Monadic second order logic

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However, we will see that *EC* is **MSO**-expressible.

MSO games

Preliminaries

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Analogously, we define $\mathfrak{B}' := (\mathfrak{B}, \vec{b}_0, \vec{U}_0)$.

Rules of the game

Two players, the spoiler and the duplicator, play on $\mathfrak{A}'$ and $\mathfrak{B}'$ with two different kind of moves:

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Point move. This is the same move as in the Ehrenfeucht-Fraïssé game for **FO**. The spoiler chooses a structure, $\mathfrak{A}'$ or $\mathfrak{B}'$ and an element of that structure. The duplicator responds with an element in the other structure.

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Set move. The spoiler chooses a structure, $\mathfrak{A}'$ or $\mathfrak{B}'$ and a subset of that structure. The duplicator responds with a subset of the other structure.

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The duplicator wins the k-round game if the function:

$$(c_1^{\mathfrak{A}}, \ldots, c_n^{\mathfrak{A}}, \vec{a}_0, \vec{a}) \mapsto (c_1^{\mathfrak{B}}, \ldots, c_n^{\mathfrak{B}}, \vec{b}_0, \vec{b})$$

is a partial isomorphism between $(\mathfrak{A}', \vec{V})$ and $(\mathfrak{B}', \vec{U})$.

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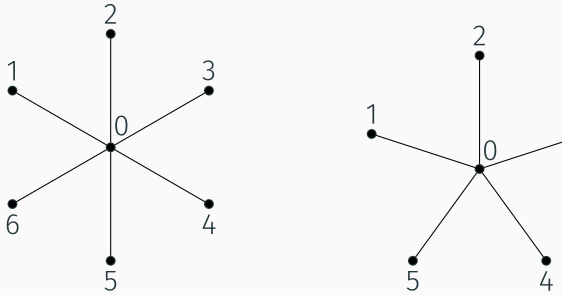
is a partial isomorphism between $(\mathfrak{A}', \vec{V})$ and $(\mathfrak{B}', \vec{U})$.

A player has a winning strategy for the k-round game if he can guarantee he wins regardless of how the other player plays.

Example

We will play games on these two graphs.



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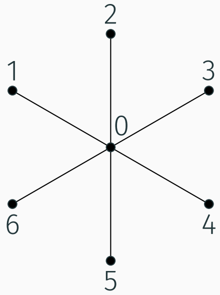

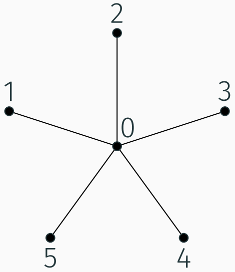

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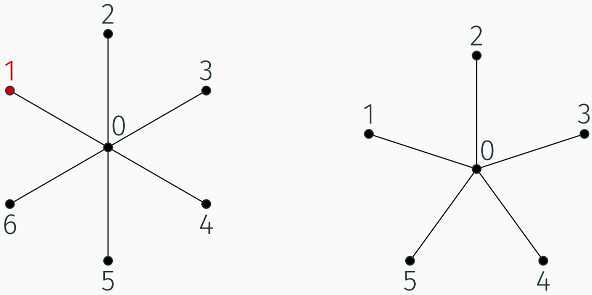

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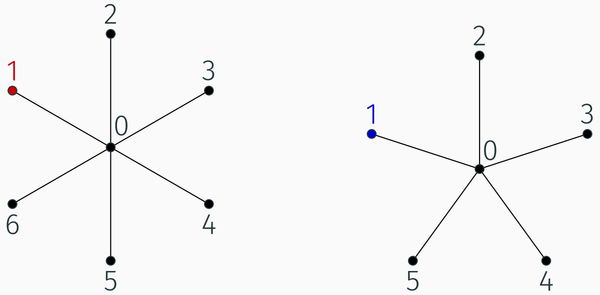

