# Proof theory in Multiplicative-Additive Linear Logic

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Non-permanent members' seminar

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# Formulas of $\mu$MALL

$$A, B ::= A \parr B \mid A \otimes B \mid A \& B \mid A \oplus B \mid \top \mid 0 \mid \bot \mid 1 \mid X \in \mathcal{V} \mid \mu X.A \mid \nu X.A.$$

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$$A := \mu X.\nu Y.(Y \oplus X)$$

$$\rightarrow B := \nu Y.(Y \oplus A)$$

$$\rightarrow (B \oplus A) \rightarrow (B \oplus A) \oplus (B \oplus A) \rightarrow \dots$$

# Negation

## De Morgan's Law...

$$(A \otimes B)^\perp := A^\perp \parr B^\perp \qquad (A \parr B)^\perp := A^\perp \otimes B^\perp$$

$$(A \,\&\, B)^\perp := A^\perp \oplus B^\perp \qquad (A \oplus B)^\perp := A^\perp \,\&\, B^\perp$$

$$1^\perp := \perp \qquad \perp^\perp := 1$$

$$\top^\perp := 0 \qquad 0^\perp := \top$$

$$(\mu X.F)^\perp := \nu X.F^\perp \qquad (\nu X.F)^\perp := \mu X.F^\perp$$

$$X^\perp := X$$

# One-sided Sequent vs. Two sided-sequent

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m \quad \Leftrightarrow \quad \vdash A_1^\perp, \ldots, A_n^\perp, B_1, \ldots, B_m$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2} \vee_l \quad \rightsquigarrow \quad \frac{\vdash A, \Gamma_1^\perp, \Delta_1 \qquad \vdash B, \Gamma_2^\perp, \Delta_2}{\vdash A^\perp \wedge B^\perp, \Gamma_1, \Gamma_2, \Delta_1, \Delta_2} \wedge_r$$

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# Derivation rules of $\mu$MALL$^\infty$

$$\frac{}{\vdash A, A^\perp}\ \text{ax} \qquad\qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\frac{}{\vdash 1}\ 1 \qquad\qquad \frac{}{\vdash \top, \Gamma}\ \top \qquad\qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ \bot$$

$$\frac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\ \otimes \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \parr$$

$$\frac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^1 \qquad\qquad \frac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ \oplus^2 \qquad\qquad \frac{\vdash A_1, \Gamma \qquad \vdash A_2, \Gamma}{\vdash A_1 \mathbin{\&} A_2, \Gamma}\ \&$$

$$\frac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\ \mu \qquad\qquad \frac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\ \nu$$

# Examples of infinite proofs

## Some abstract proofs

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \nu X.X\end{array}}{\vdash \nu X.X}\,\nu}{\vdash \nu X.X}\,\nu
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ \vdash F\end{array}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mu \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ \vdash F\end{array}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mu}{\vdash F \otimes F}\,\otimes}{\vdash \mu X.X \otimes X (= F)}\,\mu$$

# Examples of infinite proofs

## Natural numbers

$$\mathsf{Nat} := \mu X.1 \oplus X$$

$$\pi_0 := \cfrac{\cfrac{\cfrac{}{\vdash 1}\ 1}{\vdash 1 \oplus \mathsf{Nat}}\ \oplus_1}{\vdash \mathsf{Nat}}\ \mu \qquad \pi_{n+1} := \cfrac{\cfrac{\overset{\pi_n}{\vdash \mathsf{Nat}}}{\vdash 1 \oplus \mathsf{Nat}}\ \oplus_2}{\vdash \mathsf{Nat}}\ \mu$$

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$$\pi_3 := \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu$$

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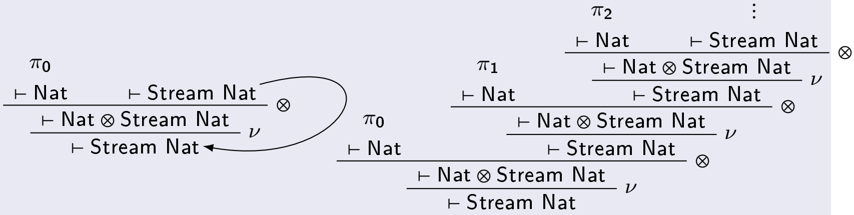
## Stream of natural numbers

$$\mathsf{Stream\ Nat} := \nu X.\mathsf{Nat} \otimes X$$



$$\dfrac{\dfrac{\overset{\pi_0}{\vdash \mathsf{Nat}} \qquad \vdash \mathsf{Stream\ Nat}}{\vdash \mathsf{Nat} \otimes \mathsf{Stream\ Nat}}\,\otimes}{\vdash \mathsf{Stream\ Nat}}\,\nu$$

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# Validity condition

The system as such is not consistent:

$$\text{For any } A, \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}}{\vdash \mu X.X}\,\mu \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, A}}{\vdash \nu X.X, A}\,\nu}{\vdash A}\,\text{cut}$$

# Ancestor relation

$$\phi, \psi ::= \phi \parr \psi \mid \phi \otimes \psi \mid \phi \,\&\, \psi \mid \phi \oplus \psi \mid \top \mid 0 \mid \bot \mid 1 \mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi.$$

$$\frac{}{\vdash A, A^{\perp}}\,\text{ax} \qquad \frac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\frac{}{\vdash 1}\,1 \qquad \frac{}{\vdash \top, \Gamma}\,\top \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,\bot$$

$$\frac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\,\otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,\parr$$

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# Threads & validity

### Thread

A thread is a sequence of sub-formulas starting in the branch of a proof and following the ancestor relation

### Validity

A branch is said to be valid if it [contains an infinite number of $\nu$-rules] it contains a thread which minimal formulas is a $\nu$. A proof is valid if each of its branches are valid.

