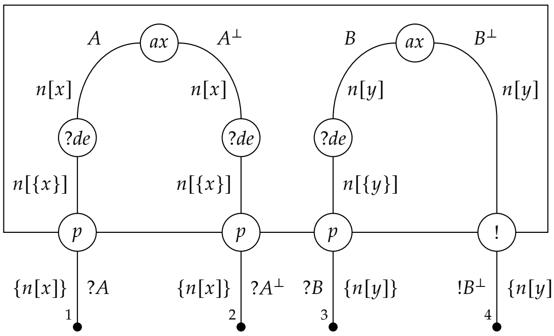

Figure 2.1: The *AC* proof net R and any experiment e of R.

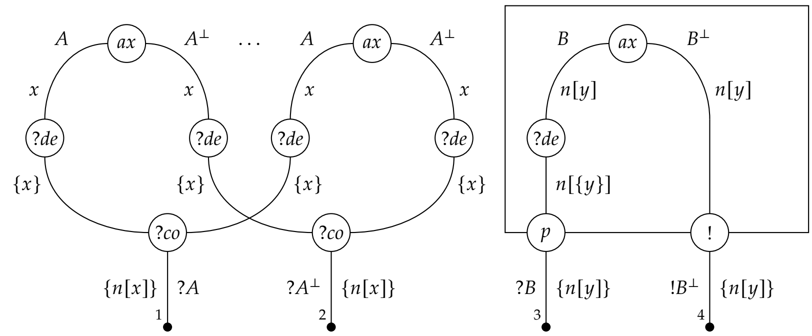

Figure 2.2: The *AC* proof net R' and a particular experiment of R' with the same result as e. The ellipsis between the axiom links on the left indicates that there are actually n copies of them and, in particular, both contraction links of R' have arity n.

R' and an experiment of R' with the same result as e, as depicted in figure 2.2. This does not contradict what we saw in the proof of theorem 2.3: if we choose $x \neq y$, then e is an injective n-obsessional experiment, but the *AC* proof net R' contains two contraction links with arity n. This argument also proves that the analogue of theorem 2.1 does not hold in the case of *AC* proof nets of *MELL*. In fact, there is an *AC* proof net R of *MELL* such that no experiment of R contains all information about R with respect to *MELL*.

However, what we have just discussed does not exclude the possibility that we could have a different answer if we restrict to *ACC* proof nets. Actually, this is an open problem. The interest of *ACC* proof nets is that they lie between the subsystem of *MELL* which corresponds to lambda calculus and the full *MELL* fragment. What we know is that the analogue of theorem 2.1 holds for lambda calculus: if R is a proof net representing a λ-term t, an injective 1-experiment e of R contains all information about R with respect to the subsystem of lambda calculus because, in this particular case, the result of e allows to rebuild t. This useful property was suggested by Laurent Regnier and a proof can be found in the paper [LTdF03].

On the other hand, in the framework of relational semantics, we know that the 2-point of the Taylor expansion of an *ACC* proof net R allows to distinguish R from all other proof nets. This was shown in the paper [GPT16]. Beware, this is a totally different setting in which the 2-point is the result not of an injective 2-obsessional experiment, but of an experiment which is not at all obsessional and is injective in a stronger sense: not only the labels associated with different axiom links are different, but also those of the same axiom link, corresponding to different copies of that axiom produced by a box, are pairwise distinct. Then the analogue of theorem 2.1 is certainly true for *ACC* proof nets in the context of relational semantics: if R is an *ACC* proof net, an experiment that is injective in the sense specified above and that takes two copies of each box of R contains all information about R with respect to *ACC* proof nets. If one could prove that the result of such an experiment belongs to the interpretation of R by coherent semantics, then one would have answered the problem in the coherent case as well.

Chapter 3

Taylor expansion of λ-terms

As suggested by the concluding remark of chapter 2, another important tool to study the question of injectivity besides obsessional experiments is the Taylor expansion of a proof net, allowing to write a proof net as an infinite series of its linear approximations. It was used by Daniel de Carvalho in his article [Car15] to conclude that the relational model is injective for the proof nets of *MELL*, by showing that different proof nets of this fragment are associated with different Taylor expansions.

In this chapter, though, we take a step back and study the Taylor expansion of λ-terms, which was introduced for the first time by Ehrhard and Regnier in their article [ER03] on differential lambda calculus. Choosing lambda calculus over proof nets seems natural if we consider that differentiation was originally understood in mathematical analysis as an operation on functions and lambda calculus claims to be a general theory of functions. Obviously, this choice does not prevent, in principle, a transposition of results obtained in this framework to the case of differential proof nets.

Using this innovative tool, we prove a quite natural and expected property: the Taylor support commutes with head reduction. By using this result, we get a proof of the following fact, called the head reduction theorem: if a λ-term M is convertible to a head normal form, then the head reduction of M terminates. To our knowledge, there are only two alternative proofs of this fact: one which uses Curry's standardization theorem as illustrated in [Bar84] and one using a factorization argument which involves the definition of a parallel reduction, as explained in [Tak95].

In the sequel, our main reference is the article [Vau19]. The only difference and simplification is that we restrict ourselves to the "qualitative" setting, that is, we forget the semiring of coefficients and we simply consider the support of Taylor expansions. More precisely, we choose the ring of integers modulo 2 as the semiring of coefficients, which allows us to consider sets of resource terms rather than formal linear combinations of resource terms.

On the other hand, we adopt definitions and notations of the book [Bar84] when dealing with usual λ-terms. In particular, we use the symbol $\equiv$ to denote syntactic equality of λ-terms and we extend this convention to resource terms and resource monomials.

In section 3.1, we provide some basic definitions and properties of resource lambda calculus. In section 3.2, we prove the already mentioned commutation of head reduction and Taylor support. Finally, in section 3.3, we introduce the resource reduction relation and study its properties. As a consequence, we get a proof of the head reduction theorem mentioned above.

3.1 Resource terms

We first give some basic definitions. The core idea of resource lambda calculus is that the usual application of a λ-term M to a λ-term N is replaced by a more fine grained construction in which a resource term s takes a certain number of resource terms $t_1, \ldots, t_n$ as arguments. Now, if s is an abstraction λxt and the variable x occurs precisely n times in t then, in the evaluation, each argument t_i replaces exactly one occurrence of x in t.

Convention. From now on, we denote **V** an infinite set, the elements of which are called *variables*.

Definition 3.1. Consider the alphabet **A** made up of the variables, the brackets [and], the angle brackets $\langle$ and $\rangle$, the punctuation symbol , and the symbol λ. We define inductively a set Δ of finite words over **A**, whose elements are called *resource terms*:

- If x is a variable, then $x \in \Delta$.
- If x is a variable and $s \in \Delta$, then $\lambda xs \in \Delta$.
- If $n \geq 0$ is an integer and $s, t_1, \ldots, t_n \in \Delta$, then $\langle s\rangle[t_1, \ldots, t_n] \in \Delta$.

