Injectivity in linear logic

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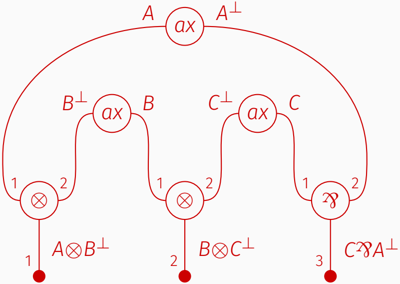
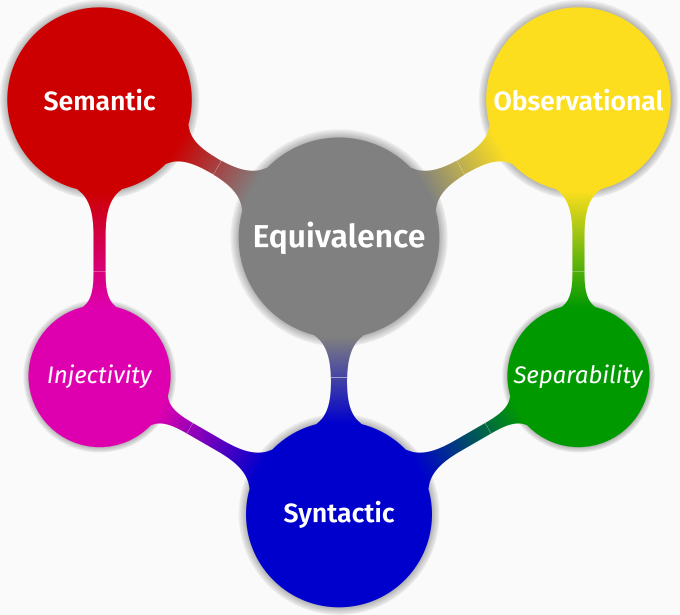
What is a proof?

Gentzen

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}} \qquad \dfrac{\dfrac{}{\vdash B, B^{\perp}} \qquad \dfrac{}{\vdash C, C^{\perp}}}{\vdash B^{\perp}, B \otimes C^{\perp}, C}}{\vdash A \otimes B^{\perp}, B \otimes C^{\perp}, C, A^{\perp}}}{\vdash A \otimes B^{\perp}, B \otimes C^{\perp}, C \parr A^{\perp}}$$

Girard

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Recent techniques

Taylor expansion

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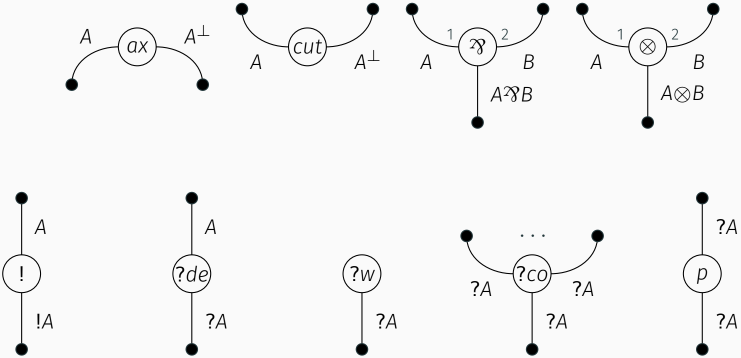
How to build a proof net?



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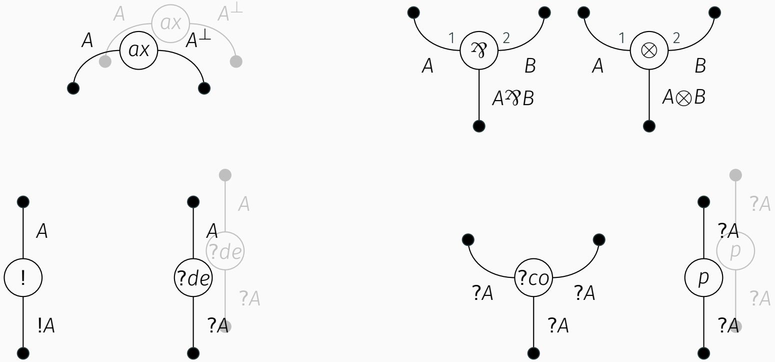
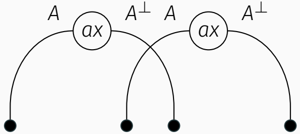