# Examples of infinite proofs

## Some abstract proofs

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \nu X.X\end{array}}{\vdash \nu X.X}\,\nu}{\vdash \nu X.X}\,\nu \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash F\end{array} \quad \begin{array}{c}\vdots\\ \vdash F\end{array}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mu \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash F\end{array} \quad \begin{array}{c}\vdots\\ \vdash F\end{array}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mu}{\vdash F \otimes F}\,\otimes}{\vdash \mu X.X \otimes X (= F)}\,\mu$$

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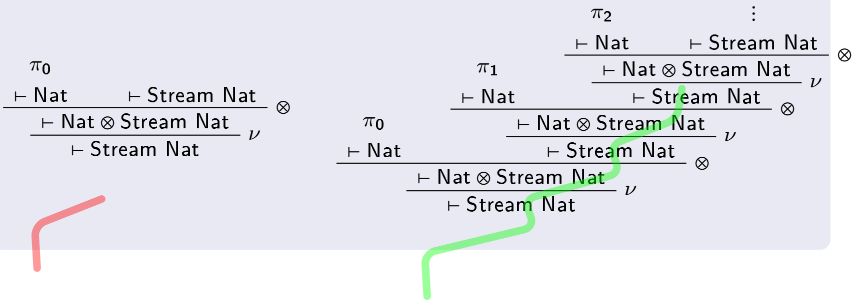

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# Cut-elimination as a rewriting system in proofs

## Reminder of the cut-rule

$$\dfrac{\vdash {\color{red}C}, \Gamma \qquad \vdash {\color{red}C^{\perp}}, \Delta}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash B, \Gamma_2}}{\vdash {\color{red}A \otimes B}, \Gamma_1, \Gamma_2}\ \otimes \qquad \dfrac{\dfrac{\pi_3}{\vdash A^{\perp}, B^{\perp}, \Delta}}{\vdash {\color{red}A^{\perp} \parr B^{\perp}}, \Delta}\ \parr}{\vdash \Gamma_1, \Gamma_2, \Delta}\ \mathrm{cut}$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\pi_1}{\vdash A, \Gamma_1} \qquad \dfrac{\dfrac{\pi_2}{\vdash B, \Gamma_2} \qquad \dfrac{\pi_3}{\vdash A^{\perp}, B^{\perp}, \Delta}}{\vdash A^{\perp}, \Gamma_2, \Delta}\ \mathrm{cut}}{\vdash \Gamma_1, \Gamma_2, \Delta}\ \mathrm{cut}$$

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$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdash F[\mu X.F/X], \Gamma\end{matrix}}{\vdash \mu X.F, \Gamma}\ \mu \qquad \dfrac{\begin{matrix}\pi_2 \\ \vdash F^{\perp}[\nu X.F^{\perp}/X], \Delta\end{matrix}}{\vdash \nu X.F^{\perp}, \Delta}\ \nu}{\vdash \Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \dfrac{\begin{matrix}\pi_1 \\ \vdash F[\mu X.F/X], \Gamma\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdash F^{\perp}[\nu X.F^{\perp}/X], \Delta\end{matrix}}{\vdash \Gamma, \Delta}\ \text{cut}$$

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$$\rightsquigarrow \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash C, A, B, \Gamma\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash C^{\perp}, \Delta\end{array}}{\vdash A, B, \Gamma, \Delta}\,\text{cut}}{\vdash A \parr B, \Gamma, \Delta}\,\parr$$

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$$\rightsquigarrow \qquad \dfrac{\dfrac{\overset{\pi_1}{\vdash C, A[\mu X.A/X], \Gamma} \qquad \overset{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash A[\mu X.A/X], \Gamma, \Delta}\,\text{cut}}{\vdash \mu X.A, \Gamma, \Delta}\,\mu$$

# Multi-cut rule

In our system, we compress many (cut)-rules, in a single one: Multi-cut (mcut) which has many hypothesis:

$$\dfrac{\vdash \Gamma_1, \Delta_1 \qquad \dots \qquad \vdash \Gamma_n, \Delta_n}{\vdash \Gamma_1, \dots, \Gamma_n}\ \text{mcut}$$

## Example

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2}}{\vdash C, \Gamma_1, \Gamma_2}\ \text{cut} \qquad \dfrac{\dfrac{\overset{\pi_3}{\vdash A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4}}{\vdash D, \Gamma_3, \Gamma_4}\ \text{cut} \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\ \text{cut}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \text{cut}$$

$$\rightsquigarrow \dfrac{\overset{\pi_1}{\vdash E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2} \qquad \overset{\pi_3}{\vdash A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4} \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \text{mcut}$$

# mcut vs. cut-reductions

$$
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  \dfrac{
    \dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\,\parr
    \qquad
    \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2}
  }{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\textbf{cut}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu
      \qquad
      \overset{\pi_4}{\vdash A^\perp, \Gamma_4}
    }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\textbf{cut}
    \qquad
    \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}
  }{\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\RightLabel{${\color{blue}\parr}$}
\UnaryInfC{$\vdash {\color{blue}G \parr H}, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^{\perp}, C, \Gamma_2$}
\RightLabel{$\mathbf{cut}$}
\BinaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, A, D, \Gamma_3$}
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\noLine
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\RightLabel{$\mathbf{cut}$}
\BinaryInfC{$\vdash \nu X.F, D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^{\perp}, C^{\perp}, \Gamma_5$}
\RightLabel{$\mathbf{cut}$}
\BinaryInfC{$\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{$\mathbf{cut}$}
\BinaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^\perp, C, \Gamma_2$}
\RightLabel{cut}
\BinaryInfC{$\vdash G, H, C, \Gamma_1, \Gamma_2$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^\perp, \Gamma_4$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^\perp, C^\perp, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^\perp, C, \Gamma_2$}
\RightLabel{cut}
\BinaryInfC{$\vdash G, H, C, \Gamma_1, \Gamma_2$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\RightLabel{$\color{blue}{\nu}$}
\UnaryInfC{$\vdash \color{blue}{\nu X.F}, A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^\perp, \Gamma_4$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^\perp, C^\perp, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^\perp, C, \Gamma_2$}
\RightLabel{cut}
\BinaryInfC{$\vdash G, H, C, \Gamma_1, \Gamma_2$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^\perp, \Gamma_4$}
\RightLabel{cut}
\BinaryInfC{$\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^\perp, C^\perp, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\,\textbf{cut}}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\parr \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\textbf{cut} \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}}{\vdash {\color{blue}\nu X.F}, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,{\color{blue}\nu}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^{\perp}, C, \Gamma_2$}
\RightLabel{$\textbf{cut}$}
\BinaryInfC{$\vdash G, H, C, \Gamma_1, \Gamma_2$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^{\perp}, \Gamma_4$}
\RightLabel{$\textbf{cut}$}
\BinaryInfC{$\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^{\perp}, C^{\perp}, \Gamma_5$}
\RightLabel{$\textbf{cut}$}
\BinaryInfC{$\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{$\textbf{cut}$}
\BinaryInfC{$\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\pi_1}{\vdash G, H, E, \Gamma_1}
      \qquad
      \dfrac{\pi_2}{\vdash E^\perp, C, \Gamma_2}
    }{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\mathbf{cut}
  }{\vdash {\color{blue}G \parr H}, C, \Gamma_1, \Gamma_2}\;{\color{blue}\parr}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}
      \qquad
      \dfrac{\pi_4}{\vdash A^\perp, \Gamma_4}
    }{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\;\mathbf{cut}
    \qquad
    \dfrac{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}
  }{\vdash F[\nu X.F/X], C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathbf{cut}
}{
  \dfrac{\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\nu
}\;\mathbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\textbf{cut} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array} \qquad \begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\;\textbf{cut} \qquad \begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}}{\vdash C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\;\textbf{cut}}{\vdash F[\nu X.F/X], G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\textbf{cut}}{\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\parr}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\nu
$$

