Injectivity of the coherent model for a fragment of connected $MELL$ proof-nets

Antonio Bucciarelli [1] **Raffaele Di Donna** [1, 2]
Lorenzo Tortora de Falco [2]

[1]Université Paris Cité

[2]Università Roma Tre

8th International Workshop on Trends in Linear Logic and Applications

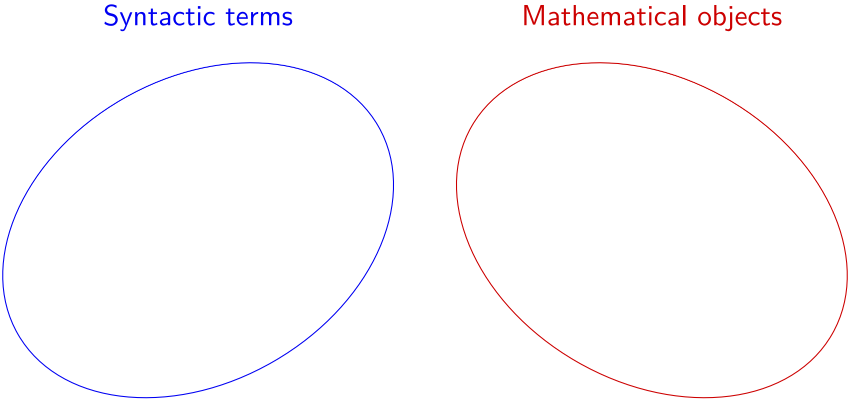
Tallinn, Estonia
8-9 July 2024

Table of contents

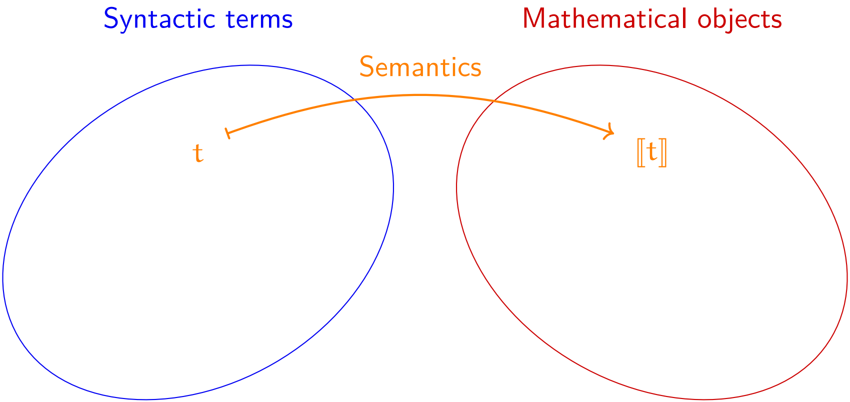
1. Introduction
2. Proof-nets and experiments
3. Injectivity for connected $(?\parr)LL_{pol}$ proof-nets