The set of *resource monomials*, denoted $!\Delta$, is the set of factors of resource terms of the form $[t_1, \ldots, t_n]$ for some integer $n \geq 0$ and for some $t_1, \ldots, t_n \in \Delta$. The integer n is called *degree* of the resource monomial $[t_1, \ldots, t_n]$ and it is denoted $deg[t_1, \ldots, t_n]$.

Convention. For all $s \in \Delta$ and for all $\bar{t}_1, \ldots, \bar{t}_n \in !\Delta$, we denote $\langle s\rangle\bar{t}_1 \ldots \bar{t}_n$ the resource term $\langle \ldots \langle s\rangle\bar{t}_1 \ldots\rangle\bar{t}_n$. Moreover, we denote s^n the resource monomial $[s, \ldots, s]$ where s occurs exactly n times. In addition, we call *resource expression* either a resource term or a resource monomial and we write $(!)\Delta$ for either Δ or $!\Delta$. Finally, if we adopt this notation several times in the same context, then we consistently refer every time to the same notion.

For all variables $x_1, \ldots, x_n$ and for all $s_1, \ldots, s_n \in \Delta$, we define as usual the resource expression $e(x_1 := s_1, \ldots, x_n := s_n)$ which is produced by performing a simultaneous substitution of s_i to every occurrence of x_i for $i = 1, \ldots, n$ in a resource expression e. Free and bound occurrences of variables are defined as expected, too. We can also say that a variable occurs free (respectively, bound) in a subset S of Δ when it occurs free (respectively, bound) in a certain resource term $s \in S$.

Convention. In the sequel, resource terms are considered up to α-equivalence and resource monomials up to permutations of their components. This means that, if x is a variable and if $s \in \Delta$, then the resource term λxs is identified with $\lambda ys(x := y)$ for all variables y which do not occur free in s and, if $t_1, \dots, t_n$ are resource terms and if σ is a permutation over $1, \dots, n$, the resource monomials $[t_1, \dots, t_n]$ and $[t_{\sigma(1)}, \dots, t_{\sigma(n)}]$ are identified.

We now introduce some notations.

Definition 3.2. Let $S_1, \dots, S_n$ be subsets of Δ. We define:

$$[S_1, \dots, S_n] := \{[s_1, \dots, s_n] \colon s_1 \in S_1, \dots, s_n \in S_n\}$$

Now consider a subset S of Δ, a subset T of $!\Delta$ and a variable x. Denote S^n the set $[S, \dots, S]$ in which S occurs precisely n times. We define the following sets:

$$\begin{aligned} \lambda xS &:= \{\lambda xs \colon s \in S\} \\ \langle S \rangle T &:= \{\langle s \rangle \bar{t} \colon s \in S, \bar{t} \in T\} \\ S^! &:= \bigcup \{S^n \colon n \in \mathbb{N}\} \end{aligned}$$

In addition, if $T_1, \dots, T_n$ are subsets of $!\Delta$, then we define the set $\langle S \rangle T_1 \dots T_n$ by induction on n as follows:

- If $n = 0$, then $\langle S \rangle T_1 \dots T_n := S$.
- If $n \geq 1$, then $\langle S \rangle T_1 \dots T_n := \langle \langle S \rangle T_1 \dots T_{n-1} \rangle T_n$.

Now, if e is a resource expression, if $x_1, \dots, x_n$ are variables and $S_1, \dots, S_n$ are subsets of Δ, we can clearly define the set $e(x_1 := S_1, \dots, x_n := S_n)$ which is obtained by simultaneously replacing all occurrences of x_i by S_i for all indices $i = 1, \dots, n$.

Definition 3.3. Let $x_1, \dots, x_n$ be variables, let $S_1, \dots, S_n$ be subsets of Δ and let E be a subset of $(!)\Delta$. We define:

$$E(x_1 := S_1, \dots, x_n := S_n) := \bigcup \{e(x_1 := S_1, \dots, x_n := S_n) \colon e \in E\}$$

We then define a notion of partial differentiation as follows.

Definition 3.4. If s and u are resource terms and x is a variable, then the *partial derivative* of s in u with respect to x, denoted $(\partial s/\partial x) \cdot u$, is a subset of Δ given by the following inductive definition:

- If s is the variable x, then $(\partial s/\partial x) \cdot u := \{u\}$.
- If s is a variable y different from x, then $(\partial s/\partial x) \cdot u$ is the empty set.
- If s is an abstraction, we can assume that $s \equiv \lambda yr$, where the variable y is distinct from x and not occurring free in u without loss of generality. We can then define $(\partial s/\partial x) \cdot u := \lambda y(\partial r/\partial x) \cdot u$.

- If $s \equiv \langle r \rangle [t_1, \ldots, t_n]$, then $(\partial s / \partial x) \cdot u$ is the set:

$$\bigcup \left\{ \left\langle \frac{\partial r}{\partial x} \cdot u \right\rangle [t_1, \ldots, t_n], \bigcup \left\{ \langle r \rangle \left[t_1, \ldots, \frac{\partial t_i}{\partial x} \cdot u, \ldots, t_n \right] : i = 1, \ldots, n \right\} \right\}$$

Also, if $t_1, \ldots, t_n$ are resource terms, we define the following subset of $!\Delta$:

$$\frac{\partial [t_1, \ldots, t_n]}{\partial x} := \bigcup \left\{ \left[t_1, \ldots, \frac{\partial t_i}{\partial x} \cdot u, \ldots, t_n \right] : i = 1, \ldots, n \right\}$$

Finally, if U is a subset of Δ and E is a subset of $(!)\Delta$, we define:

$$\frac{\partial E}{\partial x} \cdot U := \bigcup \left\{ \frac{\partial e}{\partial x} \cdot u : e \in E, u \in U \right\}$$

Remark 3.1. If r and u are resource terms, if $\bar{t}$ is a resource monomial and x is a variable then, as an immediate consequence of the previous definition, we get:

$$\frac{\partial \langle r \rangle \bar{t}}{\partial x} \cdot u = \bigcup \left\{ \left\langle \frac{\partial r}{\partial x} \cdot u \right\rangle \bar{t}, \langle r \rangle \left(\frac{\partial \bar{t}}{\partial x} \cdot u \right) \right\}$$

Remark 3.2. Let u be a resource term, let e be a resource expression and let x be a variable. If x does not occur free in e, then $(\partial e / \partial x) \cdot u$ is the empty set. This is proven by induction on e when e is a resource term and then it is immediately extended to the case of resource expressions.

Remark 3.3. Let u be a resource term, let $\bar{t}$ be a resource monomial and let x be a variable. Then all resource monomials in $(\partial \bar{t} / \partial x) \cdot u$ have the same degree as $\bar{t}$.

The following result is easily established: one gives a proof by induction in the case of a resource term and then the result is easily generalized to the case of resource expressions.