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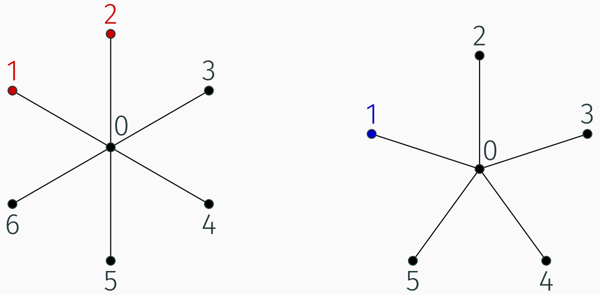

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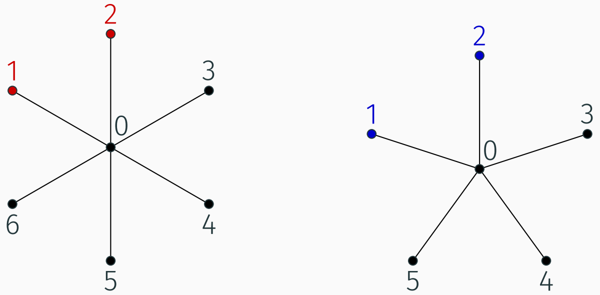

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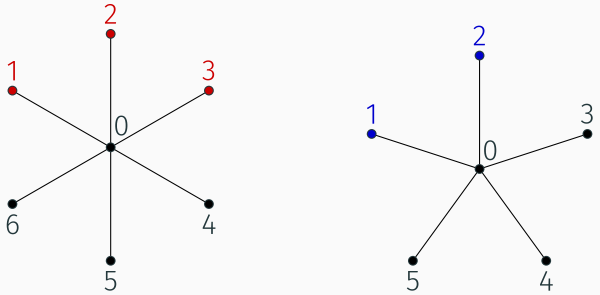

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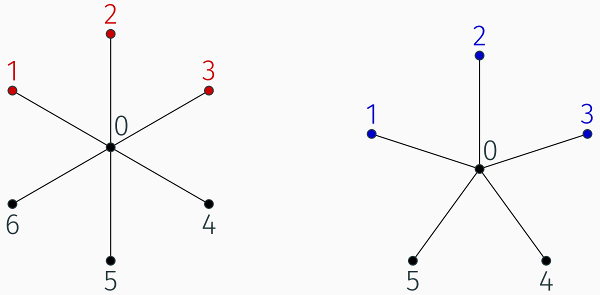

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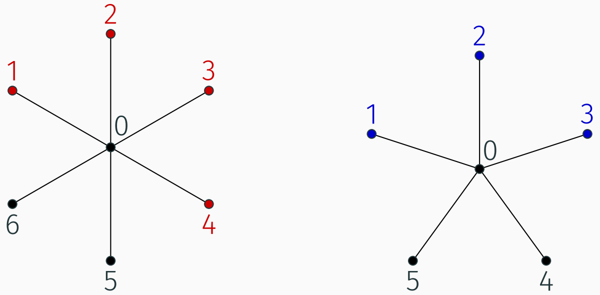

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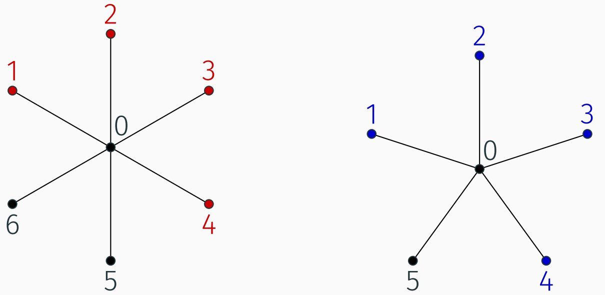

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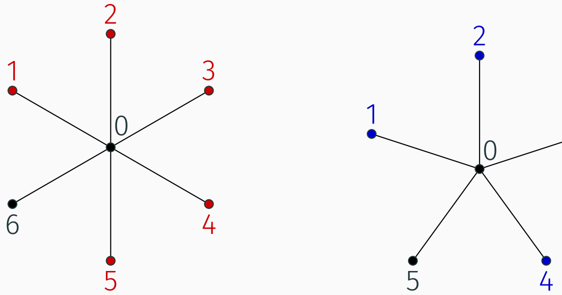

