Cut-elimination for the circular modal mu-calculus: linear logic and super exponentials to the rescue

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Formulas and derivation rules of $\mu LK^\infty_\square$

$$\phi, \psi ::= \phi \vee \psi \mid \phi \wedge \psi \mid \phi \to \psi \mid T \mid F \mid X \in \mathcal{V} \mid \square\phi \mid \Diamond\phi \mid \mu X.\phi \mid \nu X.\phi.$$

Derivation rules are those of LK together with two rules for the modalities:

$$\frac{\Gamma, A \vdash \Delta}{\square\Gamma, \Diamond A \vdash \Diamond\Delta}\ \Diamond \qquad \frac{\Gamma \vdash A, \Delta}{\square\Gamma \vdash \square A, \Diamond\Delta}\ \square$$

and four rules for fixpoints:

$$\frac{\Gamma, A[X := \mu X.A] \vdash \Delta}{\Gamma, \mu X.A \vdash \Delta}\ \mu_l \qquad \frac{\Gamma \vdash A[X := \mu X.A], \Delta}{\Gamma \vdash \mu X.A, \Delta}\ \mu_r$$

$$\frac{\Gamma, A[X := \nu X.A] \vdash \Delta}{\Gamma, \nu X.A \vdash \Delta}\ \nu_l \qquad \frac{\Gamma \vdash A[X := \nu X.A], \Delta}{\Gamma \vdash \nu X.A, \Delta}\ \nu_r$$

Some example of infinite proofs

$$\mathrm{Nat} := \mu X.\top \vee X$$

Inhabitant of natural number type

$$\pi_0 := \dfrac{\dfrac{\dfrac{}{\vdash \top}\,\top}{\vdash \top \vee \mathrm{Nat}}\,\vee_1}{\vdash \mathrm{Nat}}\,\mu \qquad \pi_{n+1} := \dfrac{\dfrac{\begin{matrix}\pi_n \\ \vdash \mathrm{Nat}\end{matrix}}{\vdash \top \vee \mathrm{Nat}}\,\vee_2}{\vdash \mathrm{Nat}}\,\mu$$

$\pi_{\text{successeur}} :=$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\top \vdash \top}\,\mathrm{ax}}{\top \vdash \top \vee \mathrm{Nat}}\,\vee_1^r}{\top \vdash \mathrm{Nat}}\,\mu_r}{\top \vdash \top \vee \mathrm{Nat}}\,\vee_2^r}{\top \vdash \mathrm{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash \top \vee \mathrm{Nat}}\,\vee_2^r}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_r}{\dfrac{\top \vee \mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_l}\,\vee_l$$

$\pi_{\text{double}} :=$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\top \vdash \top}\,\mathrm{ax}}{\top \vdash \top \vee \mathrm{Nat}}\,\vee_1^r}{\top \vdash \mathrm{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash \top \vee \mathrm{Nat}}\,\vee_2^r}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_r}{\mathrm{Nat} \vdash \top \vee \mathrm{Nat}}\,\vee_2^r}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_r}{\dfrac{\top \vee \mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_l}\,\vee_l$$

Example with modalities

Taking $F := \nu X.\Diamond X$:

$$\dfrac{\dfrac{\vdash F^\perp, F}{\vdash \square F^\perp, \Diamond F}\ \square_p}{\vdash F^\perp, F}\ \mu, \nu$$

Cut-elimination for $\mu LK^{\infty}_{\square}$

Several results:

Mints & Studer '12, Brünnler & Studer '12

Cut-elimination for fragment modal μ-calculus with ω-rule.

Niwiński & Walukiewicz '96

Cut-free non-wellfounded system with a completeness theorem.

Afshari & Leigh '16

Cut-free cyclic proof system without cut-elimination procedure.

Kloibhofer '23

Incompleteness of Afshari & Leigh's system.