Introduction

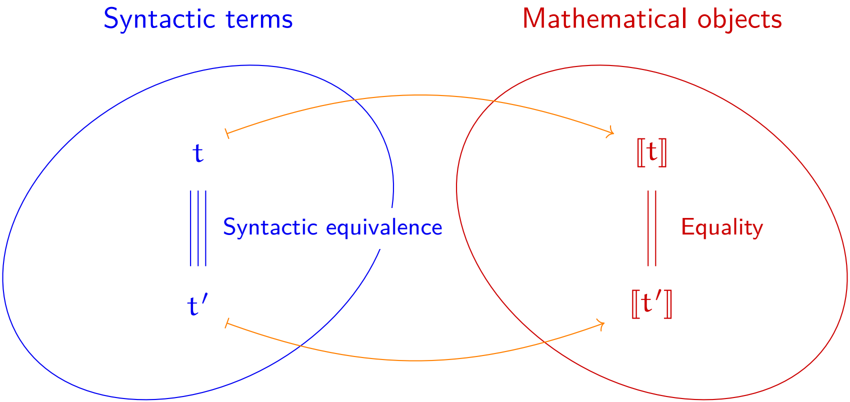
Denotational semantics



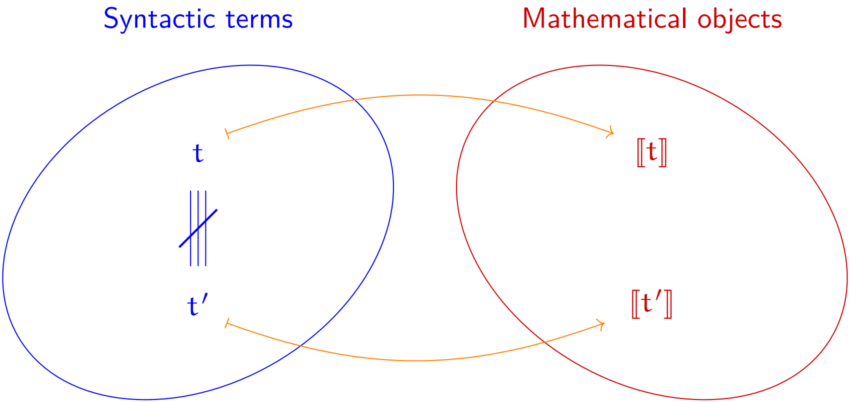
Denotational semantics



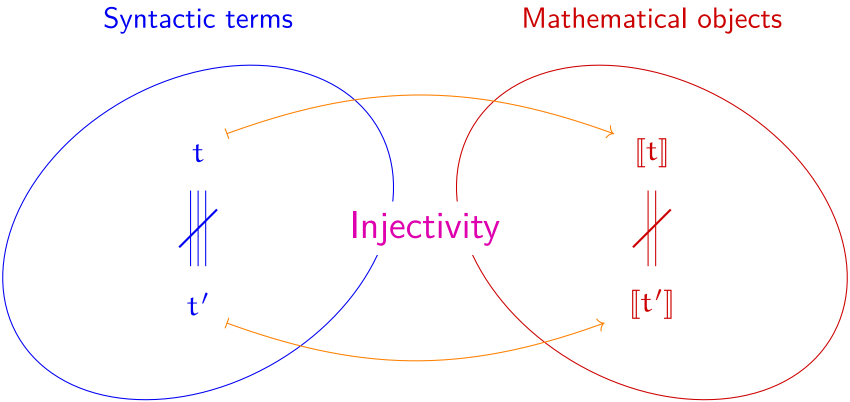
Denotational semantics



Denotational semantics



Denotational semantics



Observational equivalence

$$t \equiv_{\mathsf{obs}} t'$$

Observational equivalence

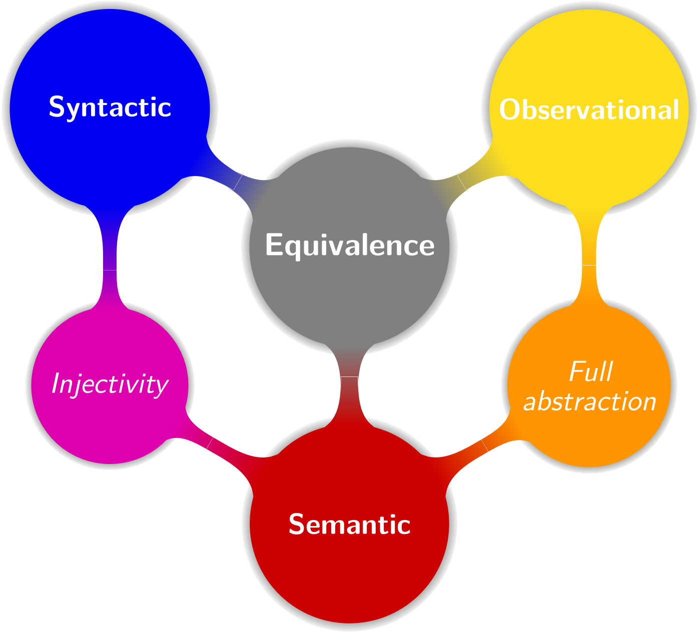
$$t \equiv_{\text{obs}} t'$$

$$\forall C[\;] : \quad C[t] \qquad\qquad C[t']$$

Observational equivalence

$$\mathtt{t} \equiv_{\mathsf{obs}} \mathtt{t}'$$

$$\forall\, C[\ \ ] : \quad C[\,\mathtt{t}\,] \xrightarrow{*} n \xleftarrow{*} C[\,\mathtt{t}'\,]$$


Syntactic
Observational
Equivalence
Injectivity
Full abstraction
Semantic

Historically at the heart of theoretical computer science, but...

PROGRAMS = PROOFS

(Curry-Howard's correspondence)

LINEAR LOGIC

$\otimes$ $\parr$ $\oplus$ $\&$ $!$ $?$

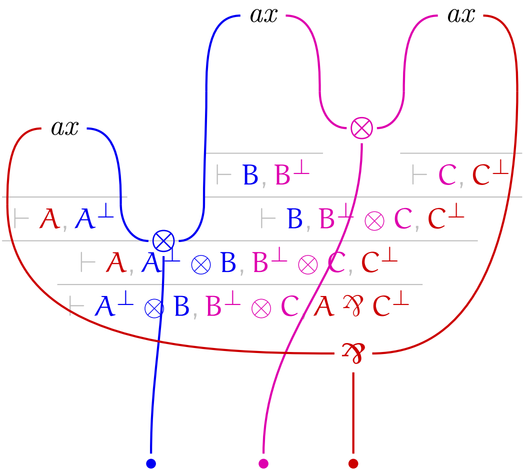
A resource-sensitive logic with non-trivial denotational semantics

(Girard, 1987)

Identity of proofs

$$
\cfrac{\cfrac{\cfrac{}{\vdash A, A^{\perp}}\,ax \qquad \cfrac{\cfrac{}{\vdash B, B^{\perp}}\,ax \qquad \cfrac{}{\vdash C, C^{\perp}}\,ax}{\vdash B, B^{\perp} \otimes C, C^{\perp}}\,\otimes}{\vdash A, A^{\perp} \otimes B, B^{\perp} \otimes C, C^{\perp}}\,\otimes}{\vdash A^{\perp} \otimes B, B^{\perp} \otimes C, A \parr C^{\perp}}\,\parr
$$

Identity of proofs



Injectivity

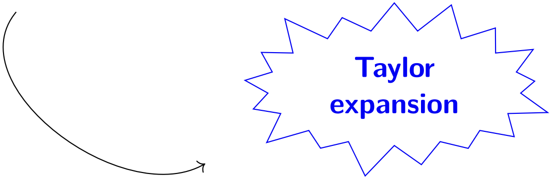
- The coherent model is not injective for $MELL$;
  (Tortora de Falco, 2003)

Injectivity

- The coherent model is not injective for $MELL$;
  (Tortora de Falco, 2003)
- The relational model is injective for $MELL$.
  (de Carvalho, 2016)

Injectivity

- The coherent model is not injective for $MELL$;
  (Tortora de Falco, 2003)
- The relational model is injective for $MELL$.
  (de Carvalho, 2016)



The coherent framework

Conjecture (Tortora de Falco, 2003)**.**
The coherent model is injective for connected $MELL$ proof-nets.

The coherent framework

Conjecture (Tortora de Falco, 2003).
The coherent model is injective for connected $MELL$ proof-nets.

Sufficient condition: there exists an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions.

The coherent framework

Conjecture (Tortora de Falco, 2003)**.**
The coherent model is injective for connected $MELL$ proof-nets.

Sufficient condition: there exists an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions.

The difficulty comes from contractions. Partial results:

The coherent framework

Conjecture (Tortora de Falco, 2003).
The coherent model is injective for connected $MELL$ proof-nets.

Sufficient condition: there exists an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions.

The difficulty comes from contractions. Partial results:

- Terminal contractions: all contractions are terminal nodes; (Tortora de Falco, 2003)

The coherent framework

Conjecture (Tortora de Falco, 2003).
The coherent model is injective for connected $MELL$ proof-nets.