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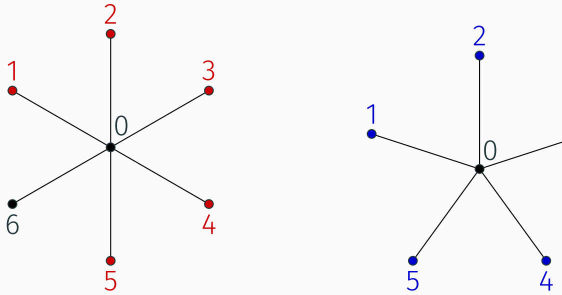

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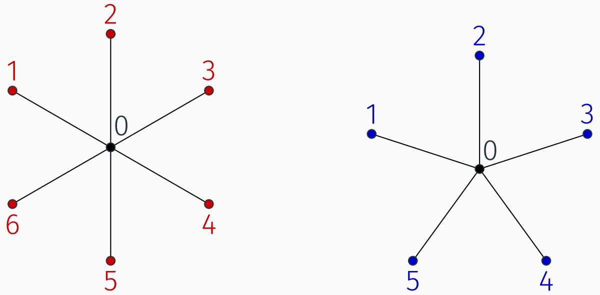

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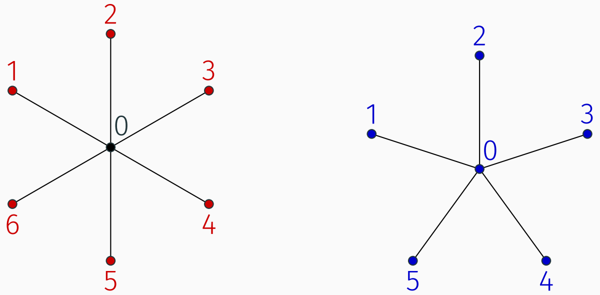

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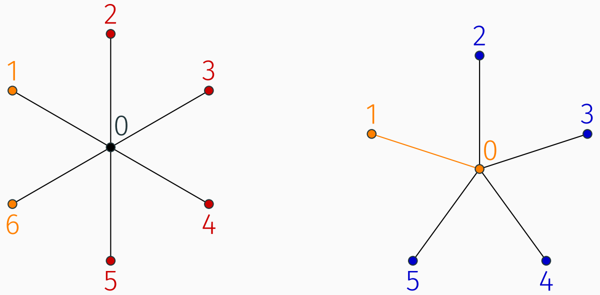
This tells us that $\mathfrak{A} \equiv_5 \mathfrak{B}$ but $\mathfrak{A} \not\equiv_6 \mathfrak{B}$.



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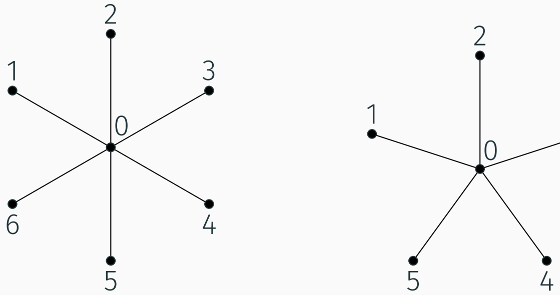

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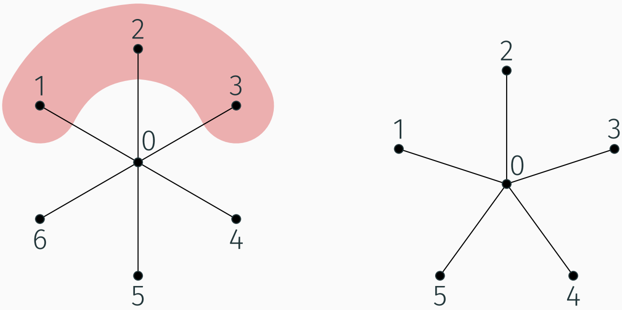