Formulas and derivation rules of $\mu MALL^{\infty}$

$$\phi, \psi ::= \phi \parr \psi \mid \phi \otimes \psi \mid \phi \,\&\, \psi \mid \phi \oplus \psi \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi.$$

$$\dfrac{}{\vdash A, A^{\perp}}\ \mathsf{ax} \qquad \dfrac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \mathsf{cut}$$

$$\dfrac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\ \otimes \qquad \dfrac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \parr$$

$$\dfrac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^1 \qquad \dfrac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^2 \qquad \dfrac{\vdash A_1, \Gamma \qquad \vdash A_2, \Gamma}{\vdash A_1 \,\&\, A_2, \Gamma}\ \&$$

$$\dfrac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\ \mu \qquad \dfrac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\ \nu$$

Multi-cut rule

Multi-cut

In order to prove the cut-elimination theorem, we use the multi-cut rule:

$$\frac{\vdash \Gamma_1, \Delta_1 \qquad \dots \qquad \vdash \Gamma_n, \Delta_n}{\vdash \Gamma_1, \dots, \Gamma_n} \text{ mcut}$$

with $\Delta_1, \dots, \Delta_n$ being the *cut-formulas* and satisfying an acyclicity and connexity condition (relative to the cut relation).

$$\cfrac{\mathcal{C} \qquad \cfrac{\cfrac{\pi_1}{\vdash C, \Gamma_1} \qquad \cfrac{\pi_2}{\vdash C^\perp, \Gamma_2}}{\vdash \Gamma_1, \Gamma_2} \text{ cut}}{\vdash \Delta} \text{ mcut} \rightsquigarrow \cfrac{\mathcal{C} \qquad \cfrac{\pi_1}{\vdash C, \Gamma_1} \qquad \cfrac{\pi_2}{\vdash C^\perp, \Gamma_2}}{\vdash \Delta} \text{ mcut}$$

Multi-cut reduction rule

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash B, \Gamma_2}}{\vdash A \otimes B, \Gamma_1, \Gamma_2}\otimes \qquad \dfrac{\dfrac{\pi_3}{\vdash A^{\perp}, B^{\perp}, \Gamma_3}}{\vdash A^{\perp} \parr B^{\perp}, \Gamma_3}\parr}{\Gamma_1, \Gamma_2, \Gamma_3}\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\dfrac{\pi_2}{\vdash B, \Gamma_2} \qquad \dfrac{\pi_3}{\vdash B^{\perp}, A^{\perp}, \Gamma_3}}{\vdash A^{\perp}, \Gamma_2, \Gamma_3}\text{cut}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}\text{cut}$$

$$\dfrac{\mathcal{C} \qquad \dfrac{\dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash B, \Gamma_2}}{\vdash A \otimes B, \Gamma_1, \Gamma_2}\otimes \qquad \dfrac{\dfrac{\pi_3}{\vdash A^{\perp}, B^{\perp}, \Gamma_3}}{\vdash A^{\perp} \parr B^{\perp}, \Gamma_3}\parr}{\vdash \Delta}\text{mcut}$$

$$\rightsquigarrow \dfrac{\mathcal{C} \qquad \dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash B, \Gamma_2} \qquad \dfrac{\pi_3}{\vdash A^{\perp}, B^{\perp}, \Gamma_3}}{\vdash \Delta}\text{mcut}$$

Multi-cut in action

Taking $F := \nu X.X \otimes X$

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\pi := \quad
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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Cut-elimination theorems

Cut-elimination of μALL^∞ (Fortier & Santocanale 2013)

The cut-rule is admissible for Additive Linear Logic with fixpoints.

Cut-elimination of $\mu MALL^\infty$ (Baelde et al. 2016)

Each fair multi-cut reduction sequences of $\mu MALL^\infty$ are converging to a $\mu MALL^\infty$-cut-free proof.

Cut-elimination of μLL^∞ (Saurin 2023)

Each fair multi-cut reduction sequences of μLL^∞ are converging to a μLL^∞-cut-free proof.

Goal

We want to prove cut-elimination in a syntactic way

$$\begin{tikzcd}
\mu LK^\infty_\square \arrow[r, "(-)^\bullet"] & \mu LL^\infty_\square \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & \mu LL^\infty \arrow[l, "\text{TABLEAUX '23}"']
\end{tikzcd}$$

$$
\begin{array}{ccc}
\mu LK_\square^\infty & \xrightarrow{(-)^\bullet} & \mu LL_\square^\infty \\
 & & \downarrow (-)^\bullet \\
\mu MALL^\infty & \xleftarrow[\text{TABLEAUX '23}]{} & \mu LL^\infty
\end{array}
$$

Exponentials

We add exponentials to $\mu MALL^\infty$:

$$?A \quad \text{and} \quad !A$$

As well as the corresponding rules:

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ ?_w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ ?_c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ ?_d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !_p$$