Lemma 3.1. *Let u and v be resource terms, let e be a resource expression, let x and y be variables. If x does not occur free in v, then:*

$$\frac{\partial}{\partial y} \left(\frac{\partial e}{\partial x} \cdot u \right) \cdot v = \bigcup \left\{ \frac{\partial}{\partial x} \left(\frac{\partial e}{\partial y} \cdot v \right) \cdot u, \frac{\partial e}{\partial x} \cdot \left(\frac{\partial u}{\partial y} \cdot v \right) \right\}$$

We now have the analogous of Schwarz's lemma stating that, under certain assumptions, the order of partial derivatives is irrelevant.

Proposition 3.1. *Assume that u and v are resource terms, e is a resource expression, x and y are variables. If x does not occur free in v and y does not occur free in u, then:*

$$\frac{\partial}{\partial y} \left(\frac{\partial e}{\partial x} \cdot u \right) \cdot v = \frac{\partial}{\partial x} \left(\frac{\partial e}{\partial y} \cdot v \right) \cdot u$$

Proof. Immediate consequence of lemma 3.1 and remark 3.2. □

Definition 3.5. Let $u_1, \ldots, u_n$ be resource terms, let e be a resource expression and let y be a variable. If y does not occur free in $u_1, \ldots, u_n$, then we define the set $(\partial^n e / \partial y^n) \cdot (u_1, \ldots, u_n)$ by induction on n as follows:

- If $n = 0$, then $(\partial^n e/\partial y^n) \cdot (u_1, \ldots, u_n) = \{e\}$.
- If $n \geq 1$, we have:

$$\frac{\partial^n e}{\partial y^n} \cdot (u_1, \ldots, u_n) := \frac{\partial}{\partial y}\Big(\frac{\partial^{n-1} e}{\partial y^{n-1}} \cdot (u_1, \ldots, u_{n-1})\Big) \cdot u_n$$

Now, if y does not occur free in $e, u_1, \ldots, u_n$ and if x is any variable, we define:

$$\frac{\partial^n e}{\partial x^n} \cdot (u_1, \ldots, u_n) := \Big(\frac{\partial^n e[x := y]}{\partial y^n} \cdot (u_1, \ldots, u_n)\Big)[y := x]$$

The previous definition is correct: exactly as in the quantitative framework, it does not depend on the choice of the variable y. Recall that, in mathematical analysis, Taylor expansions involve iterated derivatives in zero. The following definition is then justified.

Definition 3.6. Let $\bar{t} = [t_1, \ldots, t_n]$ be a resource monomial, let e be a resource expression and let x be a variable. Then we define the partial derivative of e in $\bar{t}$ with respect to x as the following set:

$$\frac{\partial^n e}{\partial x^n} \cdot \bar{t} := \frac{\partial^n e}{\partial x^n} \cdot (t_1, \ldots, t_n)$$

The *n-linear substitution* of $\bar{t}$ for x in e is the set:

$$\partial_x e \cdot \bar{t} := \Big(\frac{\partial^n e}{\partial x^n} \cdot \bar{t}\Big)[x := \varnothing]$$

Finally, if T is a subset of $!\Delta$ and E is a subset of $(!)\Delta$, we define:

$$\partial_x E \cdot T := \bigcup\{\partial_x e \cdot \bar{t} \colon e \in E, \bar{t} \in T\}$$

By proposition 3.1, the previous definition is correct.

Convention. If $S_1, \ldots, S_n$ are subsets of Δ and if $s_1, \ldots, s_m$ are resource terms, we denote $[S_1, \ldots, S_n, s_1, \ldots, s_m]$ the set $[S_1, \ldots, S_n, \{s_1\}, \ldots, \{s_m\}]$. Similarly, if s is a resource term, $\bar{t}$ is a resource monomial, e is a resource expression, x is a variable, S is a subset of Δ and T is a subset of $!\Delta$, then the sets $\langle S\rangle\{\bar{t}\}$, $\langle\{s\}\rangle T$ and $\partial_x\{e\} \cdot T$ are denoted $\langle S\rangle\bar{t}$, $\langle s\rangle T$ and $\partial_x e \cdot T$ respectively.

We now recall the following properties of multilinear substitutions. We do not provide the proofs, which follow the same pattern described in [Vau19].

Lemma 3.2. *Let S and T be subsets of Δ and let x be a variable. Then:*

$$S^![x := T] = (S[x := T])^!$$

Lemma 3.3. *If R is a subset of Δ and x is a variable, we have the following conditions:*

(i) The identity $\partial_x x \cdot R^! = R$ holds.

(ii) If y is a variable different from x, then $\partial_x y \cdot R^! = \{y\}$.

(iii) If S is a subset of Δ and if y is a variable different from x and not occurring free in R, then $\partial_x \lambda y S \cdot R^! = \lambda y \partial_x S \cdot R^!$.

(iv) If S is a set of resource terms and T set of resource monomials, then the identity $\partial_x \langle S \rangle T \cdot R^! = \langle \partial_x S \cdot R^! \rangle \partial_x T \cdot R^!$ holds.

Lemma 3.4. *If S is a subset of Δ, if E is a subset of $(!)\Delta$ and if x is a variable, we have:*

$$E[x := S] = \partial_x E \cdot S^!$$

As a consequence, we get the following result.

Lemma 3.5. *Let S and T be subsets of Δ and let x be a variable. Then:*

$$\partial_x S^! \cdot T^! = (\partial_x S \cdot T^!)^!$$

Proof. We have:

$$\begin{aligned} \partial_x S^! \cdot T^! &= S^![x := T] && \text{by lemma 3.4,} \\ &= (S[x := T])^! && \text{by lemma 3.2,} \\ &= (\partial_x S \cdot T^!)^! && \text{by lemma 3.4.} \end{aligned}$$ □

3.2 Head reduction and Taylor support

In the sequel we rely on the following well known result on the syntactic form of usual λ-terms, which can easily be generalized to the case of resource terms.

Proposition 3.2. *If M is a λ-term then, for some variables $x, x_1, \ldots, x_n$ and for some λ-terms $N_1, \ldots, N_k$, exactly one of the two following possibilities holds:*

- *$M \equiv \lambda x_1 \ldots \lambda x_n x N_1 \ldots N_k$.*
- *$M \equiv \lambda x_1 \ldots \lambda x_n (\lambda x P) N N_1 \ldots N_k$ for some λ-terms P and N.*

Proof. See corollary 8.3.8 of [Bar84]. □

Definition 3.7. The *one step head reduction* on λ-terms is the function $\mathbf{H}$ which maps λ-terms to λ-terms and is defined as follows:

$$\begin{aligned} \mathbf{H}(\lambda x_1 \ldots \lambda x_n (\lambda x P) N N_1 \ldots N_k) &:= \lambda x_1 \ldots \lambda x_n P[x := N] N_1 \ldots N_k \\ \mathbf{H}(\lambda x_1 \ldots \lambda x_n x N_1 \ldots N_k) &:= \lambda x_1 \ldots \lambda x_n x N_1 \ldots N_k \end{aligned}$$

We can then define the one step head reduction function on resource terms as follows.