Sufficient condition: there exists an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions.

The difficulty comes from contractions. Partial results:

- Terminal contractions: all contractions are terminal nodes; (Tortora de Falco, 2003)
- Atomic contractions: their premises are conclusions of axioms; (Part of this talk)

The coherent framework

Conjecture (Tortora de Falco, 2003).
The coherent model is injective for connected $MELL$ proof-nets.

Sufficient condition: there exists an injective experiment for every connected proof-net which only consists of axioms, tensors, derelictions and contractions.

The difficulty comes from contractions. Partial results:

- Terminal contractions: all contractions are terminal nodes; (Tortora de Falco, 2003)
- Atomic contractions: their premises are conclusions of axioms; (Part of this talk)
- Connected $(?\invamp)LL_{pol}$ proof-nets. (Part of this talk)

Proof-nets and experiments

Logical system

A subsystem of cut-free $MELL$ proof-nets.

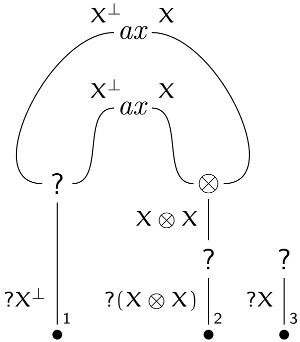
Formulas are generated by the grammar:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid ?A$$

where $X, X^{\perp}$ denote dual atomic formulas.

Proof-structures

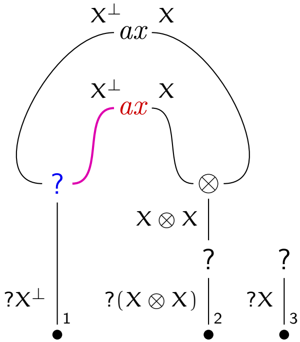
Definition 1. A *proof-structure* is a non-empty labelled directed graph R such that its nodes have exactly one label among ax, $\otimes$, ?, •, arcs are labelled by formulas and:



Proof-structures

Definition 1. A *proof-structure* is a non-empty labelled directed graph R such that its nodes have exactly one label among ax, $\otimes$, ?, $\bullet$, arcs are labelled by formulas and:

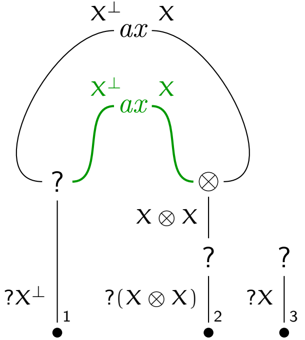
- Every arc of R is directed from top to bottom and is called a *premise* of its head, a *conclusion* of its tail;



Proof-structures

Definition 1. A *proof-structure* is a non-empty labelled directed graph $\mathcal{R}$ such that its nodes have exactly one label among ax, $\otimes$, ?, $\bullet$, arcs are labelled by formulas and:

- Every arc of $\mathcal{R}$ is directed from top to bottom and is called a *premise* of its head, a *conclusion* of its tail;
- Every node of $\mathcal{R}$ labelled by ax is called an *axiom*, has no premises and exactly two conclusions, labelled by dual atomic formulas;

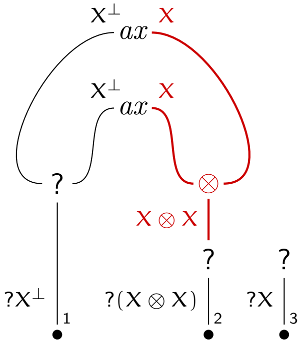


Proof-structures

Definition 1. A *proof-structure* is a non-empty labelled directed graph $\mathcal{R}$ such that its nodes have exactly one label among ax, $\otimes$, ?, $\bullet$, arcs are labelled by formulas and:

...

- Every node of $\mathcal{R}$ labelled by $\otimes$ is called a tensor, has exactly one conclusion, labelled by a formula $A \otimes B$ and exactly two premises, one of which is called its *left premise* and is labelled by A, whereas the other is called its *right premise* and is labelled by B;

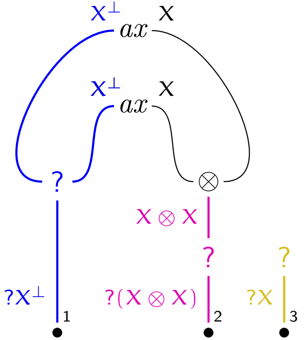


Proof-structures

Definition 1. A *proof-structure* is a non-empty labelled directed graph $\mathcal{R}$ such that its nodes have exactly one label among ax, $\otimes$, ?, $\bullet$, arcs are labelled by formulas and:

...

- Every node of $\mathcal{R}$ labelled by ? is called a *why not* and has exactly one conclusion, labelled by a formula $?A$. Such a node has all of its premises labelled by A and is called a *weakening* when it has no premises, a *dereliction* if it has exactly one premise, a *contraction* otherwise;



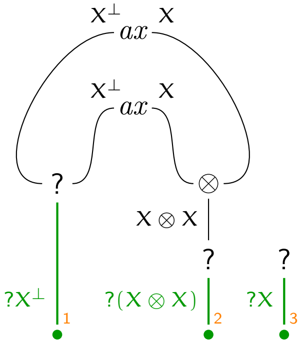
Proof-structures

Definition 1. A *proof-structure* is a non-empty labelled directed graph $\mathcal{R}$ such that its nodes have exactly one label among $ax, \otimes, ?, \bullet$, arcs are labelled by formulas and:

...

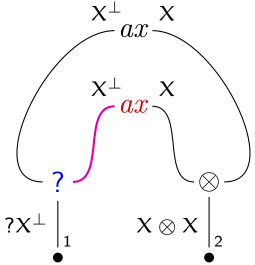
- Every node of $\mathcal{R}$ labelled by $\bullet$ is called a *conclusion* and possesses exactly one premise and no conclusions.

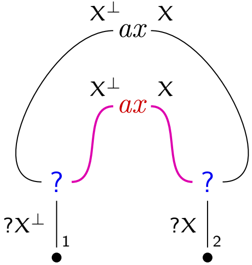
Moreover, a proof-structure is equipped with a total order of its conclusions, called its *interface*.