This system is called μLL^∞

Cut-elimination steps of $\mu\mathsf{LL}^\infty$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2,}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2}\,\mathsf{cut}}{\vdash \Gamma_1, ?\Gamma_2}\,?_c^{\#\Gamma_2}$$

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,?_w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,?_w^{\#\Gamma_2}$$

Translation of μLL$^\infty$ in μMALL$^\infty$

$$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\,?_w \qquad \dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,?_c \qquad \dfrac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,?_d \qquad \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)) \quad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of weakening

$$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\,?_w \rightsquigarrow \dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma^\bullet}{\vdash \bot, \Gamma^\bullet}\,\bot}{\vdash \bot \oplus ((?A)^\bullet \parr (?A)^\bullet), \Gamma^\bullet}\,\oplus^1}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet \parr (?A)^\bullet)), \Gamma^\bullet}\,\oplus^2}{\vdash \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)), \Gamma^\bullet}\,\mu$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ ?_w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ ?_c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ ?_d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

Translation of formulas

$$(?A)^\bullet = \mu X. A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X. A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of contraction

$$\frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ ?_c \rightsquigarrow \dfrac{\dfrac{\dfrac{\dfrac{\vdash (?A)^\bullet, (?A)^\bullet, \Gamma^\bullet}{\vdash (?A)^\bullet \parr (?A)^\bullet, \Gamma^\bullet}\ \parr}{\vdash \bot \oplus ((?A)^\bullet \parr (?A)^\bullet), \Gamma^\bullet}\ \oplus^2}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet \parr (?A)^\bullet)), \Gamma^\bullet}\ \oplus^2}{\vdash \mu X. A^\bullet \oplus (\bot \oplus (X \parr X)), \Gamma^\bullet}\ \mu$$

Translation of $\mu\mathsf{LL}^\infty$ in $\mu\mathsf{MALL}^\infty$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,?_w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,?_c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,?_d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X ⅋ X)) \qquad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of dereliction

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,?_d \rightsquigarrow \dfrac{\dfrac{\vdash A^\bullet, \Gamma^\bullet}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet ⅋ (?A)^\bullet)), \Gamma^\bullet}\,\oplus^1}{\vdash \mu X.A^\bullet \oplus (\bot \oplus (X ⅋ X)), \Gamma^\bullet}\,\mu$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ ?_w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ ?_c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ ?_d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

Translation of formulas

$$(?A)^\bullet = \mu X. A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X. A^\bullet \mathbin{\&} (1 \mathbin{\&} (X \otimes X))$$

Translation of promotion

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ ! \rightsquigarrow \dfrac{\dfrac{\vdash A^\bullet, (?\Gamma)^\bullet \qquad \dfrac{\dfrac{}{\vdash 1}\ 1}{\vdash 1, (?\Gamma)^\bullet}\ ?_w{}^\bullet \qquad \dfrac{\dfrac{\vdash (?A)^\bullet, (?\Gamma)^\bullet \qquad \vdash (?A)^\bullet, (?\Gamma)^\bullet}{\vdash (?A)^\bullet \otimes (?A)^\bullet, (?\Gamma)^\bullet, (?\Gamma)^\bullet}\ \otimes}{\vdash (?A)^\bullet \otimes (?A)^\bullet, (?\Gamma)^\bullet}\ ?_c{}^\bullet}{\vdash A^\bullet \mathbin{\&} (1 \mathbin{\&} ((?A)^\bullet \otimes (?A)^\bullet)), (?\Gamma)^\bullet}\ \mathbin{\&} \times 2}{\vdash \nu X. A^\bullet \mathbin{\&} (1 \mathbin{\&} (X \otimes X)), (?\Gamma)^\bullet}\ \nu$$

Translation of μLK$^\infty$ in μLL$^\infty$

$$
\begin{aligned}
(\mu X.A)^\bullet &:= !\mu X.?A^\bullet & (\nu X.A)^\bullet &:= !\nu X.?A^\bullet \\
(A_1 \vee A_2)^\bullet &:= !(?A_1{}^\bullet \oplus ?A_2{}^\bullet) & (A_1 \wedge A_2)^\bullet &:= !(?A_1{}^\bullet \mathbin{\&} ?A_2{}^\bullet) \\
F^\bullet &:= !0 & \top^\bullet &:= !\top \\
X^\bullet &:= !X & a^\bullet &:= !a \\
(A_1 \to A_2)^\bullet &:= !(?A_1{}^\bullet \multimap ?A_2{}^\bullet) & (\Gamma \vdash \Delta)^\bullet &:= \Gamma^\bullet \vdash ?\Delta^\bullet
\end{aligned}
$$

