Paradox and normalization failure

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Table of contents

An informal introduction

What is a paradox?

A paradox is usually related to the concepts of:

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Famous examples include:

- Russel's paradox
- Liar's paradox ...

Paradoxical $\neq$ Inconsistent

$$\dfrac{\dfrac{A \wedge \neg A}{A} \qquad \dfrac{A \wedge \neg A}{\neg A}}{\bot}$$

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Paradoxical $=$ Inconsistent $+$???

An attempt of formalization

We consider the implicative fragment of natural deduction:

$$\dfrac{\begin{array}{c}\overset{n}{[A]}\\ \vdots\\ \mathcal{D}\\ \vdots\\ B\end{array}}{A \rightarrow B}\ (\rightarrow \mathrm{I})\,(n) \qquad\qquad \dfrac{A \rightarrow B \qquad A}{B}\ (\rightarrow \mathrm{E})$$

We can then define $\neg A := A \rightarrow \bot$.

Assume that:

$$\begin{array}{ccc} A & \quad & \neg A \\ \vdots & & \vdots \\ \mathcal{D} & & \mathcal{E} \\ \vdots & & \vdots \\ \neg A & & A \end{array}$$

$$
\dfrac{
\begin{matrix}
\dfrac{\dfrac{[A]^{1} \qquad \begin{matrix}[A]^{1}\\ \mathcal{D}\\ \neg A\end{matrix}}{\bot}\,(\to E)}{\neg A}\,(\to I)(1)\\
\mathcal{E}\\
A
\end{matrix}
\qquad
\dfrac{\dfrac{[A]^{2} \qquad \begin{matrix}[A]^{2}\\ \mathcal{D}\\ \neg A\end{matrix}}{\bot}\,(\to E)}{\neg A}\,(\to I)(2)
}{\bot}\,(\to E)
$$

$$
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\overset{1}{[A]} \qquad \begin{array}{c}\overset{1}{[A]}\\ \mathcal{D}\\ \neg A\end{array}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})\,(1)\\
\mathcal{E}\\
A
\end{array}
\qquad
\dfrac{\dfrac{\overset{2}{[A]} \qquad \begin{array}{c}\overset{2}{[A]}\\ \mathcal{D}\\ \neg A\end{array}}{\bot}\,(\to \mathrm{E})}{\neg A}\,{\color{red}(\to \mathrm{I})}\,(2)
}{\bot}\,{\color{red}(\to \mathrm{E})}
$$

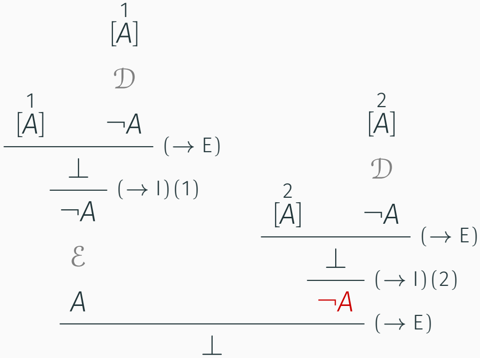
The flavor of a paradox

$$
\dfrac{
\begin{array}{c}
\dfrac{
\dfrac{
\overset{1}{[A]} \qquad \begin{array}{c}\overset{1}{[A]} \\ \mathcal{D} \\ \neg A\end{array}
}{\bot}\,(\to \mathrm{E})
}{\neg A}\,(\to \mathrm{I})\,(1) \\
\mathcal{E} \\
A
\end{array}
\qquad
\dfrac{
\dfrac{
\overset{2}{[A]} \qquad \begin{array}{c}\overset{2}{[A]} \\ \mathcal{D} \\ \neg A\end{array}
}{\bot}\,(\to \mathrm{E})
}{\neg A}\,(\to \mathrm{I})\,(2)
}{\bot}\,(\to \mathrm{E})
$$

$$
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\overset{1}{[A]} \qquad \begin{array}{c}\overset{1}{[A]} \\ \mathcal{D} \\ \neg A\end{array}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})\,(1) \\ \mathcal{E} \\ A
\end{array}
\qquad
\begin{array}{c}
\dfrac{\dfrac{\overset{2}{[A]} \qquad \begin{array}{c}\overset{2}{[A]} \\ \mathcal{D} \\ \neg A\end{array}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})\,(2) \\ \mathcal{E} \\ A \\ \mathcal{D} \\ \neg A
\end{array}
}{\bot}\,(\to \mathrm{E})
$$

The flavor of a paradox

$$
\dfrac{
\begin{matrix}
\dfrac{\dfrac{[A]^1 \qquad \begin{matrix}[A]^1 \\ \mathcal{D} \\ \neg A\end{matrix}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})(1) \\
\mathcal{E} \\
A
\end{matrix}
\qquad
\begin{matrix}
\dfrac{\dfrac{[A]^2 \qquad \begin{matrix}[A]^2 \\ \mathcal{D} \\ \neg A\end{matrix}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})(2) \\
\mathcal{E} \\
A \\
\mathcal{D} \\
\neg A
\end{matrix}
}{\bot}\,(\to \mathrm{E})
$$


1
[A]
𝒟
1
[A]
¬A
(→ E)
⊥
(→ I) (1)
¬A
ℰ
A
2
[A]
𝒟
2
[A]
¬A
(→ E)
⊥
(→ I) (2)
¬A
(→ E)
⊥

A possible answer

Paradoxical = Inconsistent + Not normalizing

An example in set theory

Prawitz extended implicative natural deduction to express the unrestricted axiom of comprehension in naive set theory:

$$\frac{A[y/x]}{y \in \{x : A\}}\ (\in \mathrm{I}) \qquad \frac{y \in \{x : A\}}{A[y/x]}\ (\in \mathrm{E})$$

These rules display the harmony property but the natural deduction systems containing them fail to normalize.

Russel's paradox

By picking $A(x) := x \notin x$ and $y := \{x : x \notin x\}$ we get:

$$\frac{y \notin y}{y \in y}\,(\in \mathrm{I}) \qquad \frac{y \in y}{y \notin y}\,(\in \mathrm{E})$$

As we saw, this leads to normalization failure.

Objections

A hidden step

In the normalization loop a step of $\in$-reduction is hidden:

$$\dfrac{\dfrac{A[y/x]}{y \in \{x : A\}}\,(\in \mathrm{I})}{A[y/x]}\,(\in \mathrm{E}) \quad \rightsquigarrow_{\in} \quad A[y/x]$$

We can mimic this behavior in propositional logic:

$$\dfrac{\dfrac{\neg A \qquad \neg A \to A}{A}\,(\to E) \qquad A \to \neg A}{\neg A}\,(\to E) \qquad \rightsquigarrow_E \qquad \neg A$$

The looping normalization would not depend on the extra-logical possibility to move, for a certain formula A, from A to $\neg A$ and vice versa, but on the logical feature that we can move, for any formula A, from $A \leftrightarrow \neg A$ to absurdity.

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Then, is it actually the case that non-normalizability is the distinctive feature of paradoxical expressions?

- Peter Schroeder-Heister and Luca Tranchini. "Ekman's paradox". In: *Notre Dame Journal of Formal Logic* 58.4 (2017), pp. 567–581.
- Neil Tennant. "Proof and paradox". In: *Dialectica* (1982), pp. 265–296.