# mcut vs. cut-reductions

$$
\dfrac{
  \dfrac{\dfrac{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\,\parr
  \qquad
  \dfrac{\pi_2}{\vdash E^{\perp}, C, \Gamma_2}
}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\mathbf{cut}
\qquad
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu
      \qquad
      \dfrac{\pi_4}{\vdash A^{\perp}, \Gamma_4}
    }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\mathbf{cut}
    \qquad
    \dfrac{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}
  }{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{cut}
}{}
$$

$$
\dfrac{\vdash G \parr H, C, \Gamma_1, \Gamma_2 \qquad \vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}}{\vdash G \parr H, E, \Gamma_1}\parr
\quad
\begin{array}{c}\pi_2\\ \vdash E^\perp, C, \Gamma_2\end{array}
}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\mathbf{cut}
\quad
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}}{\vdash \nu X.F, A, D, \Gamma_3}\nu
\quad
\begin{array}{c}\pi_4\\ \vdash A^\perp, \Gamma_4\end{array}
}{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\mathbf{cut}
\quad
\begin{array}{c}\pi_5\\ \vdash D^\perp, C^\perp, \Gamma_5\end{array}
}{\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\mathbf{cut}
}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\mathbf{cut}
$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\,\parr \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2}}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\text{cut} \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\text{cut}}{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\nu \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\,\text{cut}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\text{cut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\,\parr \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2}}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\textbf{cut} \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\textbf{cut} \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash F[\nu X.F/X], C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}}{\vdash {\color{blue}\nu X.F}, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\,{\color{blue}\nu}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}}{\vdash {\color{blue}G \parr H}, E, \Gamma_1}\,{\color{blue}\parr} \qquad \begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\mathbf{cut} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array} \qquad \begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\mathbf{cut} \qquad \begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{cut}}{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\nu}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{cut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\textbf{cut}}{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\;\parr \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\;\textbf{cut} \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash F[\nu X.F/X], C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\;\textbf{cut}}{\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\;\nu}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\textbf{cut}$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^\perp, C, \Gamma_2$}
\RightLabel{cut}
\BinaryInfC{$\vdash G, H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^\perp, \Gamma_4$}
\RightLabel{cut}
\BinaryInfC{$\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^\perp, C^\perp, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash F[\nu X.F/X], C^\perp, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\dfrac{
\dfrac{
\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}
\qquad
\begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}
}{\vdash G, H, C, \Gamma_1, \Gamma_2}\,\textbf{cut}
\qquad
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}
\qquad
\begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}
}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\textbf{cut}
\qquad
\begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}
}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
}{\vdash {\color{blue}\nu X.F}, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,{\color{blue}\nu}
}{
\dfrac{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr
}\,\textbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\mathbf{cut} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array} \qquad \begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\;\mathbf{cut} \qquad \begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathbf{cut}}{\vdash F[\nu X.F/X], G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathbf{cut}}{\vdash {\color{blue}\nu X.F}, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;{\color{blue}\nu}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\parr
$$

# mcut vs. cut-reductions

$$
\dfrac{
  \dfrac{
    \dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}}{\vdash G \parr H, E, \Gamma_1}\,\parr
    \qquad
    \begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}
  }{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\,\textbf{cut}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu
      \qquad
      \begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}
    }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\textbf{cut}
    \qquad
    \begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}
  }{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
$$

# mcut vs. cut-reductions

$$
\begin{prooftree}
\AxiomC{$\pi_1$}
\noLine
\UnaryInfC{$\vdash G, H, E, \Gamma_1$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash G \parr H, E, \Gamma_1$}
\AxiomC{$\pi_2$}
\noLine
\UnaryInfC{$\vdash E^{\perp}, C, \Gamma_2$}
\RightLabel{cut}
\BinaryInfC{$\vdash G \parr H, C, \Gamma_1, \Gamma_2$}
\AxiomC{$\pi_3$}
\noLine
\UnaryInfC{$\vdash F[\nu X.F/X], A, D, \Gamma_3$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash \nu X.F, A, D, \Gamma_3$}
\AxiomC{$\pi_4$}
\noLine
\UnaryInfC{$\vdash A^{\perp}, \Gamma_4$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, D, \Gamma_3, \Gamma_4$}
\AxiomC{$\pi_5$}
\noLine
\UnaryInfC{$\vdash D^{\perp}, C^{\perp}, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5$}
\RightLabel{cut}
\BinaryInfC{$\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$}
\end{prooftree}
$$

# mcut vs. cut-reductions

$$
\dfrac{
 \dfrac{
  \dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\mathsf{cut}
 }{\vdash G \parr H, C, \Gamma_1, \Gamma_2}\;\parr
 \qquad
 \dfrac{
  \dfrac{
   \dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\;\nu
   \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4}
  }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\;\mathsf{cut}
  \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}
 }{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathsf{cut}
}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathsf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{
  \dfrac{
    \dfrac{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash E^\perp, C, \Gamma_2}
  }{\vdash G, H, C, \Gamma_1, \Gamma_2}\ \mathbf{cut}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\ \nu
        \qquad \dfrac{\pi_4}{\vdash A^\perp, \Gamma_4}
      }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\ \mathbf{cut}
      \qquad \dfrac{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}
    }{\vdash \nu X.F, C^\perp, \Gamma_3, \Gamma_4, \Gamma_5}\ \mathbf{cut}
  }{}
}{
  \dfrac{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \parr
}\ \mathbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{
  \dfrac{
    \dfrac{\pi_1}{\vdash G, H, E, \Gamma_1}
    \qquad
    \dfrac{\pi_2}{\vdash E^{\perp}, C, \Gamma_2}
  }{\vdash G, H, C, \Gamma_1, \Gamma_2}\,\textbf{cut}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash {\color{blue}\nu X.F}, A, D, \Gamma_3}\,{\color{blue}\nu}
        \qquad
        \dfrac{\pi_4}{\vdash A^{\perp}, \Gamma_4}
      }{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\textbf{cut}
      \qquad
      \dfrac{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}
    }{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}
  }{}
}{
  \dfrac{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr
}\,\textbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}
\qquad
\begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}
}{\vdash G, H, C, \Gamma_1, \Gamma_2}\,\mathsf{cut}
\qquad
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}
\qquad
\begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}
}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\mathsf{cut}
}{\vdash \nu X.F, D, \Gamma_3, \Gamma_4}\,\nu
\qquad
\begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}
}{\vdash \nu X.F, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathsf{cut}
}{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathsf{cut}
}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr
$$