$$
\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma^\bullet \vdash ?A^\bullet, ?\Delta^\bullet \qquad \Gamma^\bullet \vdash ?B^\bullet, ?\Delta^\bullet}{\Gamma^\bullet \vdash ?A^\bullet \mathbin{\&} ?B^\bullet, ?\Delta^\bullet}\ \&_r}{\Gamma^\bullet \vdash ?!(?A^\bullet \mathbin{\&} ?B^\bullet), ?\Delta^\bullet}\ ?_{\mathsf{d}}, !_{\mathsf{p}}
$$

Cut-elimination for μLK$^\infty$

Cut-elimination for μLK$^\infty$ Saurin 2023

The cut-elimination system of μLK$^\infty$ is weakly normalizing.

$$\begin{tikzcd}
\mu LK^\infty & \mu LL^\infty \\
\pi{=}\pi_0 \arrow[r] \arrow[d, "*"] & \pi^\bullet = \theta = \theta_0 \arrow[d, "*"] \\
U(\theta_n) \arrow[d, "\infty"] & \theta_n \arrow[l] \arrow[d, "\infty"] \\
U(\theta_\infty) & \theta_\infty \arrow[l]
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \mu LK_\square^\infty \arrow[r, "(-)^\bullet"] & \mu LL_\square^\infty \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & & \mu LL^\infty \arrow[ll]
\end{tikzcd}
$$

Naïve extension of the translation and issue with it

Let's consider the two-sided system μLL$^\infty$ together with the two modal rules:

$$\frac{\Gamma, A \vdash \Delta}{\square\Gamma, \lozenge A \vdash \lozenge\Delta}\,\lozenge \qquad \frac{\Gamma \vdash A, \Delta}{\square\Gamma \vdash \square A, \lozenge\Delta}\,\square$$

We extend the translation, to get a translation from μLK$^\infty_\square$ to μLL$^\infty_\square$:

$$(\square A)^\bullet := !\,\square\,?A^\bullet \qquad\qquad (\lozenge A)^\bullet := !\lozenge ?A^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \Box A, \Diamond B}\ \Box$$

We start with the sequent:

$$\vdash ?! \Box ?A^\bullet, ?!\Diamond ?B^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \lozenge B}\ \square$$

We apply a dereliction a promotion:

$$\frac{\vdash ?!\,\square\,?A^\bullet, \lozenge ?B^\bullet}{\vdash ?!\,\square\,?A^\bullet, ?!\lozenge ?B^\bullet}\ ?_d, !$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \lozenge B}\,\square$$

We apply a dereliction again:

$$\dfrac{\dfrac{\vdash !\square ?A^\bullet, \lozenge ?B^\bullet}{\vdash ?!\square ?A^\bullet, \lozenge ?B^\bullet}\,?_d}{\vdash ?!\square ?A^\bullet, ?!\lozenge ?B^\bullet}\,?_d, !$$

And we are blocked.

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

We apply a dereliction again:

$$\dfrac{\dfrac{\vdash !\square ?A^\bullet, \Diamond ?B^\bullet}{\vdash ?!\square ?A^\bullet, \Diamond ?B^\bullet}\ ?_d}{\vdash ?!\square ?A^\bullet, ?!\Diamond ?B^\bullet}\ ?_d, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \lozenge\Delta}{\vdash !A, ?\Gamma, \lozenge\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \lozenge B}\ \square$$

Now, we can apply our promotion:

$$\cfrac{\cfrac{\cfrac{\vdash \square ?A^\bullet, \lozenge ?B^\bullet}{\vdash ! \square ?A^\bullet, \lozenge ?B^\bullet}\ !}{\vdash ?! \square ?A^\bullet, \lozenge ?B^\bullet}\ ?_\mathrm{d}}{\vdash ?! \square ?A^\bullet, ?!\lozenge ?B^\bullet}\ ?_\mathrm{d}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \Box A, \Diamond B}\ \Box$$

And finally our modal rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?A^\bullet, ?B^\bullet}{\vdash \Box ?A^\bullet, \Diamond ?B^\bullet}\ \Box}{\vdash !\Box ?A^\bullet, \Diamond ?B^\bullet}\ !}{\vdash ?!\Box ?A^\bullet, \Diamond ?B^\bullet}\ ?_d}{\vdash ?!\Box ?A^\bullet, ?!\Diamond ?B^\bullet}\ ?_d, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

But what does it imply for this system?