Definition 3.8. We call *one step head reduction* on resource terms the function $\mathbf{H}$ which maps resource terms to sets of resource terms and is defined as follows:

$$\begin{aligned} \mathbf{H}(\lambda x_1 \ldots \lambda x_n \langle \lambda x s \rangle \bar{t} \bar{t}_1 \ldots \bar{t}_k) &:= \lambda x_1 \ldots \lambda x_n \langle \partial_x s \cdot \bar{t} \rangle \bar{t}_1 \ldots \bar{t}_k \\ \mathbf{H}(\lambda x_1 \ldots \lambda x_n x \bar{t}_1 \ldots \bar{t}_k) &:= \{\lambda x_1 \ldots \lambda x_n x \bar{t}_1 \ldots \bar{t}_k\} \end{aligned}$$

In addition, if S is a subset of Δ, we define:

$$\mathbf{H}(S) := \bigcup \{\mathbf{H}(s) : s \in S\}$$

Finally, we define the Taylor support function on λ-terms.

Definition 3.9. The *Taylor support* is the function $\mathbf{T}$ which maps λ-terms to sets of resource terms and is defined inductively as follows:

$$\begin{aligned}\mathbf{T}(x) &:= \{x\}\\ \mathbf{T}(\lambda x M) &:= \lambda x \mathbf{T}(M)\\ \mathbf{T}(MN) &:= \langle \mathbf{T}(M)\rangle \mathbf{T}(N)^!\end{aligned}$$

We now prove the following key result.

Lemma 3.6. *If M and N are λ-terms and if x is a variable, then we get the condition:*

$$\mathbf{T}(M[x := N]) = \partial_x \mathbf{T}(M) \cdot \mathbf{T}(N)^!$$

Proof. We reason by induction on the structure of M as λ-term:

- If M is the variable x, then we have:

$$\begin{aligned}\mathbf{T}(M[x := N]) &= \mathbf{T}(N)\\ &= \partial_x x \cdot \mathbf{T}(N)^! && \text{by lemma 3.3, item (i)}\\ &= \partial_x \mathbf{T}(M) \cdot \mathbf{T}(N)^! && \text{by definition 3.9}\end{aligned}$$

- If M is a variable y different from x, then:

$$\begin{aligned}\mathbf{T}(M[x := N]) &= \{y\} && \text{by definition 3.9}\\ &= \partial_x y \cdot \mathbf{T}(N)^! && \text{by lemma 3.3, item (ii)}\\ &= \partial_x \mathbf{T}(M) \cdot \mathbf{T}(N)^! && \text{by definition 3.9}\end{aligned}$$

- If M is an abstraction, we can suppose $M \equiv \lambda y P$ with y different from x and not occurring free in $\mathbf{T}(N)$ without loss of generality. Hence, we get:

$$\begin{aligned}\mathbf{T}(M[x := N]) &= \mathbf{T}(\lambda y P[x := N])\\ &= \lambda y \mathbf{T}(P[x := N]) && \text{by definition 3.9}\\ &= \lambda y \partial_x \mathbf{T}(P) \cdot \mathbf{T}(N)^! && \text{by inductive hypothesis}\\ &= \partial_x \lambda y \mathbf{T}(P) \cdot \mathbf{T}(N)^! && \text{by lemma 3.3, item (iii)}\\ &= \partial_x \mathbf{T}(M) \cdot \mathbf{T}(N)^! && \text{by definition 3.9}\end{aligned}$$

- If $M \equiv PQ$, then:

$$\begin{aligned}&\mathbf{T}(M[x := N])\\ &= \mathbf{T}(P[x := N]Q[x := N])\\ &= \langle \mathbf{T}(P[x := N])\rangle \mathbf{T}(Q[x := N])^! && \text{by definition 3.9}\\ &= \langle \partial_x \mathbf{T}(P) \cdot \mathbf{T}(N)^!\rangle (\partial_x \mathbf{T}(Q) \cdot \mathbf{T}(N)^!)^! && \text{by inductive hypothesis}\\ &= \langle \partial_x \mathbf{T}(P) \cdot \mathbf{T}(N)^!\rangle \partial_x \mathbf{T}(Q)^! \cdot \mathbf{T}(N)^! && \text{by lemma 3.5}\\ &= \partial_x \langle \mathbf{T}(P)\rangle \mathbf{T}(Q)^! \cdot \mathbf{T}(N)^! && \text{by lemma 3.3, item (iv)}\\ &= \partial_x \mathbf{T}(M) \cdot \mathbf{T}(N)^! && \text{by definition 3.9}\end{aligned}$$

□

We can finally prove that the Taylor support "commutes" with the one step head reduction. Notice that, in the following statement, the symbol $\mathbf{H}$ denotes two distinct functions: one acting on resource terms, the other acting on usual λ-terms.

Proposition 3.3. *Let M be a λ-term. Then we have* $\mathbf{H}(\mathbf{T}(M)) = \mathbf{T}(\mathbf{H}(M))$.

Proof. By corollary 8.3.8 of [Bar84], there exist variables $x, x_1, \ldots, x_n$ and there are λ-terms $P, N, N_1, \ldots, N_k$ such that one of the following possibilities holds:

- We have $M \equiv \lambda x_1 \ldots \lambda x_n x N_1 \ldots N_k$. In this case, we immediately obtain:

$$\begin{aligned}
&\mathbf{H}(\mathbf{T}(M)) = \mathbf{H}(\lambda x_1 \ldots \lambda x_n \langle x\rangle \mathbf{T}(N_1)^! \ldots \mathbf{T}(N_k)^!) \\
&= \bigcup\{\mathbf{H}(\lambda x_1 \ldots \lambda x_n \langle x\rangle \bar{t}_1 \ldots \bar{t}_k) \colon \bar{t}_1 \in \mathbf{T}(N_1)^!, \ldots, \bar{t}_k \in \mathbf{T}(N_k)^!\} \\
&= \bigcup\{\{\lambda x_1 \ldots \lambda x_n \langle x\rangle \bar{t}_1 \ldots \bar{t}_k\} \colon \bar{t}_1 \in \mathbf{T}(N_1)^!, \ldots, \bar{t}_k \in \mathbf{T}(N_k)^!\} \\
&= \lambda x_1 \ldots \lambda x_n \langle x\rangle \mathbf{T}(N_1)^! \ldots \mathbf{T}(N_k)^! = \mathbf{T}(M) = \mathbf{T}(\mathbf{H}(M))
\end{aligned}$$