# mcut vs. cut-reductions

$$
\dfrac{
\dfrac{
\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array}
\qquad
\begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array}
}{\vdash G, H, C, \Gamma_1, \Gamma_2}\;\mathbf{cut}
\qquad
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}
\qquad
\begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array}
}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\;\mathbf{cut}
\qquad
\begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}
}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\;\mathbf{cut}
}{\vdash \color{blue}{\nu X.F}, C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\;\color{blue}{\nu}
}{
\dfrac{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\parr
}\;\mathbf{cut}
$$

# mcut vs. cut-reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2}}{\vdash G, H, C, \Gamma_1, \Gamma_2}\,\textbf{cut} \qquad \dfrac{\dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4}}{\vdash F[\nu X.F/X], D, \Gamma_3, \Gamma_4}\,\textbf{cut} \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}}{\vdash F[\nu X.F/X], C^{\perp}, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}}{\vdash F[\nu X.F/X], G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{cut}}{\vdash {\color{blue}\nu X.F}, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,{\color{blue}\nu}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr
$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash G, H, E, \Gamma_1\end{matrix}}{\vdash G \parr H, E, \Gamma_1}\,\parr \qquad \begin{matrix}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{matrix} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{matrix}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu \qquad \begin{matrix}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{matrix} \qquad \begin{matrix}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{matrix}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\textbf{mcut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\ \parr \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2} \qquad \dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\ \nu \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4} \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \mathbf{mcut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash E^\perp, C, \Gamma_2\end{array} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu \qquad \begin{array}{c}\pi_4\\ \vdash A^\perp, \Gamma_4\end{array} \qquad \begin{array}{c}\pi_5\\ \vdash D^\perp, C^\perp, \Gamma_5\end{array}}{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\begin{array}{ccccc} \begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array} & \begin{array}{c}\pi_2\\ \vdash E^\perp, C, \Gamma_2\end{array} & \begin{array}{c}\pi_3\\ \dfrac{\vdash F[\nu X.F/X], A, D, \Gamma_3}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu\end{array} & \begin{array}{c}\pi_4\\ \vdash A^\perp, \Gamma_4\end{array} & \begin{array}{c}\pi_5\\ \vdash D^\perp, C^\perp, \Gamma_5\end{array}\end{array}}{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\parr$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1} \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2} \qquad \overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4} \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}}{\vdash F[\nu X.F/X], G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \mathbf{mcut}}{\vdash \nu X.F, G, H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \nu}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\ \parr$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash G \parr H, E, \Gamma_1}\,\parr \qquad \overset{\pi_2}{\vdash E^\perp, C, \Gamma_2} \qquad \dfrac{\overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3}}{\vdash \nu X.F, A, D, \Gamma_3}\,\nu \qquad \overset{\pi_4}{\vdash A^\perp, \Gamma_4} \qquad \overset{\pi_5}{\vdash D^\perp, C^\perp, \Gamma_5}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash G, H, E, \Gamma_1\end{matrix}}{\vdash G \parr H, E, \Gamma_1}\parr \qquad \begin{matrix}\pi_2\\ \vdash E^\perp, C, \Gamma_2\end{matrix} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{matrix}}{\vdash \nu X.F, A, D, \Gamma_3}\nu \qquad \begin{matrix}\pi_4\\ \vdash A^\perp, \Gamma_4\end{matrix} \qquad \begin{matrix}\pi_5\\ \vdash D^\perp, C^\perp, \Gamma_5\end{matrix}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash G, H, E, \Gamma_1\end{matrix}}{\vdash G \parr H, E, \Gamma_1}\,\parr \qquad \begin{matrix}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{matrix} \qquad \begin{matrix}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{matrix} \qquad \begin{matrix}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{matrix} \qquad \begin{matrix}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{matrix}}{\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\nu$$

# mcut vs. cut-reductions

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash G, H, E, \Gamma_1}}{\vdash \color{blue}{G \parr H}\color{black}, E, \Gamma_1}\,\color{blue}{\parr}\color{black} \qquad \overset{\pi_2}{\vdash E^{\perp}, C, \Gamma_2} \qquad \overset{\pi_3}{\vdash F[\nu X.F/X], A, D, \Gamma_3} \qquad \overset{\pi_4}{\vdash A^{\perp}, \Gamma_4} \qquad \overset{\pi_5}{\vdash D^{\perp}, C^{\perp}, \Gamma_5}}{\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\mathbf{mcut}}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\,\nu$$

# mcut vs. cut-reductions

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash G, H, E, \Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash E^{\perp}, C, \Gamma_2\end{array} \qquad \begin{array}{c}\pi_3\\ \vdash F[\nu X.F/X], A, D, \Gamma_3\end{array} \qquad \begin{array}{c}\pi_4\\ \vdash A^{\perp}, \Gamma_4\end{array} \qquad \begin{array}{c}\pi_5\\ \vdash D^{\perp}, C^{\perp}, \Gamma_5\end{array}}{\vdash F[\nu X.F/X], G, `H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\textbf{mcut}}{\vdash F[\nu X.F/X], G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\parr}{\vdash \nu X.F, G \parr H, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5}\;\nu
$$