Pseudo proof structure



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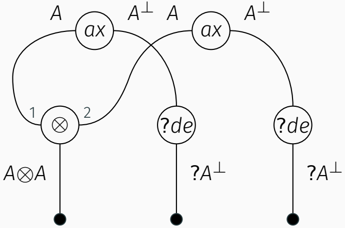

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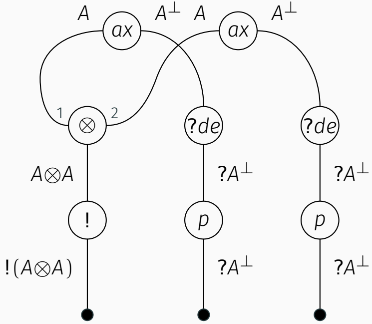

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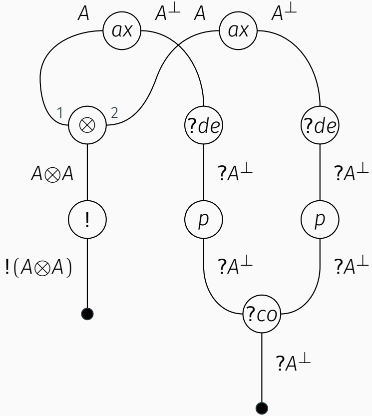

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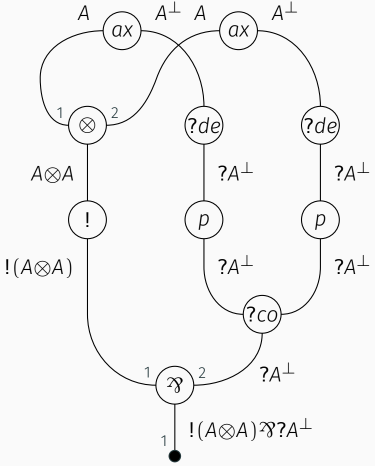
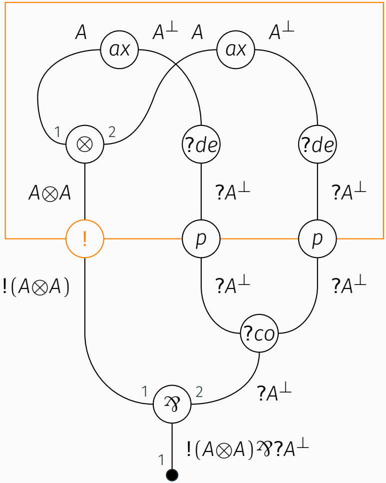


Proof structure



AC proof net

All correctness graphs are acyclic

ACC proof net

All correctness graphs are acyclic and connected

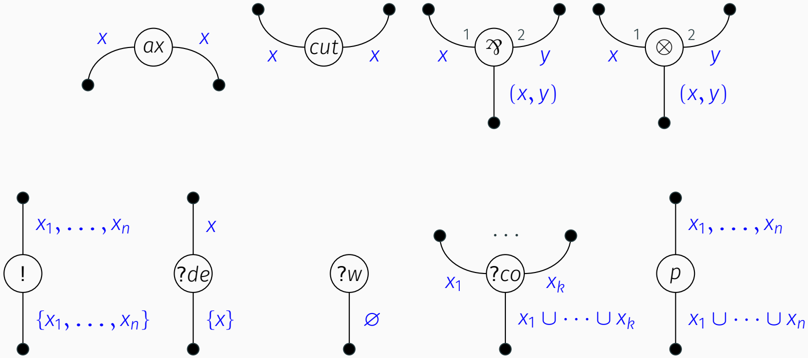
Standard proof structure

There are no cut links, conclusions of axiom links have atomic types and conclusions of weakening and contraction links are not premises of pax or contraction links.

Coherent space

Undirected graph $\mathcal{A} = (|\mathcal{A}|, \coh)$ associated with a formula *A*. The set $|\mathcal{A}|$ is called web, the binary relation $\coh$ on the web is called coherence.