Super exponential system (TLLA '21)

Exponential signatures

An exponential signature is a boolean function on the set of rule names:

$$\{?_{\mathsf{m}_i}, ?_{\mathsf{c}_i} \mid i \in \mathbb{N}\}.$$

Formulas

Given a set of exponential signatures $\mathcal{E}$, we get the exponential formulas of μsuperLL$^\infty(\mathcal{E})$ (with $\sigma \in \mathcal{E}$):

$$\phi, \psi ::= \phi \parr \psi \mid \phi \otimes \psi \mid \phi \& \psi \mid \phi \oplus \psi \mid \bot \mid 1 \mid \top \mid 0$$

$$\mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi \mid ?_\sigma \phi \mid !_\sigma \phi.$$

$$\dfrac{\vdash \overbrace{A, \ldots, A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{m}_i})}{\vdash ?_\sigma A, \Gamma}\, ?_{\mathsf{m}_i} \qquad \dfrac{\vdash \overbrace{?_\sigma A, \ldots, ?_\sigma A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{c}_i})}{\vdash ?_\sigma A, \Gamma}\, ?_{\mathsf{c}_i}$$

Promotion rules

Given three relations $\leq_{g}, \leq_{f}$ and $\leq_{u}$ on $\mathcal{E}$, we have the promotion rules of $\mu superLL^{\infty}(\mathcal{E}, \leq_{g}, \leq_{f}, \leq_{u})$:

$$\dfrac{\vdash A, ?_{\vec{\sigma'}}\Delta \qquad \sigma \leq_{g} \vec{\sigma'}}{\vdash !_{\sigma}A, ?_{\vec{\sigma'}}\Delta}\ !_{g} \qquad \dfrac{\vdash A, \Delta \qquad \sigma \leq_{f} \vec{\sigma'}}{\vdash !_{\sigma}A, ?_{\vec{\sigma'}}\Delta}\ !_{f} \qquad \dfrac{\vdash A, B \qquad \sigma_1 \leq_{u} \sigma_2}{\vdash !_{\sigma_1}A, ?_{\sigma_2}B}\ !_{u}$$

Instances of superLL

ELL

Elementary Linear Logic (ELL) is a variant of LL where we remove ($?_{d}$) and ($!_{g}$) and add the functorial promotion:

$$\frac{\vdash A, \Gamma}{\vdash !A, ?\Gamma}\ !_{f}$$

superLL($\mathcal{E}, \leq_{g}, \leq_{f}, \leq_{u}$) is ELL where:

$\mathcal{E} = \{\bullet\}$;

$\bullet(?_{c_2}) = \bullet(?_{m_0}) = \text{true}$ (and $(\bullet)(r) = \text{false}$ otherwise);

$\leq_{g} = \leq_{u} = \varnothing$ and $\bullet \leq_{f} \bullet$.

superLL subsume many existing system of linear logic such as LL, LL with shifts, ELL, LLL, SLL or seLL.

Cut-elimination axioms

$\sigma \leq_{\mathbf{g}} \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{m}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{c}_i})$	$i \geq 0$	(axgmpx)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{m}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{m}_i})$	$i \geq 0$ and $s \neq g$	(axfumpx)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma(?_{\mathbf{c}_i})$	$\Rightarrow$	$\bar{\sigma}'(?_{\mathbf{c}_i})$	$i \geq 2$	(axcontr)
$\sigma \leq_s \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_s \sigma''$		(axTrans)
$\sigma \leq_{\mathbf{g}} \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_{\mathbf{g}} \sigma''$		(axleqgs)
$\sigma \leq_{\mathbf{f}} \sigma'$	$\Rightarrow$	$\sigma' \leq_{\mathbf{u}} \sigma''$	$\Rightarrow$	$\sigma \leq_{\mathbf{f}} \sigma''$		(axleqfu)
$\sigma \leq_{\mathbf{f}} \sigma'$	$\Rightarrow$	$\sigma' \leq_{\mathbf{g}} \sigma''$	$\Rightarrow$	$(\sigma \leq_{\mathbf{g}} \sigma'' \wedge (\sigma \leq_{\mathbf{f}} \sigma''' \Rightarrow (\sigma \leq_{\mathbf{g}} \sigma''' \wedge \sigma'''(?_{\mathbf{m}_1})))$		(axleqfg)
$\sigma \leq_{\mathbf{u}} \sigma'$	$\Rightarrow$	$\sigma' \leq_s \sigma''$	$\Rightarrow$	$\sigma \leq_s \sigma''$		(axlequs)

with $s \in \{g, f, u\}$, all the axioms are universally quantified.