# Examples of reductions

## Natural numbers

$$\mathrm{Nat} := \mu X.1 \oplus X$$

$$\pi_0 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1 \oplus \mathrm{Nat}}\,\oplus_1}{\vdash \mathrm{Nat}}\,\mu \qquad \pi_{n+1} := \dfrac{\dfrac{\begin{array}{c}\pi_n\\ \vdash \mathrm{Nat}\end{array}}{\vdash 1 \oplus \mathrm{Nat}}\,\oplus_2}{\vdash \mathrm{Nat}}\,\mu$$

$$\pi_{\mathrm{succ}} := \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1}\,\mathrm{ax}}{1 \vdash 1 \oplus \mathrm{Nat}}\,\oplus_1^r}{1 \vdash \mathrm{Nat}}\,\mu_r}{1 \vdash 1 \oplus \mathrm{Nat}}\,\oplus_2^r}{1 \vdash \mathrm{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\mathrm{Nat} \vdash \mathrm{Nat}}{\mathrm{Nat} \vdash 1 \oplus \mathrm{Nat}}\,\oplus_2^r}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_r}{1 \oplus \mathrm{Nat} \vdash \mathrm{Nat}}\,\oplus_l}{\mathrm{Nat} \vdash \mathrm{Nat}}\,\mu_l$$

# Examples of reductions

## Natural numbers

$$\mathsf{Nat} := \mu X.1 \oplus X$$

$$\pi_1 := \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\;1}{\vdash 1 \oplus \mathsf{Nat}}\;\oplus_1}{\vdash \mathsf{Nat}}\;\mu}{\vdash 1 \oplus \mathsf{Nat}}\;\oplus_2}{\vdash \mathsf{Nat}}\;\mu$$

$$\pi_{\mathsf{succ}} := \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1}\;\mathsf{ax}}{1 \vdash 1 \oplus \mathsf{Nat}}\;\oplus_1^r}{1 \vdash \mathsf{Nat}}\;\mu_r}{1 \vdash 1 \oplus \mathsf{Nat}}\;\oplus_2^r}{1 \vdash \mathsf{Nat}}\;\mu_r \qquad \dfrac{\dfrac{\mathsf{Nat} \vdash \mathsf{Nat}}{\mathsf{Nat} \vdash 1 \oplus \mathsf{Nat}}\;\oplus_2^r}{\mathsf{Nat} \vdash \mathsf{Nat}}\;\mu_r}{1 \oplus \mathsf{Nat} \vdash \mathsf{Nat}}\;\oplus_l}{\mathsf{Nat} \vdash \mathsf{Nat}}\;\mu_l$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_1}{\vdash \mathbf{Nat}}\,\mu}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_2}{\vdash \mathbf{Nat}}\,\mu \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_1}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_2}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\begin{array}{c}\pi_{\mathbf{succ}}\\ \mathbf{Nat} \vdash \mathbf{Nat}\end{array}}{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_2}{\mathbf{Nat} \vdash \mathbf{Nat}}\,\mu_r}{\mathbf{1} \oplus \mathbf{Nat} \vdash \mathbf{Nat}}\,\oplus_l}{\mathbf{Nat} \vdash \mathbf{Nat}}\,\mu_l}{\vdash \mathbf{Nat}}\,\mathbf{cut}$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\;1}{\vdash 1 \oplus \mathrm{Nat}}\;\oplus_1}{\vdash \mathrm{Nat}}\;\mu}{\vdash 1 \oplus \mathrm{Nat}}\;\oplus_2}{\vdash \mathrm{Nat}}\;\mu \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1}\;\mathrm{ax}}{1 \vdash 1 \oplus \mathrm{Nat}}\;\oplus^r_1}{1 \vdash \mathrm{Nat}}\;\mu_r}{1 \vdash 1 \oplus \mathrm{Nat}}\;\oplus^r_2}{1 \vdash \mathrm{Nat}}\;\mu_r \qquad \dfrac{\dfrac{\begin{array}{c}\pi_{\mathrm{succ}} \\ \mathrm{Nat} \vdash \mathrm{Nat}\end{array}}{\mathrm{Nat} \vdash 1 \oplus \mathrm{Nat}}\;\oplus^r_2}{\mathrm{Nat} \vdash \mathrm{Nat}}\;\mu_r}{1 \oplus \mathrm{Nat} \vdash \mathrm{Nat}}\;\oplus_l}{\mathrm{Nat} \vdash \mathrm{Nat}}\;\mu_l}{\vdash \mathrm{Nat}}\;\mathrm{cut}}{\vdash \mathrm{Nat}}\;\mathrm{mcut}$$

# Examples of reductions

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathbf{1}$}
\UnaryInfC{$\vdash \mathbf{1}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash \mathbf{Nat}$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash \mathbf{Nat}$}
\AxiomC{}
\RightLabel{$\mathbf{ax}$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1}$}
\RightLabel{$\oplus^r_1$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus^r_2$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{Nat}$}
\AxiomC{$\pi_{\mathbf{succ}}$}
\noLine
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus^r_2$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus_l$}
\BinaryInfC{$\mathbf{1} \oplus \mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\mu_l$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\mathbf{cut}$}
\BinaryInfC{$\vdash \mathbf{Nat}$}
\RightLabel{$\mathbf{mcut}$}
\UnaryInfC{$\vdash \mathbf{Nat}$}
\end{prooftree}
$$

# Examples of reductions

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\;\mu}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus_{\mathbf{2}}}{\vdash {\color{blue}\mathbf{Nat}}}\;{\color{blue}\mu}
\qquad
\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1}\vdash \mathbf{1}}\;\mathbf{ax}}{\mathbf{1}\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^{r}_{\mathbf{1}}}{\mathbf{1}\vdash\mathbf{Nat}}\;\mu_r}{\mathbf{1}\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^{r}_{\mathbf{2}}}{\mathbf{1}\vdash\mathbf{Nat}}\;\mu_r
\qquad
\dfrac{\dfrac{\dfrac{\pi_{\mathbf{succ}}}{\mathbf{Nat}\vdash\mathbf{Nat}}}{\mathbf{Nat}\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^{r}_{\mathbf{2}}}{\mathbf{Nat}\vdash\mathbf{Nat}}\;\mu_r
}{\mathbf{1}\oplus\mathbf{Nat}\vdash\mathbf{Nat}}\;\oplus_l}{{\color{blue}\mathbf{Nat}}\vdash\mathbf{Nat}}\;{\color{blue}\mu_l}
}{\vdash \mathbf{Nat}}\;\mathbf{mcut}
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\oplus_{\mathbf{2}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1}\vdash\mathbf{1}}\,\mathbf{ax}}{\mathbf{1}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{1}}}{\mathbf{1}\vdash\mathbf{Nat}}\,\mu_r}{\mathbf{1}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{1}\vdash\mathbf{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\begin{array}{c}\pi_{\mathbf{succ}}\\ \mathbf{Nat}\vdash\mathbf{Nat}\end{array}}{\mathbf{Nat}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{Nat}\vdash\mathbf{Nat}}\,\mu_r}{\mathbf{1}\oplus\mathbf{Nat}\vdash\mathbf{Nat}}\,\oplus_l}{\vdash\mathbf{Nat}}\,\mathbf{mcut}$$