Experiment



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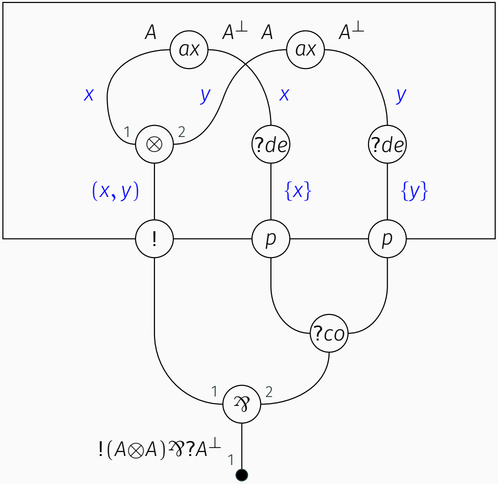
Building an experiment.



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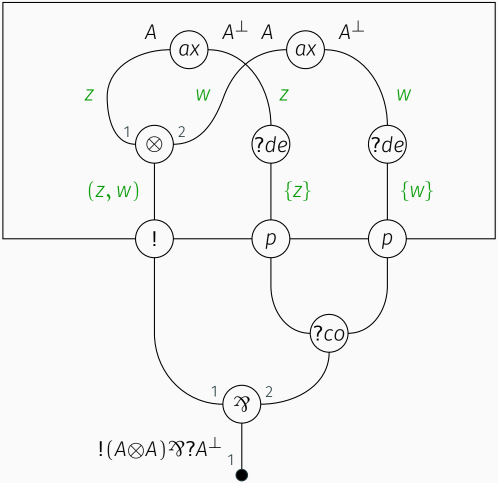

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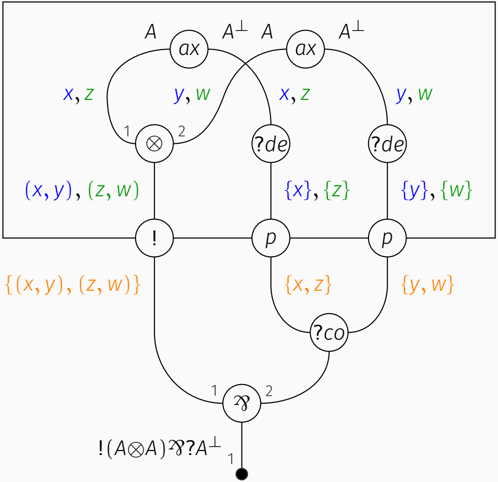

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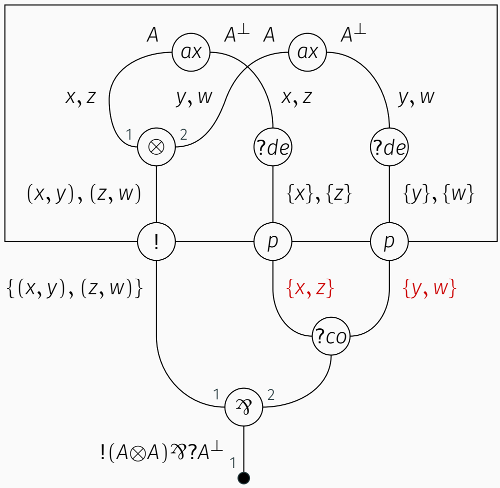

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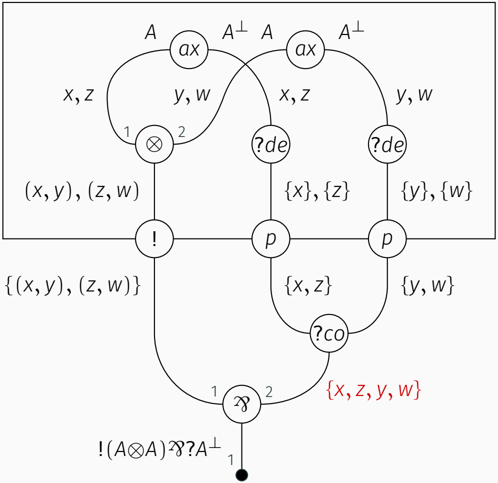
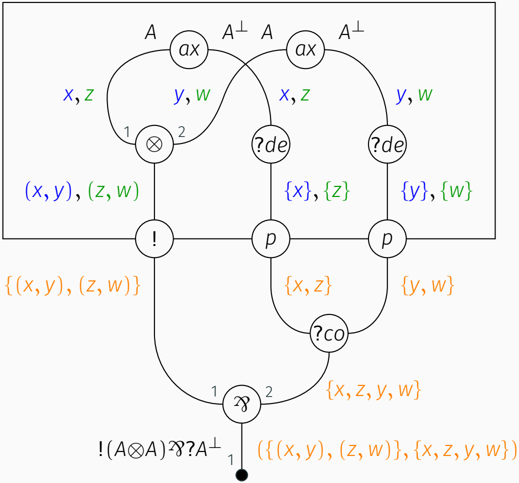


Building an experiment.