Instances of superLL satisfies cut-elimination axioms

LL, LL with shifts, ELL, LLL, SLL, seLL satisfy cut-elimination axioms.

Cut-elimination for superLL

Let's consider the following axiom:

$$\sigma \leq_g \sigma' \quad \Rightarrow \quad \sigma' \leq_f \sigma'' \quad \Rightarrow \quad \sigma \leq_g \sigma'' \qquad \text{(axleqgs)}$$

We use it for the following cut-elimination step:

$$\dfrac{\dfrac{\vdash A, ?_{\vec{\tau}}\Gamma \qquad \sigma \leq_g \vec{\tau}, \tau}{\vdash !_\sigma A, ?_{\vec{\tau}}\Gamma, ?_\tau C}\,!_g \qquad \dfrac{\vdash C, \Delta \qquad \tau \leq_f \vec{\rho}}{\vdash !_\tau C^\perp, ?_{\vec{\rho}}\Delta}\,!_f}{\vdash !_\sigma A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,\text{cut} \rightsquigarrow$$

$$\dfrac{\dfrac{\vdash A, ?_{\vec{\tau}}\Gamma \qquad \dfrac{\vdash C, \Delta \qquad \tau \leq_f \vec{\rho}}{\vdash !_\tau C^\perp, ?_{\vec{\rho}}\Delta}\,!_f}{\vdash A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,\text{cut} \qquad \sigma \leq_g \vec{\tau}, \vec{\rho}}{\vdash !_\sigma A, ?_{\vec{\tau}}\Gamma, ?_{\vec{\rho}}\Delta}\,!_g$$

superLL eliminates cuts

Cut-eliminations (B. & Laurent '21)

As soon as the 8 cut-elimination axioms are satisfied, cut elimination holds for superLL$(\mathcal{E}, \leq_{\mathsf{g}}, \leq_{\mathsf{f}}, \leq_{\mathsf{u}})$.

Cut-elimination steps

Example

If

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash A, ?_{\vec{\tau}}\Delta} \qquad \sigma \leq_g \vec{\tau}}{\vdash !_\sigma A, ?_{\vec{\tau}}\Delta}\,!_g \qquad \mathcal{C}^!}{\vdash !_\sigma A, ?_{\vec{\rho}}\Gamma}\,\mathrm{mcut}(\iota, \perp\!\!\!\perp)$$

is a $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$-proof then

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash A, ?_{\vec{\tau}}\Delta} \qquad \mathcal{C}^!}{\vdash A, ?_{\vec{\rho}}\Gamma}\,\mathrm{mcut}(\iota, \perp\!\!\!\perp) \qquad \sigma \leq_g \vec{\rho}}{\vdash !_\sigma A, ?_{\vec{\rho}}\Gamma}\,!_g$$

is also a $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$-proof.

Translation of $\mu superLL^\infty$ into μLL^∞

Translation of formulas

We translate formulas by induction using:

$$(!_\sigma A)^\circ := !A^\circ \qquad\qquad (?_\sigma A)^\circ := ?A^\circ$$

$$\dfrac{\vdash \overbrace{A, \ldots, A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{m}_i})}{\vdash ?_\sigma A, \Gamma}\, ?_{\mathsf{m}_i} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \overbrace{A^\circ, \ldots, A^\circ}^{i}, \Gamma^\circ}{\vdash ?A^\circ, \ldots, ?A^\circ, \Gamma^\circ}\, ?_{\mathsf{d}} \times i}{\vdash ?A^\circ, \Gamma^\circ}\, ?_{\mathsf{c}} \times i$$

$$\dfrac{\vdash \overbrace{?_\sigma A, \ldots, ?_\sigma A}^{n}, \Gamma \qquad \sigma(?_{\mathsf{c}_i})}{\vdash ?_\sigma A, \Gamma}\, ?_{\mathsf{c}_i} \quad \rightsquigarrow \quad \dfrac{\vdash \overbrace{?A^\circ, \ldots, ?A^\circ}^{i}, \Gamma^\circ}{\vdash ?A^\circ, \Gamma^\circ}\, ?_{\mathsf{c}} \times i$$