- We have $M \equiv \lambda x_1 \ldots \lambda x_n (\lambda x P) N N_1 \ldots N_k$. Then, by lemma 3.6, we have the following identities, where it is intended that the index i ranges from 1 to k:

$$\begin{aligned}
&\mathbf{H}(\mathbf{T}(M)) = \mathbf{H}(\lambda x_1 \ldots \lambda x_n \langle \lambda x \mathbf{T}(P)\rangle \mathbf{T}(N)^! \mathbf{T}(N_1)^! \ldots \mathbf{T}(N_k)^!) \\
&= \bigcup\{\mathbf{H}(\lambda x_1 \ldots \lambda x_n \langle \lambda x s\rangle \bar{t}\bar{t}_1 \ldots \bar{t}_k) \colon s \in \mathbf{T}(P), \bar{t} \in \mathbf{T}(N)^!, \bar{t}_i \in \mathbf{T}(N_i)^!\} \\
&= \bigcup\{\lambda x_1 \ldots \lambda x_n \langle \partial_x s \cdot \bar{t}\rangle \bar{t}_1 \ldots \bar{t}_k \colon s \in \mathbf{T}(P), \bar{t} \in \mathbf{T}(N)^!, \bar{t}_i \in \mathbf{T}(N_i)^!\} \\
&= \lambda x_1 \ldots \lambda x_n \langle \partial_x \mathbf{T}(P) \cdot \mathbf{T}(N)^!\rangle \mathbf{T}(N_1)^! \ldots \mathbf{T}(N_k)^! \\
&= \lambda x_1 \ldots \lambda x_n \langle \mathbf{T}(P[x := N])\rangle \mathbf{T}(N_1)^! \ldots \mathbf{T}(N_k)^! \\
&= \mathbf{T}(\lambda x_1 \ldots \lambda x_n P[x := N] N_1 \ldots N_k) = \mathbf{T}(\mathbf{H}(M))
\end{aligned}$$

□

3.3 The head reduction theorem

We recall the some standard notions.

Definition 3.10. A λ-term $\lambda x_1 \ldots \lambda x_n x N_1 \ldots N_k$ (respectively, a resource term $\lambda x_1 \ldots \lambda x_n x \bar{t}_1 \ldots \bar{t}_k$), where $N_1, \ldots, N_k$ are λ-terms (respectively, $\bar{t}_1, \ldots, \bar{t}_k$ are resource terms) and $x, x_1, \ldots, x_n$ are variables, is called a *head normal form*. We say that a λ-term M has a head normal form if it is convertible (in other words, β-equivalent)[1] to a head normal form. On the other hand, we say that the head reduction of M terminates if there is a non negative integer k such that $\mathbf{H}^k(M)$ is a head normal form.

Remark 3.4. Obviously, if the head reduction of a λ-term M terminates, then M has a head normal form.

The converse of the previous remark holds. Our goal is to prove this using the tools we developed in this chapter. We first provide a definition of resource reduction in the qualitative setting.

[1] See proposition 3.2.1 of [Bar84].

Convention. In what follows, if s is a resource term and S is a subset of Δ, we write $s \to_\partial S$ actually meaning $\{s\} \to_\partial S$.

Definition 3.11. The relation $\to_\partial$ on the set of resource terms is determined by the following inductive definition:

- If $s \in \Delta$ and $\bar{t} \in !\Delta$, then $\langle \lambda xs\rangle \bar{t} \to_\partial \partial_x s \cdot \bar{t}$.
- If $s \in \Delta$, if $t \in !\Delta$ and if S is a subset of Δ such that $s \to_\partial S$, then we have the conditions $\lambda xs \to_\partial \lambda xS$ and $\langle s\rangle \bar{t} \to_\partial \langle S\rangle \bar{t}$.
- If $s, t_0, t_1, \dots, t_n \in \Delta$ and if T is a subset of Δ such that $t_0 \to_\partial T$, then the condition $\langle s\rangle[t_0, t_1, \dots, t_n] \to_\partial \langle s\rangle[T, t_1, \dots, t_n]$ holds.

The *one step resource reduction*, still denoted $\to_\partial$ with a slight abuse of notation, is the binary relation on the set of subsets of Δ which is given by the following condition: whenever $s_0, \dots, s_n$ are resource terms (not distinct, in general) and whenever $S_0, \dots, S_n$ are subsets of Δ such that $s_0 \to_\partial S_0$ and $s_i \to_\partial{}^? S_i$ for each index $i = 1, \dots, n$, we get $\{s_0, \dots, s_n\} \to_\partial \bigcup\{S_i : i = 0, \dots, n\}$. We call *resource reduction* the reflexive transitive closure of the one step resource reduction.

Moreover, we say that a resource term is *normal* or a *normal form* if it has no factor of the shape $\langle \lambda xs\rangle \bar{t}$, with x a variable, s a resource term and $\bar{t}$ a resource monomial. Such a factor is called a *redex* and $\partial_x s \cdot \bar{t}$ is its *contractum*. A normal subset of Δ is a set of normal resource terms.

Remark 3.5. Let s be a resource term. One can immediately check, by induction on s, that $s \to_\partial S$ for some set S of resource terms if and only if s is not normal. In addition, that set S is finite. Similarly, if T is a subset of Δ, we get $T \to_\partial S$ for some set S of resource terms if and only if T is not normal. Moreover, the set S contains all normal resource terms of T.

Remark 3.6. The condition $s_0 \to_\partial S_0$ appearing in the definition of the one step resource reduction is not crucial in the sense that we could also ask $s_0 \to_\partial{}^? S_0$ and go on with pretty much the same arguments. We can justify our choice by noticing that the alternative requirement $s_0 \to_\partial{}^? S_0$ would make the reduction relation on subsets of Δ reflexive. This seems quite unpleasant: intuitively, the computational process never terminates. On the other hand, as we observed in the previous remark, the definition we actually gave ensures that, when T is a normal subset of Δ, there is no set of resource terms S such that $T \to_\partial S$.

Remark 3.7. Let $s \in \Delta$ and let $T, T', T_1, \dots, T_n$ be finite subsets of Δ. If $T \to_\partial T'$, then we have $\langle s\rangle[T, T_1, \dots, T_n] \to_\partial \langle s\rangle[T', T_1, \dots, T_n]$. In particular, if $t_1, \dots, t_n$ are resource terms such that $t_i \to_\partial{}^* T_i$ for every $i = 1, \dots, n$, then the condition $\langle s\rangle[t_1, \dots, t_n] \to_\partial{}^* \langle s\rangle[T_1, \dots, T_n]$ holds.

Remark 3.8. If S, S', T, T' are finite subsets of Δ such that $S \to_\partial S'$ and $T \to_\partial T'$, then we get $\bigcup\{S, T\} \to_\partial \bigcup\{S', T'\}$. Evidently, the same holds for the reflexive closure of the one step resource reduction and this can be generalized to finite unions of subsets of Δ.

Remark 3.9. If S is a finite subset of Δ, then we have $S \to_\partial{}^? \mathbf{H}(S)$. In particular, the set $\mathbf{H}(S)$ is finite as well, by remark 3.5.

The following result is easily established as in the quantitative case. See, for instance, the proof of lemma 3.11 of [Vau19].