# Examples of reductions

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus_1}{\vdash \mathbf{Nat}}\;\mu}{{\color{blue}\vdash \mathbf{1} \oplus \mathbf{Nat}}}\;{\color{blue}\oplus_2}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\;\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^r_1}{\mathbf{1} \vdash \mathbf{Nat}}\;\mu_r}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^r_2}{\mathbf{1} \vdash \mathbf{Nat}}\;\mu_r
\qquad
{\color{blue}\dfrac{\dfrac{\begin{matrix}\pi_{\mathbf{succ}}\\ \mathbf{Nat} \vdash \mathbf{Nat}\end{matrix}}{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^r_2}{\mathbf{Nat} \vdash \mathbf{Nat}}\;\mu_r}
}{{\color{blue}\mathbf{1} \oplus \mathbf{Nat}} \vdash \mathbf{Nat}}\;{\color{blue}\oplus_l}
}{\vdash \mathbf{Nat}}\;\mathbf{mcut}
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\;\mu \qquad \dfrac{\dfrac{\begin{array}{c}\pi_{\mathbf{succ}}\\ \mathbf{Nat}\vdash\mathbf{Nat}\end{array}}{\mathbf{Nat}\vdash\mathbf{1}\oplus\mathbf{Nat}}\;\oplus_{\mathbf{2}}^{r}}{\mathbf{Nat}\vdash\mathbf{Nat}}\;\mu_r}{\vdash\mathbf{Nat}}\;\mathbf{mcut}$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu \qquad \dfrac{\dfrac{\begin{matrix}\pi_{\mathbf{succ}}\\ \mathbf{Nat} \vdash \mathbf{Nat}\end{matrix}}{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{Nat} \vdash \color{blue}{\mathbf{Nat}}}\,\color{blue}{\mu_r}}{\vdash \mathbf{Nat}}\,\mathbf{mcut}$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu \qquad \dfrac{\overset{\pi_{\mathbf{succ}}}{\mathbf{Nat} \vdash \mathbf{Nat}}}{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu \qquad \dfrac{\overset{\pi_{\mathbf{succ}}}{\mathbf{Nat} \vdash \mathbf{Nat}}}{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}^{r}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus_{\mathbf{1}}}{\vdash \mathbf{Nat}}\;\mu \qquad \begin{array}{c}\pi_{\mathbf{succ}}\\ \mathbf{Nat}\vdash\mathbf{Nat}\end{array}}{\vdash \mathbf{Nat}}\;\mathbf{mcut}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^{r}_{\mathbf{2}}}{\vdash \mathbf{Nat}}\;\mu_r$$

# Examples of reductions

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathbf{1}$}
\UnaryInfC{$\vdash \mathbf{1}$}
\RightLabel{$\oplus_{\mathbf{1}}$}
\UnaryInfC{$\vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash \mathbf{Nat}$}
\AxiomC{}
\RightLabel{$\mathbf{ax}$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1}$}
\RightLabel{$\oplus^r_{\mathbf{1}}$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus^r_{\mathbf{2}}$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{1} \vdash \mathbf{Nat}$}
\AxiomC{$\pi_{\mathbf{succ}}$}
\noLine
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus^r_{\mathbf{2}}$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\oplus_l$}
\BinaryInfC{$\mathbf{1} \oplus \mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\mu_l$}
\UnaryInfC{$\mathbf{Nat} \vdash \mathbf{Nat}$}
\RightLabel{$\mathbf{mcut}$}
\BinaryInfC{$\vdash \mathbf{Nat}$}
\RightLabel{$\oplus^r_{\mathbf{2}}$}
\UnaryInfC{$\vdash \mathbf{1} \oplus \mathbf{Nat}$}
\RightLabel{$\mu_r$}
\UnaryInfC{$\vdash \mathbf{Nat}$}
\end{prooftree}
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1}\oplus \mathsf{Nat}}\,\oplus_{\mathbf{1}}}{\vdash {\color{blue}\mathsf{Nat}}}\,{\color{blue}\mu} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1}\vdash \mathbf{1}}\,\mathsf{ax}}{\mathbf{1}\vdash \mathbf{1}\oplus \mathsf{Nat}}\,\oplus^{r}_{\mathbf{1}}}{\mathbf{1}\vdash \mathsf{Nat}}\,\mu_r}{\mathbf{1}\vdash \mathbf{1}\oplus \mathsf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{1}\vdash \mathsf{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\overset{\pi_{\mathsf{succ}}}{\mathsf{Nat}\vdash \mathsf{Nat}}}{\mathsf{Nat}\vdash \mathbf{1}\oplus \mathsf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathsf{Nat}\vdash \mathsf{Nat}}\,\mu_r}{\mathbf{1}\oplus \mathsf{Nat}\vdash \mathsf{Nat}}\,\oplus_l}{{\color{blue}\mathsf{Nat}}\vdash \mathsf{Nat}}\,{\color{blue}\mu_l}}{\vdash \mathsf{Nat}}\,\mathsf{mcut}}{\vdash \mathbf{1}\oplus \mathsf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\vdash \mathsf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\oplus_{\mathbf{1}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1}\vdash\mathbf{1}}\,\mathbf{ax}}{\mathbf{1}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{1}}}{\mathbf{1}\vdash\mathbf{Nat}}\,\mu_{r}}{\mathbf{1}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{1}\vdash\mathbf{Nat}}\,\mu_{r} \qquad \dfrac{\dfrac{\overset{\pi_{\mathbf{succ}}}{\mathbf{Nat}\vdash\mathbf{Nat}}}{\mathbf{Nat}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\mathbf{Nat}\vdash\mathbf{Nat}}\,\mu_{r}}{\mathbf{1}\oplus\mathbf{Nat}\vdash\mathbf{Nat}}\,\oplus_{l}}{\vdash\mathbf{Nat}}\,\mathbf{mcut}}{\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\vdash\mathbf{Nat}}\,\mu_{r}$$