Proof-nets

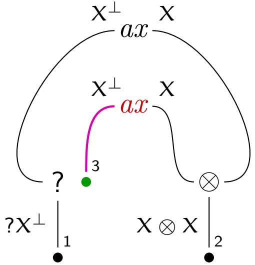
Definition 2. A *switching graph* of $\mathcal{R}$ is a proof-structure obtained by replacing every premise p of a contraction except one with an arc having the same tail as p and a fresh $\bullet$ as head, for all contractions of $\mathcal{R}$.





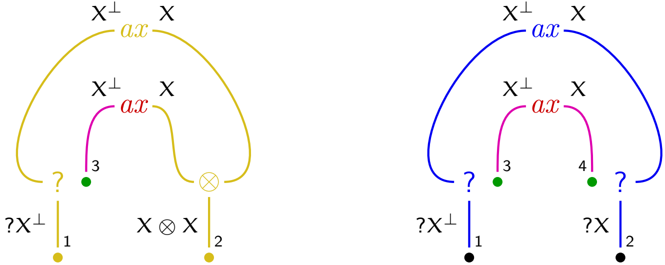
Proof-nets

Definition 2. A *switching graph* of $\mathcal{R}$ is a proof-structure obtained by replacing every premise p of a contraction except one with an arc having the same tail as p and a fresh • as head, for all contractions of $\mathcal{R}$.



Proof-nets

Definition 2. A *switching graph* of R is a proof-structure obtained by replacing every premise p of a contraction except one with an arc having the same tail as p and a fresh • as head, for all contractions of R. We say that R is a *proof-net* if the underlying undirected graph of every switching graph is acyclic, a *connected proof-net* if such graphs are also connected.



Coherence spaces

Definition 3. A *coherence space $\mathcal{A}$* is an ordered pair $(|\mathcal{A}|, \frown_{\mathcal{A}})$, where:

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \frown_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \frown_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\frown_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \frown\!\!\!\!\smile_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\frown\!\!\!\!\smile_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \frown\!\!\!\smile_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\frown\!\!\!\smile_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

A *clique* of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \frown\!\!\!\smile_{\mathcal{A}} y$ for every $x, y \in C$.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \coh_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\coh_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

A *clique* of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \coh_{\mathcal{A}} y$ for every $x, y \in C$.

The coherence space $\mathcal{A}^{\perp}$ is defined by $|\mathcal{A}^{\perp}| := |\mathcal{A}|$ and $\coh_{\mathcal{A}^{\perp}} := |\mathcal{A}|^2 \setminus \frown_{\mathcal{A}}$.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \coh_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\coh_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

A *clique* of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \coh_{\mathcal{A}} y$ for every $x, y \in C$.

The coherence space $\mathcal{A}^{\perp}$ is defined by $|\mathcal{A}^{\perp}| := |\mathcal{A}|$ and $\coh_{\mathcal{A}^{\perp}} := |\mathcal{A}|^2 \setminus \frown_{\mathcal{A}}$.

We then define *incoherence* in $\mathcal{A}$ as $\asymp_{\mathcal{A}} := \coh_{\mathcal{A}^{\perp}}$.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \coh_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\coh_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

A *clique* of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \coh_{\mathcal{A}} y$ for every $x, y \in C$.

The coherence space $\mathcal{A}^{\perp}$ is defined by $|\mathcal{A}^{\perp}| := |\mathcal{A}|$ and $\coh_{\mathcal{A}^{\perp}} := |\mathcal{A}|^2 \setminus \frown_{\mathcal{A}}$.

We then define *incoherence* in $\mathcal{A}$ as $\asymp_{\mathcal{A}} := \coh_{\mathcal{A}^{\perp}}$.

We write $\smile_{\mathcal{A}}$ for strict incoherence.

Coherence spaces

Definition 3. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \frown\!\!\!\!\smile_{\mathcal{A}})$, where:

- $|\mathcal{A}|$ is a set, called *web*;
- $\frown\!\!\!\!\smile_{\mathcal{A}}$ is a binary reflexive and symmetric relation on the web, *coherence*.

Strict coherence is written $\frown_{\mathcal{A}}$.

A *clique* of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \frown\!\!\!\!\smile_{\mathcal{A}} y$ for every $x, y \in C$.

The coherence space $\mathcal{A}^{\perp}$ is defined by $|\mathcal{A}^{\perp}| := |\mathcal{A}|$ and $\frown\!\!\!\!\smile_{\mathcal{A}^{\perp}} := |\mathcal{A}|^2 \setminus \frown_{\mathcal{A}}$.

We then define *incoherence* in $\mathcal{A}$ as $\asymp_{\mathcal{A}} := \frown\!\!\!\!\smile_{\mathcal{A}^{\perp}}$.

We write $\smile_{\mathcal{A}}$ for strict incoherence.

Lastly, an *anticlique* of $\mathcal{A}$ is just a clique of $\mathcal{A}^{\perp}$.

Interpretation of formulas

An interpretation of *atomic* formulas by:

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets

is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:

$$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$$

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets

is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:

$$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$$

coherence spaces

is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:

$$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^{\perp}$$

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^{\perp}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$			
$?A$			

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]^{\perp}_{\mathbf{Coh}}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert [\![\cdot]\!]_{\mathbf{Coh}} \rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$		
$?A$			

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^{\perp}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$\boldsymbol{x \frown y}$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$		
$?A$	$\mathcal{M}_{\mathsf{fin}}([\![A]\!]_{\mathbf{Rel}})$		

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^\perp]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^\perp]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^\perp$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$	$\lvert[\![A]\!]_{\mathbf{Coh}}\rvert \times \lvert[\![B]\!]_{\mathbf{Coh}}\rvert$	
$?A$	$\mathcal{M}_{\mathrm{fin}}([\![A]\!]_{\mathbf{Rel}})$		