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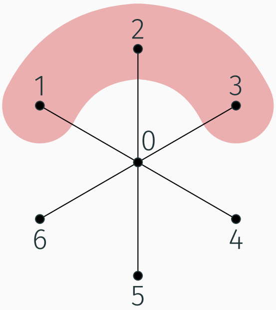

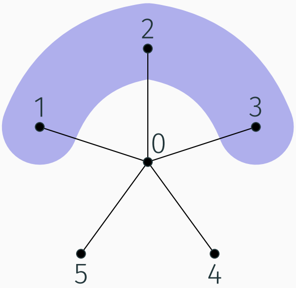

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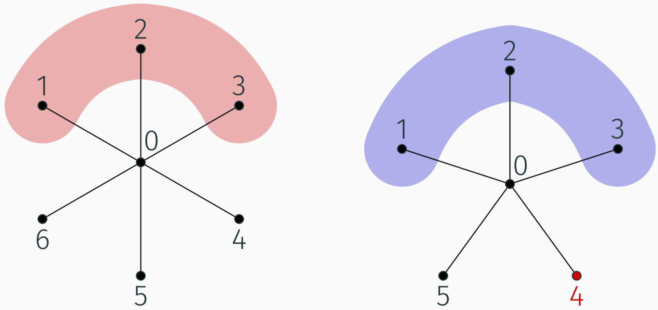

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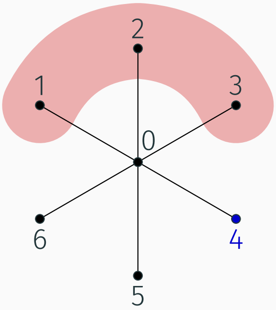

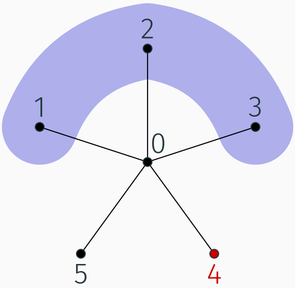

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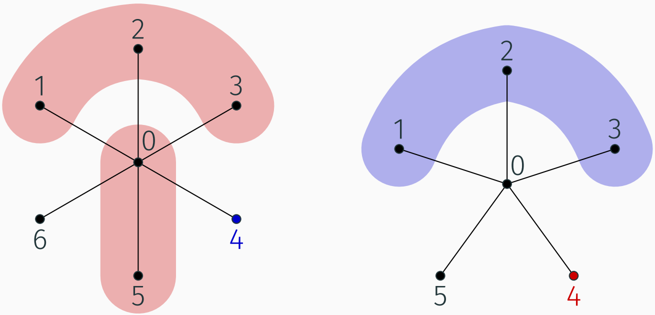

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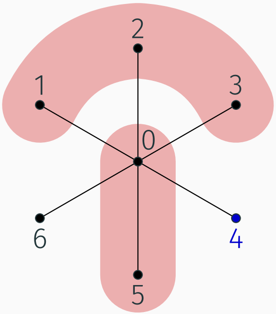

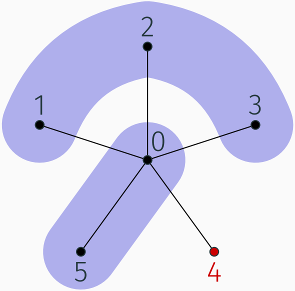

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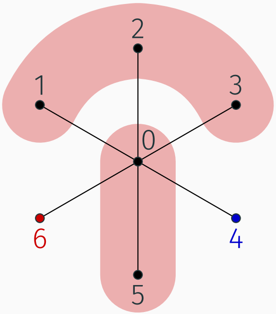