Translation of μsuperLL$^\infty$ into μLL$^\infty$

$$\dfrac{\vdash ?_{\sigma_1}A_1,\dots,?_{\sigma_n}A_n, A \qquad \begin{array}{c} i \in [\![1,n]\!] \\ \sigma \leq_{\mathrm{g}} \sigma_i \end{array}}{\vdash ?_{\sigma_1}A_1,\dots,?_{\sigma_n}A_n, !_\sigma A}\,!_{\mathrm{g}} \quad\rightsquigarrow\quad \dfrac{\vdash ?A_1^\circ,\dots,?A_n^\circ, A^\circ}{\vdash ?A_1^\circ,\dots,?A_n^\circ, !A^\circ}\,!_{\mathrm{p}}$$

$$\dfrac{\vdash A_1,\dots,A_n, A \qquad \begin{array}{c} i \in [\![1,n]\!] \\ \sigma \leq_{\mathrm{f}} \sigma_i \end{array}}{\vdash ?_{\sigma_1}A_1,\dots,?_{\sigma_n}A_n, !_\sigma A}\,!_{\mathrm{f}} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash A_1^\circ,\dots,A_n^\circ, A^\circ}{\vdash ?A_1^\circ,\dots,?A_n^\circ, A^\circ}\,?_{\mathrm{d}}}{\vdash ?A_1^\circ,\dots,?A_n^\circ, !A^\circ}\,!_{\mathrm{p}}$$

$$\dfrac{\vdash B, A \qquad \sigma_1 \leq_{\mathrm{u}} \sigma_2}{\vdash ?_{\sigma_2}B, !_{\sigma_1}A}\,!_{\mathrm{u}} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash B^\circ, A^\circ}{\vdash ?B^\circ, A^\circ}\,?_{\mathrm{d}}}{\vdash ?B^\circ, !A^\circ}\,!_{\mathrm{p}}$$

Cut-elimination for $\mu superLL^\infty$

Cut-elimination reduction system correctness

For every $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$ reduction sequences $(\pi_i)_{i\in\mathbb{N}}$, there exists a μLL^∞ reduction sequence $(\theta_i)_{i\in\mathbb{N}}$ such that for each i, there exists j such that π_i° is equal to θ_j up to rule-permutations.

Cut-elimination reduction system completeness

If there is a μLL^∞-redex $\mathcal{R}$ sending π° to π'° then there is also a $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$-redex $\mathcal{R}'$ sending π to a proof π'', such that in the translation of $\mathcal{R}'$, $\mathcal{R}$ is reduced.

Cut-elimination theorem for $\mu superLL^\infty$

Every fair (mcut)-reduction sequence of $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$ converges to a $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$ cut-free proof.

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

But what does it imply for this system?

Solution

Promotion rule

$$\frac{\vdash A, ?_{\bullet}\Gamma, ?_{\star}\Delta}{\vdash !_{\bullet}A, ?_{\bullet}\Gamma, ?_{\star}\Delta}\ !$$

But what does it imply for this system?

Solution

Promotion rule

$$\dfrac{\vdash A, ?_{\bullet}\Gamma, ?_{\star}\Delta \qquad \bullet \leq_{g} \star}{\vdash !_{\bullet}A, ?_{\bullet}\Gamma, ?_{\star}\Delta}\ !_{g}$$

But what does it imply for this system?

Solution

Promotion rule

$$\dfrac{\vdash A, ?_\bullet\Gamma, ?_\star\Delta \qquad \bullet \leq_g \star}{\vdash !_\bullet A, ?_\bullet\Gamma, ?_\star\Delta}\ !_g$$

But what does it imply for this system?
Axiom

$$\sigma \leq_g \sigma' \quad \Rightarrow \quad \sigma(?_{m_i}) \quad \Rightarrow \quad \bar{\sigma}'(?_{c_i}) \qquad i \geq 0 \quad (\text{axgmpx})$$

for $i = 0$ and $i = 1$ give us

$$\dfrac{\vdash \Delta}{\vdash ?_\star A, \Delta}\ ?_{c_0} \qquad\qquad \dfrac{\vdash ?_\star A, \Delta}{\vdash ?_\star A, \Delta}\ ?_{c_1}$$