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]^{\perp}_{\mathbf{Coh}}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$	$\lvert[\![A]\!]_{\mathbf{Coh}}\rvert \times \lvert[\![B]\!]_{\mathbf{Coh}}\rvert$	$\mathbf{pr}_1(x) \frown \mathbf{pr}_1(y)$ and $\mathbf{pr}_2(x) \frown \mathbf{pr}_2(y)$
$?A$	$\mathcal{M}_{\mathsf{fin}}([\![A]\!]_{\mathbf{Rel}})$		

Interpretation of formulas

An interpretation of *atomic* formulas by:

sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]^{\perp}_{\mathbf{Coh}}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$	$\lvert[\![A]\!]_{\mathbf{Coh}}\rvert \times \lvert[\![B]\!]_{\mathbf{Coh}}\rvert$	$\mathbf{pr}_1(x) \frown \mathbf{pr}_1(y)$ and $\mathbf{pr}_2(x) \frown \mathbf{pr}_2(y)$
$?A$	$\mathcal{M}_{\mathrm{fin}}([\![A]\!]_{\mathbf{Rel}})$	$\mathcal{M}_{\mathrm{clfin}}([\![A]\!]^{\perp}_{\mathbf{Coh}})$	

Interpretation of formulas

An interpretation of *atomic* formulas by:

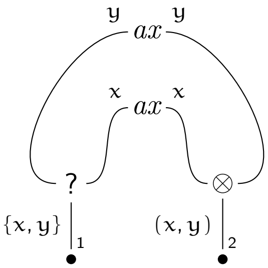
sets	coherence spaces
is a map $X \mapsto [\![X]\!]_{\mathbf{Rel}}$ such that:	is a map $X \mapsto [\![X]\!]_{\mathbf{Coh}}$ such that:
$[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$	$[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^{\perp}$

The interpretation of *non-atomic* formulas is then inductively defined by:

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$	$\lvert[\![A]\!]_{\mathbf{Coh}}\rvert \times \lvert[\![B]\!]_{\mathbf{Coh}}\rvert$	$\mathbf{pr}_1(x) \frown \mathbf{pr}_1(y)$ and $\mathbf{pr}_2(x) \frown \mathbf{pr}_2(y)$
$?A$	$\mathcal{M}_{\mathsf{fin}}([\![A]\!]_{\mathbf{Rel}})$	$\mathcal{M}_{\mathsf{clfin}}([\![A]\!]_{\mathbf{Coh}}^{\perp})$	$x = y$ or $x \cup y \notin \mathcal{M}_{\mathsf{clfin}}([\![A]\!]_{\mathbf{Coh}}^{\perp})$

Experiments

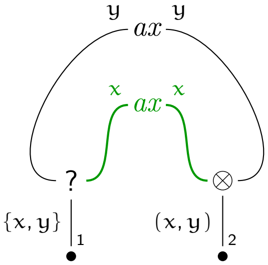
Definition 4. A *relational* (resp. *coherent*) *experiment* of a proof-structure R is a map e which associates with every arc of type A of R an element of $[\![A]\!]_{\mathbf{Rel}}$ (resp. $|[\![A]\!]_{\mathbf{Coh}}|$) and such that:



Experiments

Definition 4. A *relational* (resp. *coherent*) *experiment* of a proof-structure R is a map e which associates with every arc of type A of R an element of $[\![A]\!]_{\mathbf{Rel}}$ (resp. $|[\![A]\!]_{\mathbf{Coh}}|$) and such that:

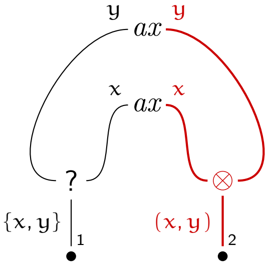
- If $\alpha, \alpha^{\perp}$ are the conclusions of an axiom of R, then $e(\alpha) = e(\alpha^{\perp})$;



Experiments

Definition 4. A *relational* (resp. *coherent*) *experiment* of a proof-structure R is a map e which associates with every arc of type A of R an element of $[\![A]\!]_{\mathbf{Rel}}$ (resp. $|[\![A]\!]_{\mathbf{Coh}}|$) and such that:

- If $\alpha, \alpha^{\perp}$ are the conclusions of an axiom of R, then $e(\alpha) = e(\alpha^{\perp})$;
- If a is the conclusion of a tensor of R with left premise b and right premise c, then $e(a) = (e(b), e(c))$;

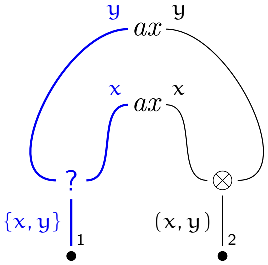


Experiments

Definition 4. A *relational* (resp. *coherent*) *experiment* of a proof-structure R is a map e which associates with every arc of type A of R an element of $[\![A]\!]_{\mathbf{Rel}}$ (resp. $|[\![A]\!]_{\mathbf{Coh}}|$) and such that:

...

- If a is the conclusion of a why not of R with premises $a_1, \ldots, a_k$, then $e(a) = \{e(a_1), \ldots, e(a_k)\}$.



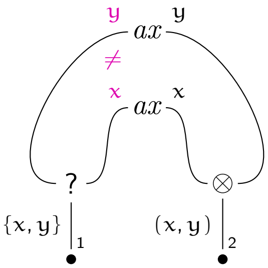
Experiments

Definition 4. A *relational* (resp. *coherent*) *experiment* of a proof-structure R is a map e which associates with every arc of type A of R an element of $[\![A]\!]_{\mathbf{Rel}}$ (resp. $|[\![A]\!]_{\mathbf{Coh}}|$) and such that:

...

- If a is the conclusion of a why not of R with premises $a_1, \ldots, a_k$, then $e(a) = \{e(a_1), \ldots, e(a_k)\}$.

We say that e is *injective* if $e(\alpha_1) \neq e(\alpha_2)$ for all $\alpha_1 \neq \alpha_2$ of the same atomic type.

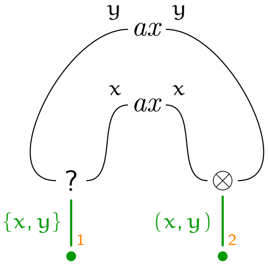


Experiments

Definition 4.