Lemma 3.7. *Let r be a resource term and let S and T be subsets of Δ such that $r \to_\partial S$ and $r \to_\partial T$. Then there is a subset U of Δ such that $S \to_\partial{}^? U$ and $T \to_\partial{}^? U$.*

Remark 3.10. One easily generalizes the previous result to the case in which we have $r \to_\partial{}^? S$ and $r \to_\partial{}^? T$. For instance, if $S = \{r\}$, then we just choose $U := T$ and we are done.

As a consequence, we obtain the following lemma.

Lemma 3.8. *Let R, S and T be sets of resource terms such that $R \to_\partial S$ and $R \to_\partial T$. Then there is a subset U of Δ such that $S \to_\partial{}^? U$ and $T \to_\partial{}^? U$.*

Proof. By definition 3.11, for some resource terms $s_0, \dots, s_n, t_0, \dots, t_m$ and for some subsets $S_0, \dots, S_n, T_0, \dots, T_m$ of Δ, we have $s_0 \to_\partial S_0$, $t_0 \to_\partial T_0$, $s_i \to_\partial{}^? S_i$ for every $i = 1, \dots, n$, $t_j \to T_j$ for every $j = 1, \dots, m$ and we get the conditions:

$$R = \{s_0, \dots, s_n\} = \{t_0, \dots, t_m\}$$
$$S = \bigcup\{S_i : i = 0, \dots, n\}$$
$$T = \bigcup\{T_j : j = 0, \dots, m\}$$

Now consider, for all indices $i = 0, \dots, n$, the set $J_i := \{j \in \{0, \dots, m\} : s_i = t_j\}$. For all indices $j \in J_i$ there is, by either lemma 3.7 or remark 3.10, a subset U_{ij} of Δ such that $S_i \to_\partial{}^? U_{ij}$ and $T_j \to_\partial{}^? U_{ij}$. Since $\{0, \dots, m\} = \bigcup\{J_i : i = 0, \dots, n\}$, the following conditions hold:

$$S = \bigcup\{S_i : j \in J_i, i = 0, \dots, n\}$$
$$T = \bigcup\{T_j : j \in J_i, i = 0, \dots, n\}$$

Then, by remark 3.8, we are done by picking:

$$U := \bigcup\{U_{ij} : j \in J_i, i = 0, \dots, n\}$$ □

Remark 3.11. The analogous of remark 3.10 holds. In other words, the reflexive closure of the one step resource reduction satisfies the diamond property: if R, S and T are sets of resource terms such that $R \to_\partial{}^? S$ and $R \to_\partial{}^? T$, there exists a subset U of Δ such that $S \to_\partial{}^? U$ and $T \to_\partial{}^? U$.

Recall that, if the diamond property is satisfied by a binary relation, then it is also satisfied by its transitive closure (lemma 3.2.2 of [Bar84]). Therefore, by the previous remark, we get the following result.

Proposition 3.4. *Resource reduction satisfies the diamond property: if R, S and T are sets of resource terms such that $R \to_\partial{}^* S$ and $R \to_\partial{}^* T$, there is a subset U of Δ such that $S \to_\partial{}^* U$ and $T \to_\partial{}^* U$.*

We now define a notion of size on resource expressions.

Definition 3.12. The *size* of a resource term s, denoted $\mathbf{s}(s)$, is a strictly positive integer defined inductively as follows:

- If s is a variable x, then $\mathbf{s}(s) := 1$.
- If $s \equiv \lambda x r$, then $\mathbf{s}(s) := 1 + \mathbf{s}(r)$.
- If $s \equiv \langle r \rangle [t_1, \ldots, t_n]$, then $\mathbf{s}(s) := 1 + \mathbf{s}(r) + \mathbf{s}(t_1) + \cdots + \mathbf{s}(t_n)$.

In addition, the size of a resource monomial $\bar{t} = [t_1, \ldots, t_n]$ is the non negative integer defined by $\mathbf{s}(\bar{t}) := \mathbf{s}(t_1) + \cdots + \mathbf{s}(t_n)$.

If we restrict to finite sets of resource terms, we can extend the definition of size.

Definition 3.13. The size of a finite set of resource terms S is the non negative integer defined by the condition:

$$\mathbf{s}(S) := \sup\{\mathbf{s}(s) \colon s \in S\}$$

In addition, we define the non negative integer $\mathbf{n}(S)$ as the number of resource terms $s \in S$ such that $\mathbf{s}(s) = \mathbf{s}(S)$.

We do not prove the following result, which is the analogous of lemma 3.12 of [Vau19].

Lemma 3.9. *Let $s \in \Delta$ and let S be a subset of Δ such that $s \to_\partial S$. Then $\mathbf{s}(S) < \mathbf{s}(s)$.*

We consequently have the following lemma.

Lemma 3.10. *Let s be a resource term. Then $\mathbf{s}(r) \leq \mathbf{s}(s)$ for all $r \in \mathbf{H}(s)$. In addition, for every such r, equality holds if and only if s is a head normal form.*

Proof. If s is a head normal form, we get $r \equiv s$ by definition 3.8. Otherwise, we must have $s \to_\partial \mathbf{H}(s)$. By lemma 3.9, we can then conclude that $\mathbf{s}(r) < \mathbf{s}(s)$. □

By remark 3.9, it is meaningful to compare the size of a finite set of resource terms S with the size of $\mathbf{H}(S)$.

Lemma 3.11. *Let S be a finite subset of Δ. Then $\mathbf{s}(\mathbf{H}(S)) \leq \mathbf{s}(S)$. Moreover, if S is not empty and contains no head normal forms, the strict inequality holds.*

Proof. The result is trivial if $\mathbf{H}(S)$ is empty. On the other hand, if this set is not empty, then S is not empty as well. Let s_0 be a resource term of maximal size in $\mathbf{H}(S)$ and let $s_1 \in S$ be a resource term such that $s_0 \in \mathbf{H}(s_1)$. By lemma 3.10 and by definition 3.13, we obtain the following inequalities:

$$\mathbf{s}(\mathbf{H}(S)) = \mathbf{s}(s_0) \leq \mathbf{s}(s_1) \leq \mathbf{s}(S)$$

Finally, if S contains no head normal forms then, for sure, its element s_1 is not a head normal form either. Hence, by lemma 3.10, the first inequality above is a strict inequality and we are done. □

We now prove that resource reduction enjoys weak normalization.

Proposition 3.5. *Let S be a finite subset of Δ. Then there exist a non negative integer n and a normal subset T of Δ such that $S \rightarrow_{\partial}{}^{n} T$.*

Proof. For any subset R of Δ, we denote $\tilde{R}$ the set of non normal resource terms of R. We reason by lexicographical induction on the pair $(\mathbf{s}(\tilde{S}), \mathbf{n}(\tilde{S}))$.