# Examples of reductions

$$
\dfrac{
\dfrac{
  \dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}
}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}
\qquad
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}
        }{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{1}}
      }{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r
    }{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}
  }{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r
  \qquad
  \dfrac{
    \dfrac{
      \begin{matrix}\pi_{\mathbf{succ}}\\ \mathbf{Nat} \vdash \mathbf{Nat}\end{matrix}
    }{\mathbf{Nat} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}
  }{\mathbf{Nat} \vdash \mathbf{Nat}}\,\mu_r
}{\mathbf{1} \oplus \mathbf{Nat} \vdash \mathbf{Nat}}\,\oplus_{l}
}{
\dfrac{
  \dfrac{\vdash \mathbf{Nat}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}
}{\vdash \mathbf{Nat}}\,\mu_r
}\,\mathbf{mcut}
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{1}}^{r}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_{r}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}^{r}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_{r}}{\vdash \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}^{r}}{\vdash \mathbf{Nat}}\,\mu_{r}$$

# Examples of reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{1}}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_1}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_2}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\mathbf{mcut}}{\vdash {\color{blue}\mathbf{Nat}}}\,{\color{blue}\mu_r}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_2}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{1}}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{1}}}{\mathbf{1} \vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{\mathbf{2}}_r}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1 \qquad \dfrac{\dfrac{\dfrac{}{1 \vdash 1}\,\mathrm{ax}}{1 \vdash 1 \oplus \mathrm{Nat}}\,\oplus^r_1}{1 \vdash \mathrm{Nat}}\,\mu_r}{\vdash \mathrm{Nat}}\,\mathrm{mcut}}{\vdash 1 \oplus \mathrm{Nat}}\,\oplus^2_r}{\vdash \mathrm{Nat}}\,\mu_r}{\vdash 1 \oplus \mathrm{Nat}}\,\oplus^r_2}{\vdash \mathrm{Nat}}\,\mu_r
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1} \qquad \dfrac{\dfrac{\dfrac{}{\mathbf{1}\vdash\mathbf{1}}\;\mathbf{ax}}{\mathbf{1}\vdash \mathbf{1}\oplus \mathbf{Nat}}\;\oplus^r_1}{\mathbf{1}\vdash {\color{blue}\mathbf{Nat}}}\;{\color{blue}\mu_r}}{\vdash \mathbf{Nat}}\;\mathbf{mcut}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^2_r}{\vdash \mathbf{Nat}}\;\mu_r}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\;\oplus^r_2}{\vdash \mathbf{Nat}}\;\mu_r$$

# Examples of reductions

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{}{\mathbf{1}\vdash\mathbf{1}}\,\mathbf{ax}}{\mathbf{1}\vdash\mathbf{1}\oplus\mathbf{Nat}}\,\oplus^r_{\mathbf{1}}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\mathbf{mcut}}{\vdash {\color{blue}\mathbf{Nat}}}\,{\color{blue}\mu_r}}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\oplus^{\mathbf{2}}_r}{\vdash\mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1}\oplus\mathbf{Nat}}\,\oplus^r_{\mathbf{2}}}{\vdash\mathbf{Nat}}\,\mu_r
$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{1}}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\mathbf{mcut}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{\mathbf{2}}_{r}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\;\mathbf{1} \qquad \dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}\;\mathbf{ax}}{\mathbf{1} \vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^{r}_{\mathbf{1}}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\mathbf{mcut}}{\vdash \mathbf{Nat}}\;\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^{\mathbf{2}}_{r}}{\vdash \mathbf{Nat}}\;\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus^{r}_{\mathbf{2}}}{\vdash \mathbf{Nat}}\;\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\vdash \mathbf{1}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}^{r}}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\vdash \mathbf{1}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^1}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^2}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_2^r}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{}{\mathbf{1} \vdash \mathbf{1}}\,\mathbf{ax}}{\vdash \mathbf{1}}\,\mathbf{mcut}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^{\mathbf{1}}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_r^{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus_{\mathbf{2}}^r}{\vdash \mathbf{Nat}}\,\mu_r$$

# Examples of reductions

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash \mathbf{1}}\;\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus_r^{\mathbf{1}}}{\vdash \mathbf{Nat}}\;\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus_r^{\mathbf{2}}}{\vdash \mathbf{Nat}}\;\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\;\oplus_{\mathbf{2}}^{r}}{\vdash \mathbf{Nat}}\;\mu_r$$

# Examples of reductions

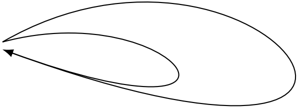
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{\mathbf{1}}_{r}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{\mathbf{2}}_{r}}{\vdash \mathbf{Nat}}\,\mu_r}{\vdash \mathbf{1} \oplus \mathbf{Nat}}\,\oplus^{r}_{\mathbf{2}}}{\vdash \mathbf{Nat}}\,\mu_r$$

$$= \pi_2$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$\pi := \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash F, F^{\perp}}\text{ax} \qquad \cfrac{}{\vdash F, F^{\perp}}\text{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\otimes}{\vdash F, F^{\perp}, F^{\perp}}\nu}{\vdash F, F^{\perp} \parr F^{\perp}}\parr}{\vdash F, F^{\perp}}\mu \qquad \cfrac{\cfrac{\vdash F \qquad \vdash F}{\vdash F \otimes F}\otimes}{\vdash F}\nu}{\vdash F}\text{cut}$$



# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr}{\vdash F, F^\perp}\,\mu
\qquad
\dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu
}{\dfrac{\vdash F}{\vdash F}\,\text{mcut}}\,\text{cut}
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\vdash F, F^{\perp}}\,\text{ax}
\qquad
\dfrac{}{\vdash F, F^{\perp}}\,\text{ax}
}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes
}{\vdash F, F^{\perp}, F^{\perp}}\,\nu
}{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr
}{\vdash F, F^{\perp}}\,\mu
\qquad\qquad
\dfrac{
\dfrac{
\overset{\pi}{\vdash F}
\qquad
\overset{\pi}{\vdash F}
}{\vdash F \otimes F}\,\otimes
}{\vdash F}\,\nu
}{\vdash F}\,\text{mcut}
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\text{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes}{\vdash F, F^{\perp}, F^{\perp}}\,\nu}{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr}{\vdash F, F^{\perp}}\,\mu \qquad \dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu}{\vdash F}\,\text{mcut}$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr \qquad \dfrac{\begin{matrix}\pi \\ \vdash F\end{matrix} \qquad \begin{matrix}\pi \\ \vdash F\end{matrix}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\text{mcut}$$