...

If $(c_1, \ldots, c_h)$ is the sequence of the premises of the conclusion nodes in the interface order, then $(e(c_1), \ldots, e(c_k))$ is called the *result* of e.



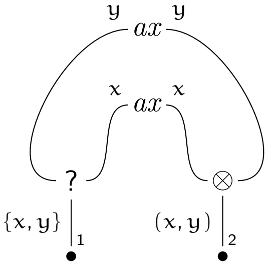
Experiments

Definition 4.

...

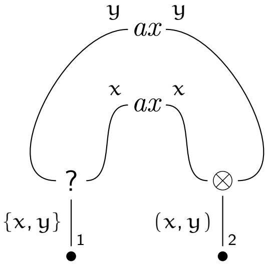
If $(c_1, \ldots, c_h)$ is the sequence of the premises of the conclusion nodes in the interface order, then $(e(c_1), \ldots, e(c_k))$ is called the *result* of e.

The *relational* (resp. *coherent*) *semantics* $[\![R]\!]_{\mathbf{Rel}}$ (resp. $[\![R]\!]_{\mathbf{Coh}}$) is the set of the results of all relational (resp. coherent) experiments of R.



Experiments

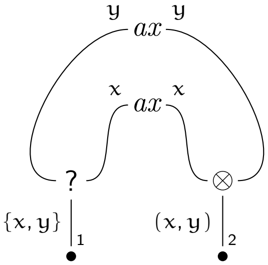
Remark 1. Let C_i be the type of c_i for all $i \in \{1, \ldots, h\}$ and $\wp\Gamma := (C_1 \wp \cdots) \wp C_h$. If R is a proof-net, then:



Experiments

Remark 1. Let C_i be the type of c_i for all $i \in \{1, \ldots, h\}$ and $\parr\Gamma := (C_1 \parr \cdots) \parr C_h$. If R is a proof-net, then:

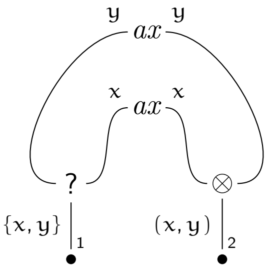
- $[\![R]\!]_{\mathbf{Coh}}$ is a clique of $[\![\parr\Gamma]\!]_{\mathbf{Coh}}$;



Experiments

Remark 1. Let C_i be the type of c_i for all $i \in \{1, \ldots, h\}$ and $\parr\Gamma := (C_1 \parr \cdots) \parr C_h$. If R is a proof-net, then:

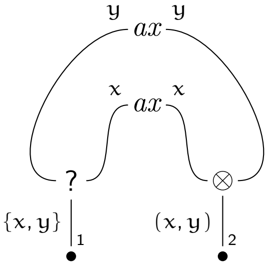
- $[\![R]\!]_{\mathbf{Coh}}$ is a clique of $[\![\parr\Gamma]\!]_{\mathbf{Coh}}$;
- $[\![R]\!]_{\mathbf{Coh}} = [\![R]\!]_{\mathbf{Rel}} \cap |[\![\parr\Gamma]\!]_{\mathbf{Coh}}|$;



Experiments

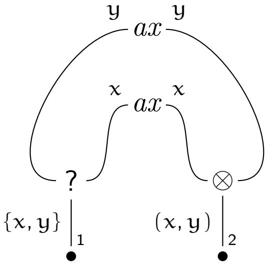
Remark 1. Let C_i be the type of c_i for all $i \in \{1, \ldots, h\}$ and $\parr\Gamma := (C_1 \parr \cdots) \parr C_h$. If R is a proof-net, then:

- $[\![R]\!]_{\mathbf{Coh}}$ is a clique of $[\![\parr\Gamma]\!]_{\mathbf{Coh}}$;
- $[\![R]\!]_{\mathbf{Coh}} = [\![R]\!]_{\mathbf{Rel}} \cap |[\![\parr\Gamma]\!]_{\mathbf{Coh}}|$;
- The injectivity of coherent semantics for a fragment of proof-nets entails the injectivity of relational semantics for the same fragment.



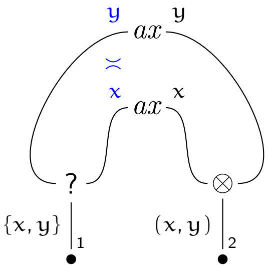
Experiments

Remark 2. Every function mapping distinct axioms of $\mathcal{R}$ to distinct points of the relational interpretations of their conclusions trivially induces an injective relational experiment of $\mathcal{R}$.



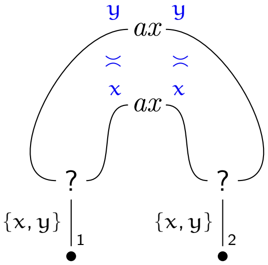
Experiments

Remark 2. Every function mapping distinct axioms of R to distinct points of the relational interpretations of their conclusions trivially induces an injective relational experiment of R. On the other hand, the existence of an injective *coherent* experiment of R is non-trivial: whenever a is the conclusion of a contraction with premises $a_1, \ldots, a_k$ of type A, we have $e(a) \in |[\![?A]\!]_{\mathbf{Coh}}|$, or equivalently $e(a_i) \asymp e(a_j)$ for all $i, j \in \{1, \ldots, k\}$.



Experiments

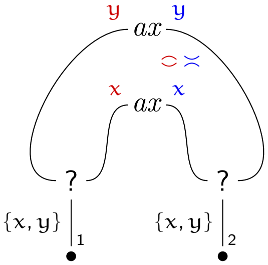
Example 1. There exists a non-connected proof-net for which there is no injective coherent experiment.