- *Base of the induction.* If $\mathbf{s}(\tilde{S}) = 0$, then S is normal. Thus, by picking $n := 0$ and $T := S$, we are done.
- *Inductive step.* We can suppose $\mathbf{s}(\tilde{S}) \geq 1$. Consider a non normal resource term $s_0 \in \tilde{S}$ of maximal size. By remark 3.5, there is a subset S_0 of Δ such that $s_0 \rightarrow_{\partial} S_0$. Then, by definition 3.11, we get $S \rightarrow_{\partial} R$, where we define:

$$R := \bigcup\{S \setminus \{s_0\}, S_0\}$$

  By lemma 3.9, the size of S_0 is strictly smaller than the size of s_0. We then have two possibilities:

  - If $\mathbf{n}(\tilde{S}) = 1$, then $\mathbf{s}(\tilde{R}) < \mathbf{s}(\tilde{S})$.
  - If $\mathbf{n}(\tilde{S}) \geq 2$, then $\mathbf{s}(\tilde{R}) = \mathbf{s}(\tilde{S})$ but $\mathbf{n}(\tilde{R}) < \mathbf{n}(\tilde{S})$.

  In both cases, we can apply the inductive hypothesis, which yields a non negative integer k and a normal subset T of Δ satisfying $R \rightarrow_{\partial}{}^{k} T$. Then, by picking $n := k + 1$, we are done. □

Remark 3.12. Resource reduction does not enjoy at all strong normalization, as we have plenty of counterexamples of non terminating reduction sequences: if we just consider a resource term s and a subset S of Δ such that $s \rightarrow_{\partial} S$, we get $T \rightarrow_{\partial} T$ by picking $T := \bigcup\{S, \{s\}\}$.

On the other hand, the following result is good news.

Proposition 3.6. *If S is a finite subset of Δ, then there exists a unique normal subset T of Δ such that $S \rightarrow_{\partial}{}^{*} T$.*

Proof. The existence of such a subset of Δ is guaranteed by proposition 3.5. If R and T are normal subsets of Δ such that $S \rightarrow_{\partial}{}^{*} R$ and $S \rightarrow_{\partial}{}^{*} T$ then, thanks to proposition 3.4, we also have a subset U of Δ such that $R \rightarrow_{\partial}{}^{*} U$ and $T \rightarrow_{\partial}{}^{*} U$. Observe that, by remark 3.5, we must have $R = U = T$ and so we are done. □

The following definition is now justified.

Definition 3.14. Let S be a finite subset of Δ. The unique normal subset T of Δ such that $S \rightarrow_{\partial}{}^{*} T$ is called *normal form* of S and denoted $\mathbf{NF}(S)$.

Convention. For the sake of simplicity, if s is a resource term, we write $\mathbf{NF}(s)$ actually meaning $\mathbf{NF}(\{s\})$.

We now have the following results.

Lemma 3.12. *Let s be a resource term such that $\mathbf{NF}(s)$ is not empty. Then there exists a non negative integer k such that $\mathbf{H}^{k}(s)$ contains a head normal form.*

Proof. We notice that, if $\mathbf{H}^n(s)$ were empty for some positive integer n, then we would have $s \to_\partial^* \mathbf{H}^n(s)$ and thus $\mathbf{NF}(s) = \mathbf{H}^n(s) = \emptyset$, which is a contradiction with our hypotheses. Therefore, the set $\mathbf{H}^n(s)$ is not empty for all non negative integers n.

We now choose k as a non negative integer such that $\mathbf{s}(\mathbf{H}^k(s))$ is minimal in the set $\{\mathbf{s}(\mathbf{H}^n(s)) \colon n \in \mathbb{N}\}$. We notice that, by lemma 3.11 and by the choice of k, the condition $\mathbf{s}(\mathbf{H}^k(s)) = \mathbf{s}(\mathbf{H}^{k+1}(s))$ holds. Hence, by using lemma 3.11 and the fact that $\mathbf{H}^k(s)$ is not empty, we can conclude that the set $\mathbf{H}^k(s)$ contains a head normal form. □

Theorem 3.1. *Let M be a λ-term. If $\mathbf{NF}(s)$ is not empty for some $s \in \mathbf{T}(M)$, the head reduction of M terminates.*

Proof. By lemma 3.12, we know that there exists a non negative integer k such that $\mathbf{H}^k(s)$ contains a head normal form. For the sake of contradiction, assume:

$$\mathbf{H}^k(M) \equiv \lambda x_1 \dots \lambda x_n (\lambda x P) N N_1 \dots N_k$$

By using the hypothesis that $s \in \mathbf{T}(M)$ and by applying proposition 3.3, we get the condition:

$$\begin{aligned} \mathbf{H}^k(s) \subseteq \mathbf{H}^k(\mathbf{T}(M)) &= \mathbf{T}(\mathbf{H}^k(M)) \\ &= \lambda x_1 \dots \lambda x_n \langle \lambda x \mathbf{T}(P) \rangle \mathbf{T}(N)^! \mathbf{T}(N_1)^! \dots \mathbf{T}(N_k)^! \end{aligned}$$

This contradicts the fact that $\mathbf{H}^k(s)$ contains a head normal form. We can then conclude that $\mathbf{H}^k(M)$ is a head normal form and thus the head reduction of M terminates. □

Lemma 3.13. *Let M and N be λ-terms. If $M \to_\beta N$ and $t \in \mathbf{T}(N)$, then there exist a resource term $s \in \mathbf{T}(M)$ and a set of resource terms S such that $s \to_\partial^* S$ and $t \in S$.*

Proof. We reason by induction on the one step β-reduction $M \to_\beta N$.

- If $M \equiv (\lambda x P)Q$ and $N \equiv P[x := Q]$ then, by lemma 3.5, we get $t \in \partial_x u \cdot \bar{v}$ for a certain $u \in \mathbf{T}(P)$ and for some $\bar{v} \in \mathbf{T}(Q)^!$. We now obtain the desired conclusion by picking $s := \langle \lambda x u \rangle \bar{v}$ and $S := \partial_x u \cdot \bar{v}$.

- If $M \equiv \lambda x P$, $N \equiv \lambda x Q$ and $P \to_\beta Q$, then $t \equiv \lambda x u$ for some $u \in \mathbf{T}(Q)$. By inductive hypothesis, there exist a resource term $r \in \mathbf{T}(P)$ and a subset R of Δ such that $r \to_\partial^* R$ and $u \in R$. Then we are done by picking $s := \lambda x r$ and $S := \lambda x R$.

- If $M \equiv PO$, $N \equiv QO$ and $P \to_\beta Q$, we get $t \equiv \langle u \rangle \bar{v}$ for a certain $u \in \mathbf{T}(Q)$ and for some $\bar{v} \in \mathbf{T}(O)^!$. The inductive hypothesis yields a resource term $r \in \mathbf{T}(P)$ and a subset R of Δ such that $r \to_\partial^* R$ and $u \in R$. We then have the desired result by picking $s := \langle r \rangle \bar{v}$ and $S := \langle R \rangle \bar{v}$.