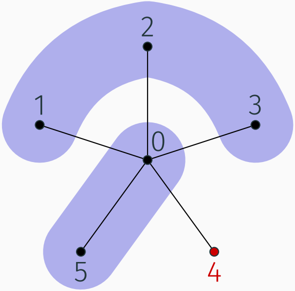

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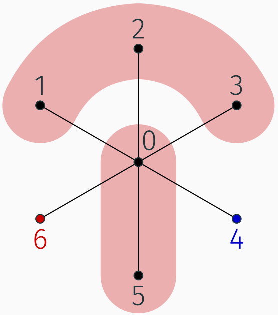

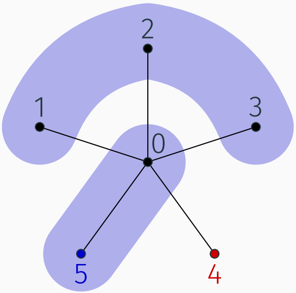

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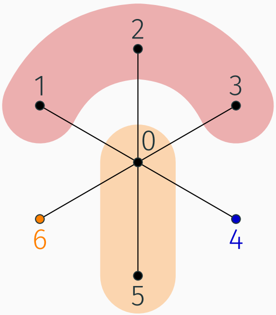

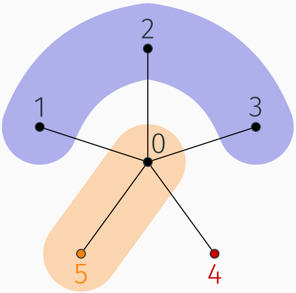

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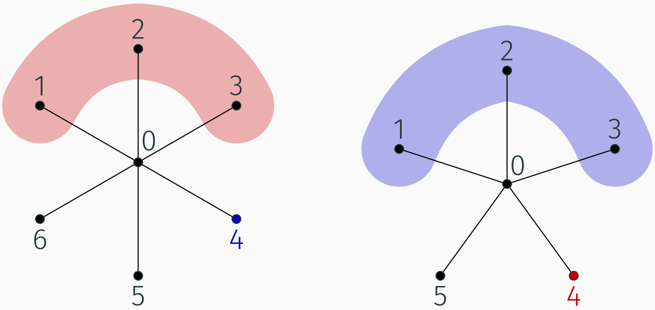

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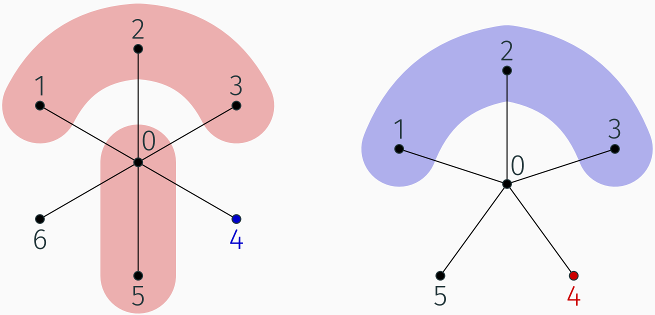

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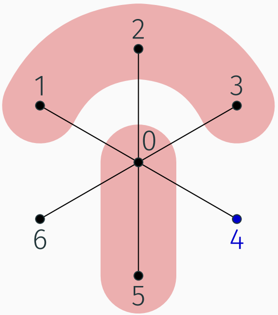

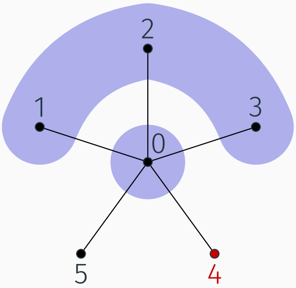

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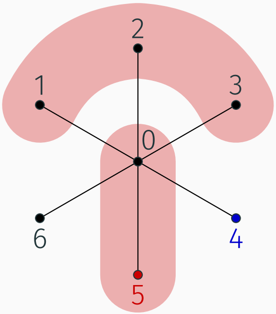