Experiments

Example 1. There exists a non-connected proof-net for which there is no injective coherent experiment. Indeed:

$$x \asymp_{[\![X^\perp]\!]_{\mathbf{Coh}}} y \iff x \frown_{[\![X]\!]_{\mathbf{Coh}}} y$$



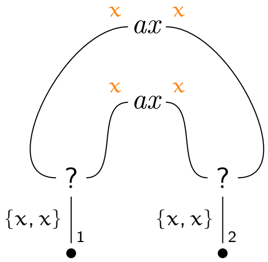
Experiments

Example 1. There exists a non-connected proof-net for which there is no injective coherent experiment. Indeed:

$$x \asymp_{[\![X^{\perp}]\!]_{\mathbf{Coh}}} y \iff x \frown_{[\![X]\!]_{\mathbf{Coh}}} y$$

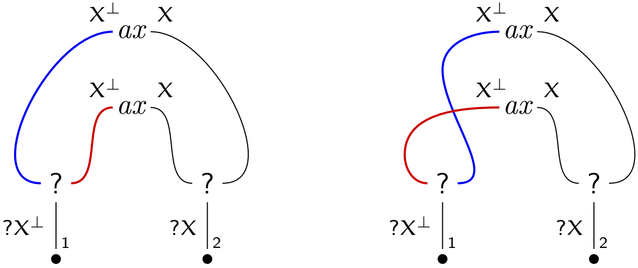
And we know that:

$$\begin{array}{l} x \asymp_{[\![X]\!]_{\mathbf{Coh}}} y \\ x \frown_{[\![X]\!]_{\mathbf{Coh}}} y \end{array} \iff x = y$$



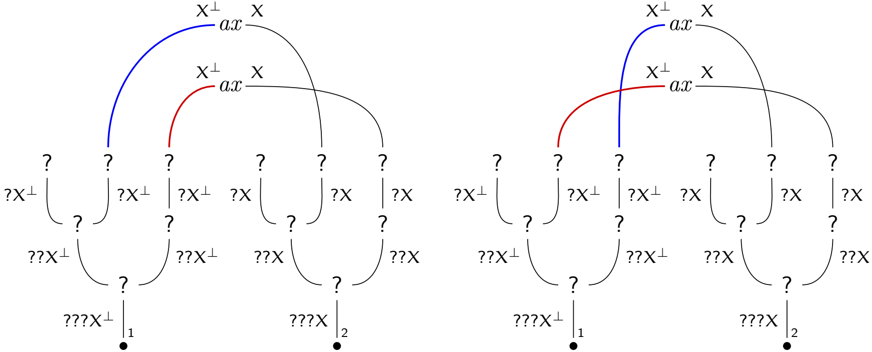
Coherent semantics is not injective for $MELL$

The previous example tells us that no coherent experiment can distinguish the following proof-nets:



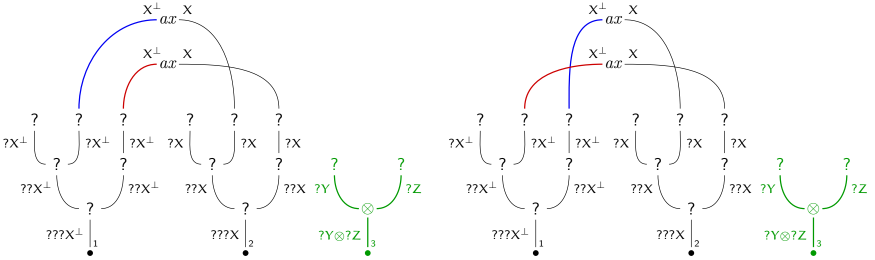
Coherent semantics is not injective for $MELL$

Two distinct proof-nets
with the same coherent interpretation:



Coherent semantics is not injective for $MELL$

Two distinct proof-nets, images of sequent calculus proofs,
with the same coherent interpretation:
(Tortora de Falco, 2003)



Coherent semantics is not injective for $MELL$

Two sequent calculus proofs whose images have the same coherent interpretation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash X^\perp, X}{\vdash X^\perp, X, ?Y}\,?we \quad \dfrac{\vdash X^\perp, X}{\vdash X^\perp, X, ?Z}\,?we}{\vdash X^\perp, X^\perp, X, X, ?Y\otimes ?Z}\,\otimes}{\vdash ?X^\perp, X^\perp, X^\perp, ?X, X, X, ?Y\otimes ?Z}\,?we\times 2}{\vdash ??X^\perp, ??X^\perp, ??X^\perp, ??X, ??X, ??X, ?Y\otimes ?Z}\,?de\times 10}{\vdash ??X^\perp, ??X^\perp, ??X, ??X, ?Y\otimes ?Z}\,?co\times 2}{\vdash ???X^\perp, ???X^\perp, ???X, ???X, ?Y\otimes ?Z}\,?de\times 4}{\vdash ???X^\perp, ???X, ?Y\otimes ?Z}\,?co\times 2}{}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash X^\perp, X}{\vdash X^\perp, X, ?Y}\,?we \quad \dfrac{\vdash X^\perp, X}{\vdash X^\perp, X, ?Z}\,?we}{\vdash X^\perp, X^\perp, X, X, ?Y\otimes ?Z}\,\otimes}{\vdash ?X^\perp, X^\perp, X^\perp, ?X, X, X, ?Y\otimes ?Z}\,?we\times 2}{\vdash ??X^\perp, ??X^\perp, ??X^\perp, ??X, ??X, ??X, ?Y\otimes ?Z}\,?de\times 10}{\vdash ??X^\perp, ??X^\perp, ??X, ??X, ?Y\otimes ?Z}\,?co\times 2}{\vdash ???X^\perp, ???X^\perp, ???X, ???X, ?Y\otimes ?Z}\,?de\times 4}{\vdash ???X^\perp, ???X, ?Y\otimes ?Z}\,?co\times 2}{}
$$

Connectedness and coherence

Conjecture

If R is a connected proof-net, then
$\exists\, e$ injective coherent experiment of R.

Connectedness and coherence

Conjecture

If $\mathcal{R}$ is a connected proof-net, then
$\exists\, e$ injective coherent experiment of $\mathcal{R}$.