- If $M \equiv OP$, $N \equiv OQ$ and $P \to_\beta Q$, then we have $t \equiv \langle u\rangle[v_1, \dots, v_n]$ for a certain $u \in \mathbf{T}(O)$ and for some $v_1, \dots, v_n \in \mathbf{T}(Q)$. For $i = 1, \dots, n$ we get, by inductive hypothesis, an element $r_i \in \mathbf{T}(P)$ and a subset R_i of Δ which satisfy $r_i \to_\partial{}^* R_i$ and $v_i \in R_i$. Thus, by remark 3.7, it is enough to choose $s := \langle u\rangle[r_1, \dots, r_n]$ and $S := \langle u\rangle[R_1, \dots, R_n]$. □

We recall one last result about usual lambda calculus.

Lemma 3.14. *If M and N are β-equivalent λ-terms, then there exists a λ-term P such that $M \to_\beta{}^* P$ and $N \to_\beta{}^* P$.*

Proof. See theorem 3.2.8 of [Bar84]. □

We can now prove the following theorem.

Theorem 3.2. *Let M be a λ-term. If M has a head normal form, then the set $\mathbf{NF}(s)$ is not empty for some resource term $s \in \mathbf{T}(M)$.*

Proof. By hypothesis, the λ-term M is convertible to a head normal form N. By lemma 3.14, we get a λ-term P such that $M \to_\beta{}^* P$ and $N \to_\beta{}^* P$. In particular, the λ-term P is a head normal form. This means that:

$$P \equiv \lambda x_1 \dots \lambda x_n (x) P_1 \dots P_k$$

We define the following resource term (it is intended that there are precisely k occurrences of resource monomials):

$$t := \lambda x_1 \dots \lambda x_n \langle x\rangle[\,]\dots[\,]$$

Then $t \in \mathbf{T}(P)$ and so, by lemma 3.13, there exist a resource term $s \in \mathbf{T}(M)$ and a set of resource terms S satisfying $s \to_\partial{}^* S$ and $t \in S$. Now, by proposition 3.4 and by definition 3.14, we must have the condition $S \to_\partial{}^n \mathbf{NF}(s)$ for some non negative integer n. Finally, since t is a normal resource term, by remark 3.5 we obtain $t \in \mathbf{NF}(s)$ and so we are done. □

We can finally establish the head reduction theorem as an easy corollary of the previous results.

Corollary 3.1. *Let M be a λ-term. If M has a head normal form, the head reduction of M terminates.*

Proof. Immediate consequence of theorems 3.1 and 3.2. □

Conclusion

Driven by the fundamental question of identity of proofs, we got interested in three equivalence relations on proof nets and the relationships between them: the syntactic, semantic and observational equivalences. The former is intrinsic to computation, whereas the others depend on the model or on the observable value we choose. If the syntactic and semantic equivalences coincide, then we say that the model is injective. If the syntactic and observational equivalences, instead, coincide, then the question of separability has a positive answer (with respect to the considered observable value).

We chose to investigate the semantic equivalences induced by the coherent multiset based model and the relational model. We turned our attention to the multiplicative and exponential fragment of linear logic. In this framework, we revisited the key notion of obsessional experiment and reviewed the results in the paper [TdF03] leading us to a sufficient condition of local injectivity. Then, motivated by the recent result of injectivity in [Car15], we shifted our attention to the Taylor expansion of λ-terms. We first established that the Taylor support commutes with head reduction, then we focused on the properties of resource reduction in a qualitative setting and we employed them to establish the head reduction theorem.

We now retrieve and update the table in section 4.3 of the paper [TdF03] to sum up the state of the art concerning the question of injectivity. First of all, we introduce the following notations:

- We denote $MELL \setminus \{?w\}$ the fragment of $MELL$ which contains every link except the weakening link. It corresponds to the subsystem of ACC proof nets.
- We define positive and negative formulas by mutual induction:
  - If X is an atomic formula, then $!X$ is a positive formula and $?X^{\perp}$ is a negative formula.
  - If A and B are positive formulas, then $A \otimes B$ is a positive formula.
  - If A and B are negative formulas, then $A \parr B$ is a negative formula.
  - If A is a negative formula, then $!A$ is a positive formula.
  - If A is a positive formula, then $?A$ is a negative formula.

	$[\![\,]\!]_{cohs}$	$[\![\,]\!]_{cohm}$	$[\![\,]\!]_{rel}$
$MELL$	NO	NO	YES
$MELL \setminus \{?w\}$	? (yes)	? (yes)	YES
LL_{pol}	? (no)	? (no)	YES
$(?\parr)LL$	? (yes)	YES	YES

Table 3.1: The question of injectivity: answers and open problems.

We say that an AC proof net R is polarized if the types of the conclusions of R are all subformulas of a positive or of a negative formula. We denote LL_{pol} the system of polarized proof nets.

- We define inductively the set of $(?\parr)LL$ formulas:
  - ⋄ If X is an atomic formula, then X is a $(?\parr)LL$ formula.
  - ⋄ If A and B are two $(?\parr)LL$ formulas, then $?A \parr B$, $A \parr ?B$, $A \parr B$ and $A \otimes B$ are $(?\parr)LL$ formulas.
  - ⋄ If A is a $(?\parr)LL$ formula, then so is $!A$.

  We say that an AC proof net is in $(?\parr)LL$ if the types of its conclusions are all subformulas of $(?\parr)LL$ formulas. We also denote $(?\parr)LL$ the system of $(?\parr)LL$ proof nets.

- The set based coherent semantics, the multiset based coherent semantics and relational semantics are denoted by $[\![\,]\!]_{cohs}$, $[\![\,]\!]_{cohm}$, $[\![\,]\!]_{rel}$ respectively.

The current situation in the study of these subsystems of $MELL$ is summed up in table 3.8. We wrote in capital letters the answers we possess and in small letters the conjectures. Notably, the state of the last column is due to Daniel de Carvalho's proof of injectivity in [Car15]. In addition, the row concerning LL_{pol} was also updated, following a private communication of Damiano Mazza and Michele Pagani. On the other hand the case of $MELL \setminus \{?w\}$, which would help to understand the relation between connectivity and coherence, is still an open problem. There are at least two strategies to prove injectivity for this fragment:

(A) Proving the existence of an injective 1-experiment for all ACC proof nets.

(B) Proving, for any ACC proof net R, the existence of a 2-point that belongs to the interpretation of R by coherent semantics and which is injective in the relational sense.

If we manage to carry out strategy (A), then we obtain a weaker result than the one we would get with strategy (B): for any ACC proof net R we would have an injective 1-experiment and in particular, for all positive integers n, an injective n-obsessional experiment. So, for each proof net R' with the same conclusions as R, we would be able to find a sufficiently large integer n such that the result of an n-obsessional experiment of R is not in the interpretation of R'. However, we would not have found a 2-point that contains all information about R with respect to *ACC* proof nets, which is something we would find instead by using strategy (B).

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