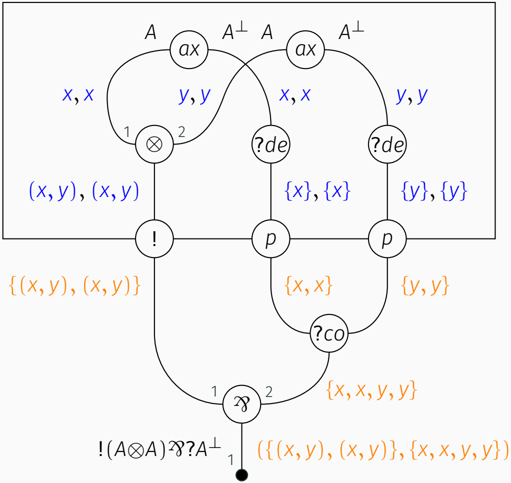
n-obsessional experiment

- Any two labels of an arc with an atomic type are the same.
- The multiset of labels associated with the conclusion of any of course link is not empty and each of its elements has cardinality n.

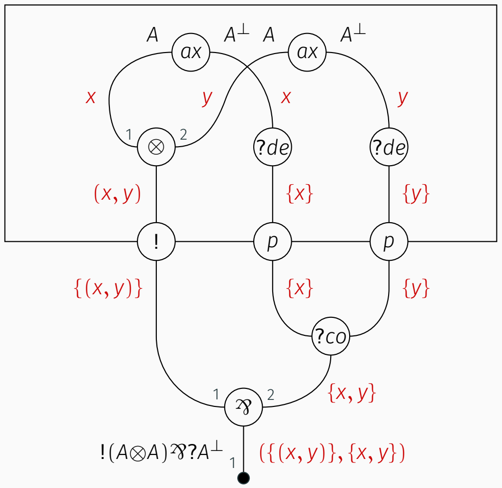
1-experiment

- Any two labels of an arc with an atomic type are the same.
- The multiset of labels associated with the conclusion of any of course link is not empty and each of its elements has cardinality 1.

Example: 2-obsessional experiment.



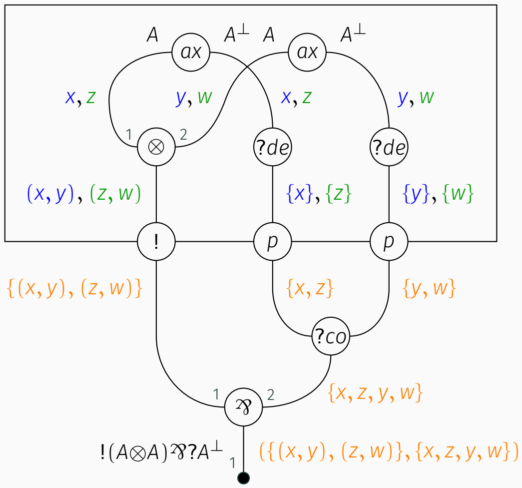
Example: 1-experiment.



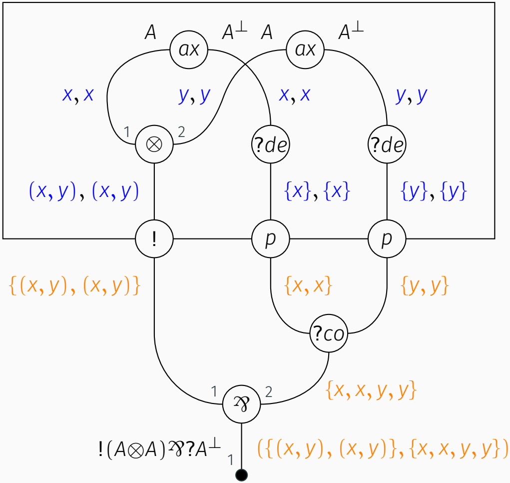
Injective experiment

Multisets of labels associated with distinct arcs having the same atomic type are different.

Example: injective experiment when $\{x, z\} \neq \{y, w\}$.



Example: injective experiment when $x \neq y$.



Let $[\![R]\!]$ be the set of the results of any experiment of an AC proof net R. Let F be a set of AC proof nets and assume $R \in F$.