Solution

Promotion rule

$$\dfrac{\vdash A, ?_\bullet\Gamma, ?_\star\Delta \qquad \bullet \leq_g \star}{\vdash !_\bullet A, ?_\bullet\Gamma, ?_\star\Delta}\ !_g$$

But what does it imply for this system?
Axiom

$$\sigma \leq_g \sigma' \quad \Rightarrow \quad \sigma(?_{c_i}) \quad \Rightarrow \quad \bar{\sigma}'(?_{c_i}) \qquad i \geq 2 \quad (\text{axcontr})$$

for $i = 2$, gives us:

$$\dfrac{\vdash ?_\star A, ?_\star A, \Delta}{\vdash ?_\star A, \Delta}\ c$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

Consequences

$$\frac{\vdash \Delta}{\vdash \Diamond A, \Delta}\ w \qquad \frac{\vdash \Diamond A, \Diamond A, \Delta}{\vdash \Diamond A, \Delta}\ c$$

$\mu LL_\square^\infty$ as an instance of $\mu superLL^\infty$

The system $\mu LL_\square^\infty$ is the system $\mu superLL^\infty(\mathcal{E}, \leq_g, \leq_f, \leq_u)$ such that:

- The set of signatures contains two elements $\mathcal{E} := \{\bullet, \star\}$.
- $?_{c_2}(\bullet) = ?_{c_2}(\star) = \text{true}$ $?_{m_1}(\bullet) = \text{true}$, $?_{m_0}(\bullet) = ?_{m_0}(\star) = \text{true}$, and all the other elements have value false for both signatures.
- $\bullet \leq_g \bullet$; $\bullet \leq_g \star$, $\star \leq_f \star$, and all couples for the three relations $\leq_g, \leq_f$ and $\leq_u$ being false.

This system is $\mu LL_\square^\infty$ when taking:

$$?_\bullet := ?, \quad !_\bullet := !, \quad ?_\star := \Diamond \quad \text{and} \quad !_\star := \square.$$

Moreover, the system satisfy cut-elimination axioms.

$$\begin{tikzcd}
\mu LK^\infty_\square \arrow[rr, "(-)^\bullet"] & & \mu LL^\infty_\square \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & & \mu LL^\infty \arrow[ll]
\end{tikzcd}$$

Translation from $\mu LL^\infty_\Box$ to μLL^∞

Translation from $\mu LL^\infty_\Box$ to μLL^∞ and cut-elimination for $\mu LL^\infty_\Box$

We define:

$$(\Diamond A)^\circ \to ?A^\circ \text{ and } (\Box A)^\circ \to !A^\circ.$$

We easily get weakening and contractions of $\Diamond$ with weakening and contraction of ?.
For the modality rule, we have:

$$\dfrac{\vdash A, \Gamma}{\vdash \Box A, \Diamond\Gamma}\ \Box \rightsquigarrow \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\ ?_d}{\vdash !A^\circ, ?\Gamma^\circ}\ !$$

Translation from $\mu LL^\infty_\square$ to μLL^∞

Translation from $\mu LL^\infty_\square$ to μLL^∞ and cut-elimination for $\mu LL^\infty_\square$

We define:

$$(\lozenge A)^\circ \to ?A^\circ \text{ and } (\square A)^\circ \to !A^\circ.$$

We easily get weakening and contractions of $\lozenge$ with weakening and contraction of $?$.
For the modality rule, we have:

$$\dfrac{\vdash A, \Gamma}{\vdash \square A, \lozenge\Gamma}\ \square \rightsquigarrow \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\ ?_d}{\vdash !A^\circ, ?\Gamma^\circ}\ !$$

Cut-elimination theorems

Using $(-)^\circ$, we obtain cut-elimination for $\mu LL^\infty_\square$.

Using $(-)^\bullet$, we get a proof of $\mu LL^\infty_\square$, from which we can eliminate cuts, we then can come back to $\mu LK^\infty_\square$ and get a cut-free proof of $\mu LL^\infty_\square$.

Conclusion & future works

Summary

We proved a (syntactic) cut-elimination theorem for a general parametrized linear logic system $\mu\mathsf{superLL}^\infty$.

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Future works

- Integrate the digging rule (axiom S4) or the co-dereliction rules (axiom T)
- Prove cut-elimination for the system that was, actually, presented in TLLA '21.