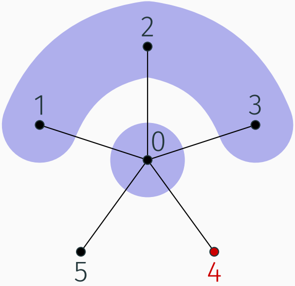

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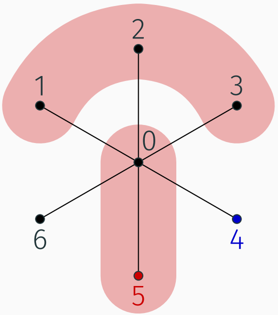
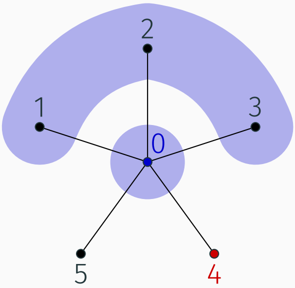
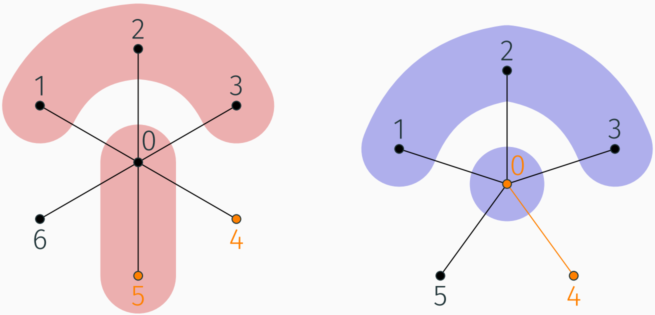




Spoiler and duplicator moves will be red and blue respectively.

This suggests that $\mathfrak{A} \equiv_3^{\mathsf{MSO}} \mathfrak{B}$ but $\mathfrak{A} \not\equiv_4^{\mathsf{MSO}} \mathfrak{B}$.



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The proof is essentially the same as for EF games.

Expressibility of queries

Let $\mathcal{L}$ be a finite relational language with no constant symbols.

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$$(\mathfrak{A}, \vec{a}_1, \vec{a}_2, V_1 \cup W_1, \ldots, V_s \cup W_s)$$

where $\mathfrak{A}$ is the structure for $\mathcal{L}$ whose domain is $A_1 \cup A_2$ and interpreting as $R^{\mathfrak{A}_1} \cup R^{\mathfrak{A}_2}$ each relation symbol R of $\mathcal{L}$.

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$\uparrow$
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Even cardinality query

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Corollary.

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By the composition lemma, we get $(A, a) \equiv_{k-1}^{\text{MSO}} (B, b)$. As usual, this implies $A \equiv_{k}^{\text{MSO}} B$. $\square$

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Let us consider a finite linear order:

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where:

1. *first*(x) stands for $\forall y(x < y \vee x = y)$.
2. *last*(x) stands for $\forall y(y < x \vee x = y)$.
3. *succ*(x, y) stands for $(x < y) \wedge \neg\exists z(x < z \wedge z < y)$. □

Graph queries and EMSO games

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For the converse, one may use Hanf-locality. $\square$

The (s, t)-reachability query asks whether there is a path from s to t in a given graph.

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Theorem.

Reachability for directed graphs is not EMSO-expressible.

The *l,k-Fagin game* on two structures $\mathfrak{A}$ and $\mathfrak{B}$ is played as follows.

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EMSO games

The l,k-Fagin game on two structures $\mathfrak{A}$ and $\mathfrak{B}$ is played as follows. The spoiler selects l subsets $U_1, \ldots, U_l$ of A. Then the duplicator selects l subsets $V_1, \ldots, V_l$ of B.

After that, the spoiler and the duplicator play k rounds of the Ehrenfeucht-Fraïssé game on $(\mathfrak{A}, U_1, \ldots, U_l)$ and $(\mathfrak{B}, V_1, \ldots, V_l)$.

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The winning condition for the duplicator is that the elements played on $(\mathfrak{A}, U_1, \ldots, U_l)$ and $(\mathfrak{B}, V_1, \ldots, V_l)$ form a partial isomorphism between these two structures.

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