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Local injectivity for F in R

For all $R' \in F$ with the same conclusions as R and $[\![R]\!] = [\![R']\!]$, we have $R = R'$.

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For all $R' \in F$ with the same conclusions as R and $[\![R]\!] = [\![R']\!]$, we have $R = R'$.

Injectivity for F

For all $R \in F$, local injectivity for F in R.

LET $\mathcal{F}$ BE THE SET OF ACC PROOF NETS. IF $R \in \mathcal{F}$ AND THERE EXISTS AN **INJECTIVE 1-EXPERIMENT** OF R, WE HAVE **LOCAL INJECTIVITY** FOR $\mathcal{F}$ IN R.

Can we find, for any $R \in F$, an experiment of R whose result is in $[\![R]\!]$ but not in $[\![R']\!]$ for any other $R' \in F$?

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F	Answer

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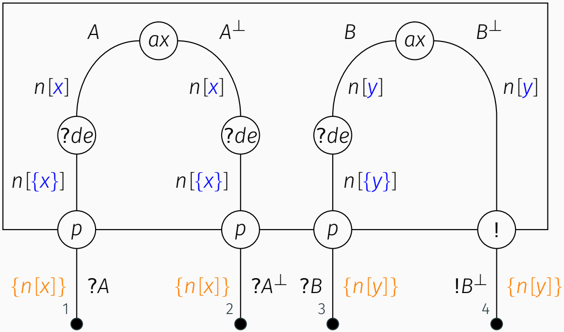
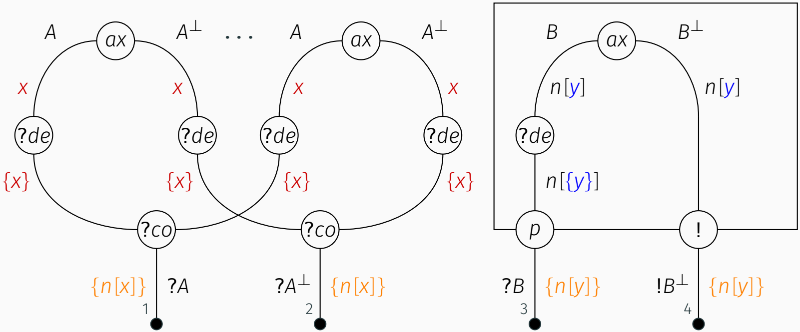
F	Answer
MELL	NO

Counterexample: a particular R and any experiment of R.



A particular R' and an experiment of R' with the same result.



Can we find, for any $R \in F$, an experiment of R whose result is in $[\![R]\!]$ but not in $[\![R']\!]$ for any other $R' \in F$?

F	Answer
MELL	NO

Can we find, for any $R \in F$, an experiment of R whose result is in $[\![R]\!]$ but not in $[\![R']\!]$ for any other $R' \in F$?

F	Answer
$MELL$	NO
MLL	YES

Can we find, for any $R \in F$, an experiment of R whose result is in $[\![R]\!]$ but not in $[\![R']\!]$ for any other $R' \in F$?

F	Answer
$MELL$	NO
λ-terms	YES
MLL	YES

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F	Answer
MELL	NO
***ACC* proof nets**	
λ-terms	YES
MLL	YES

Can we find, for any $R \in F$, an experiment of R whose result is in $[\![R]\!]$ but not in $[\![R']\!]$ for any other $R' \in F$?

F	Answer
MELL	NO
ACC **proof nets**	**?**
λ-terms	YES
MLL	YES

Taylor expansion of lambda terms

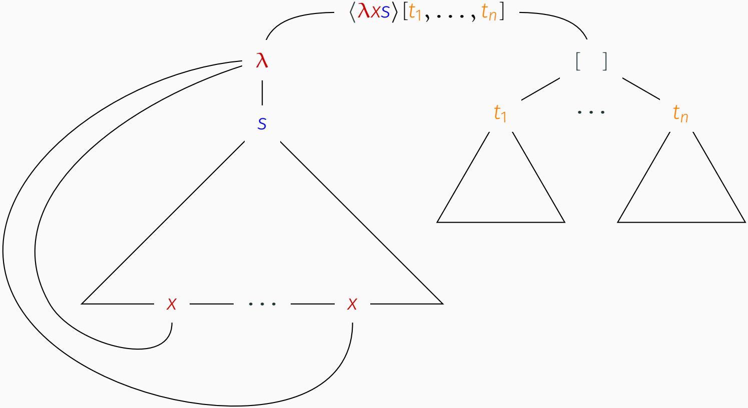
Resource terms

A set Δ defined inductively:

- If x is a variable, then $x \in \Delta$.
- If x is a variable and $s \in \Delta$, then $\lambda x s \in \Delta$.
- If $n \geqslant 0$ and $s, t_1, \ldots, t_n \in \Delta$, then $\langle s \rangle [t_1, \ldots, t_n] \in \Delta$.