If this conjecture holds, then coherent semantics is injective for connected $MELL$ proof-nets and, in particular, for $MELL$ without weakenings.
(Tortora de Falco, 2003)

Injectivity for connected $(?\parr)LL_{pol}$ proof-nets

The $(?\invamp)LL_{pol}$ fragment

$$N, M ::= X \mid ?X \mid ?P \invamp N \mid N \invamp ?P$$

$$P, Q ::= X^{\perp} \mid !X^{\perp} \mid !N \otimes P \mid P \otimes !N$$

Injectivity of coherent semantics

Theorem

Coherent semantics is injective
for connected $(?\parr)LL_{pol}$ proof-nets.

Injectivity of coherent semantics

Theorem

Coherent semantics is injective
for connected $(?\parr)LL_{pol}$ proof-nets.

Corollary. Coherent semantics is injective for the simply typed λI-calculus.

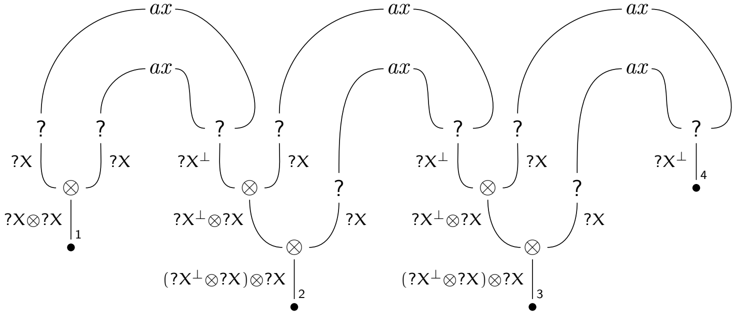
Sufficient condition

We can restrict ourselves to connected proof-nets whose conclusions are labelled by formulas of the shape ?X or:

$$?X \otimes \cdots \otimes ?X \otimes ?X^{\perp} \otimes ?X \otimes \cdots \otimes ?X$$

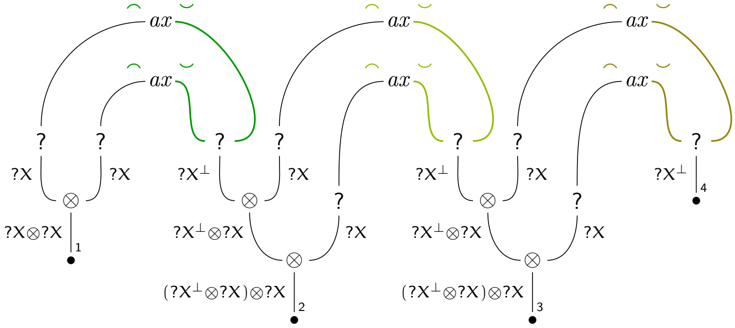
Example

We build an injective coherent experiment on a concrete example.



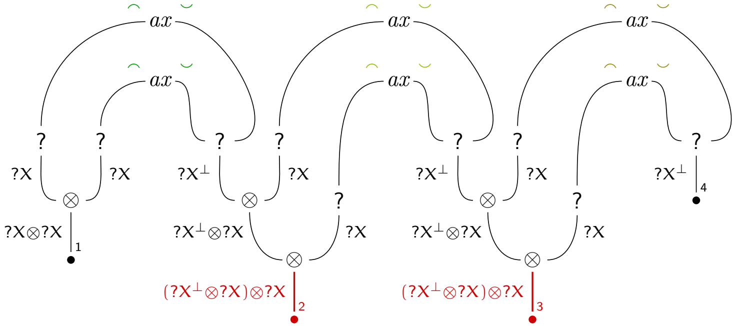
Example

We have no choice on the premises of atomic contractions.



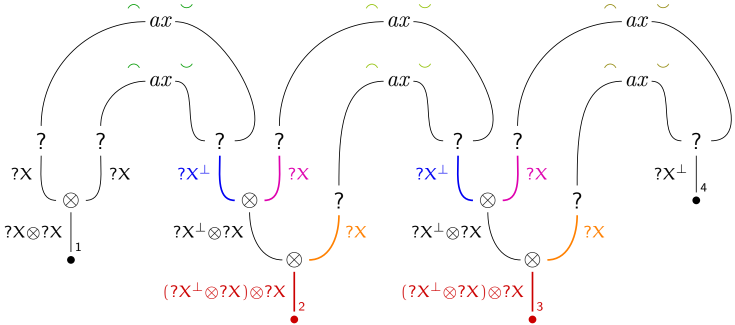
Example

Conclusions of the same type are potential premises of contractions!



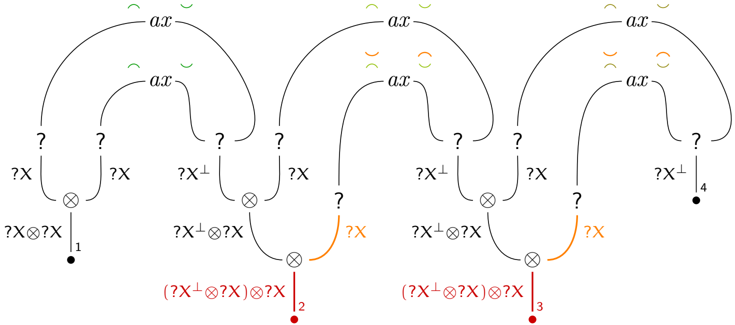
Example

We choose to assign incoherence on one of the pairs of arcs which are involved in the switching paths between 2 and 3.



Example

Because there is at most one occurrence of ?X$^{\perp}$ in the formulas, we know that we can always pick a pair of type ?X.



References

(de Carvalho, 2016)

Daniel de Carvalho. "The relational model is injective for multiplicative exponential linear logic". In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*, volume 62 of *LIPIcs*, pages 41:1-41:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

(Girard, 1987)

Jean-Yves Girard. "Linear Logic".
Theoretical Computer Science, 50:1-102, 1987.

(Tortora de Falco, 2003)

Lorenzo Tortora de Falco.
"Obsessional experiments for linear logic proof-nets".
Mathematical Structures in Computer Science, 13(6):799-855, 2003.

Thank you for your attention!