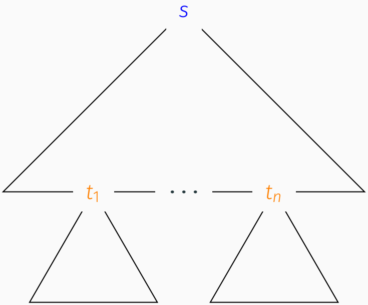
The string $[t_1, \ldots, t_n]$ is called a resource monomial.

Evaluation intuitively



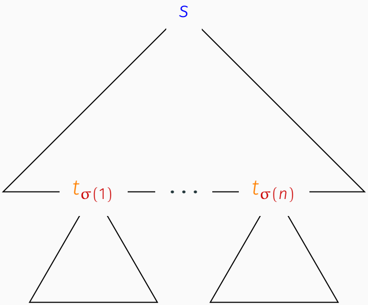
Evaluation intuitively

$s(t_1, \ldots, t_n / x)$



Evaluation intuitively

$s(t_{\sigma(1)}, \ldots, t_{\sigma(n)}/x)$



Multilinear substitution

We define $\partial_x s \cdot [t_1, \ldots, t_n]$ as the set:

$$\{s(t_{\sigma(1)}, \ldots, t_{\sigma(n)}/x) : \sigma \text{ permutation over } 1, \ldots, n\}$$

Resource reduction

The base case is $\langle \lambda x s \rangle [t_1, \ldots, t_n] \to_\partial \partial_x s \cdot [t_1, \ldots, t_n]$ and this is extended to contexts as in usual lambda calculus. In addition, whenever we have $s_0, \ldots, s_n \in \Delta$ and $S_0, \ldots, S_n \subseteq \Delta$ such that $s_0 \to_\partial S_0$ and $s_i \to_\partial{}^? S_i$ for all $i = 1, \ldots, n$, we also have $\{s_0, \ldots, s_n\} \to_\partial S_0 \cup \cdots \cup S_n$.

Properties

- Resource reduction enjoys weak normalization.
- The relation $\to_{\partial}{}^{*}$ satisfies the diamond property.

Properties

- Resource reduction enjoys weak normalization.
- The relation $\to_\partial{}^*$ satisfies the diamond property.

$\implies$ Every $S \subseteq \Delta$ has a unique normal form $\mathbf{NF}(S)$.

Taylor support

Let $S^!$ be the set of resource monomials of the form $[s_1, \ldots, s_n]$ for some $n \geqslant 0$, $s_1, \ldots, s_n \in S$. We define inductively on usual lambda terms the following set of resource terms:

$$\mathbf{T}(x) := \{x\}$$

$$\mathbf{T}(\lambda x M) := \lambda x \mathbf{T}(M)$$

$$\mathbf{T}(MN) := \langle \mathbf{T}(M) \rangle \mathbf{T}(N)^!$$

Head reduction

If $\mathbf{H}$ is the head reduction function on lambda terms or resource terms, then for all lambda terms M:

$$\mathbf{H}(\mathbf{T}(M)) = \mathbf{T}(\mathbf{H}(M))$$

Corollary

- If $\mathbf{NF}(s)$ is not empty for some $s \in \mathbf{T}(M)$, then there exists $k \geqslant 0$ such that $\mathbf{H}^k(M)$ is a head normal form.
- If M is β-equivalent to a head normal form, then $\mathbf{NF}(s)$ is not empty for some $s \in \mathbf{T}(M)$.

Corollary

- If $\mathbf{NF}(s)$ is not empty for some $s \in \mathbf{T}(M)$, then there exists $k \geqslant 0$ such that $\mathbf{H}^k(M)$ is a head normal form.
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$\implies$ If M is β-equivalent to a head normal form, then there exists $k \geqslant 0$ such that $\mathbf{H}^k(M)$ is a head normal form.

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