# Multi-cut in action

Taking $F := \nu X. X \otimes X$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
        \qquad
        \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
      }{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes
    }{\vdash F, F^{\perp}, F^{\perp}}\,\nu
  }{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr
  \qquad
  \dfrac{
    \dfrac{\pi}{\vdash F}
    \qquad
    \dfrac{\pi}{\vdash F}
  }{\vdash F \otimes F}\,\otimes
}{\vdash F}\,\mathsf{mcut}
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\text{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes}{\vdash F, F^{\perp}, F^{\perp}}\,\nu \qquad \begin{array}{c}\pi\\ \vdash F\end{array} \qquad \begin{array}{c}\pi\\ \vdash F\end{array}}{\vdash F}\,\text{mcut}$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\otimes}{\vdash F, F^\perp, F^\perp}\nu \qquad \overset{\pi}{\vdash F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\otimes}{\vdash F, F^\perp, F^\perp}\nu}{\vdash F, F^\perp \parr F^\perp}\parr}{\vdash F, F^\perp}\mu \qquad \dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\otimes}{\vdash F}\nu}{\vdash F}\text{cut}}{\vdash F}\text{mcut}$$

# Multi-cut in action

Taking $F := \nu X. X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu \qquad \overset{\pi}{\vdash F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\text{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\text{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr}{\vdash F, F^\perp}\,\mu \qquad \dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu}{\vdash F}\,\textcolor{red}{\text{cut}}}{\vdash F}\,\text{mcut}$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes}{\vdash F, F^{\perp}, F^{\perp}}\,\nu \qquad \overset{\pi}{\vdash F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes}{\vdash F, F^{\perp}, F^{\perp}}\,\nu}{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr}{\vdash F, F^{\perp}}\,\mu \qquad \dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu}{\vdash F}\,\mathsf{mcut}$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^\perp, F^\perp$}
\RightLabel{$\textcolor{red}{\nu}$}
\UnaryInfC{$\vdash F, F^\perp, F^\perp$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^\perp, F^\perp$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F, F^\perp, F^\perp$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash F, F^\perp \parr F^\perp$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash F, F^\perp$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\QuaternaryInfC{$\vdash F$}
\end{prooftree}
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^{\perp}, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^{\perp}, F^{\perp}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F, F^{\perp}, F^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash F, F^{\perp} \parr F^{\perp}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\QuaternaryInfC{$\vdash F \otimes F$}
\RightLabel{$\color{red}{\nu}$}
\UnaryInfC{$\color{red}{\vdash F}$}
\end{prooftree}
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\text{ax}}{\color{red}{\vdash F \otimes F, F^{\perp}, F^{\perp}}}\,{\color{red}{\otimes}} \qquad \overset{\pi}{\vdash F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,\text{ax} \qquad \dfrac{}{\vdash F, F^{\perp}}\,\text{ax}}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes}{\vdash F, F^{\perp}, F^{\perp}}\,\nu}{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr}{\vdash F, F^{\perp}}\,\mu \qquad \dfrac{\dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu}{\vdash F \otimes F}\,\text{mcut}}{\vdash F}\,\nu
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\BinaryInfC{$\vdash F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^{\perp}, F^{\perp}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F, F^{\perp}, F^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash F, F^{\perp} \parr F^{\perp}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\TrinaryInfC{$\vdash F$}
\RightLabel{${\color{red}\otimes}$}
\BinaryInfC{${\color{red}\vdash F \otimes F}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\end{prooftree}
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\BinaryInfC{$\vdash F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^{\perp}, F^{\perp}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F, F^{\perp}, F^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash F, F^{\perp} \parr F^{\perp}$}
\RightLabel{$\mu$}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\TrinaryInfC{$\vdash F$}
\RightLabel{$\color{red}{\otimes}$}
\BinaryInfC{$\color{red}{\vdash F \otimes F}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\end{prooftree}
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$
\dfrac{
\dfrac{
\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
\qquad
\begin{matrix}\pi \\ \vdash F\end{matrix}
}{\vdash F}\,\mathsf{mcut}
\qquad
\dfrac{
\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
\qquad
\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
}{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes
}{\vdash F, F^{\perp}, F^{\perp}}\,\nu
}{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr
}{\vdash F, {\color{red}F^{\perp}}}\,{\color{red}\mu}
\qquad
\dfrac{
\dfrac{
\begin{matrix}\pi \\ \vdash F\end{matrix}
\qquad
\begin{matrix}\pi \\ \vdash F\end{matrix}
}{\vdash F \otimes F}\,\otimes
}{\vdash {\color{red}F}}\,{\color{red}\nu}
}{\vdash F}\,\mathsf{mcut}
}{
\dfrac{\vdash F \otimes F}{\vdash F}\,\nu
}\,\otimes
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \qquad \begin{matrix}\pi\\ \vdash F\end{matrix}}{\vdash F}\,\mathsf{mcut} \qquad \dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \quad \dfrac{}{\vdash F, F^\perp}\,\mathsf{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr \quad \dfrac{\begin{matrix}\pi\\ \vdash F\end{matrix} \quad \begin{matrix}\pi\\ \vdash F\end{matrix}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mathsf{mcut}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\dfrac{
  \dfrac{
    \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
    \qquad
    \dfrac{\pi}{\vdash F}
  }{\vdash F}\,\mathsf{mcut}
  \qquad
  \dfrac{
    {\color{red}\dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
          \qquad
          \dfrac{}{\vdash F, F^{\perp}}\,\mathsf{ax}
        }{\vdash F \otimes F, F^{\perp}, F^{\perp}}\,\otimes
      }{\vdash F, F^{\perp}, F^{\perp}}\,\nu
    }{\vdash F, F^{\perp} \parr F^{\perp}}\,\parr
    \qquad
    \dfrac{
      \dfrac{\pi}{\vdash F}
      \qquad
      \dfrac{\pi}{\vdash F}
    }{\vdash F \otimes F}\,\otimes
  }{\vdash F}\,\mathsf{mcut}
}{
  \dfrac{\vdash F \otimes F}{\vdash F}\,\nu
}\,\otimes
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$
\dfrac{\dfrac{
 \dfrac{ \dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \qquad \overset{\pi}{\vdash F} }{\vdash F}\,\mathsf{mcut}
 \qquad
 \dfrac{ \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\mathsf{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr \qquad \dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mathsf{mcut}
}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu
$$

# Multi-cut in action

Taking $F := \nu X . X \otimes X$

$$
\dfrac{\dfrac{
 \dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \qquad \overset{\pi}{\vdash F}}{\vdash F}\,\textcolor{blue}{\mathsf{mcut}}
 \qquad
 \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash F, F^\perp}\,\mathsf{ax} \qquad \dfrac{}{\vdash F, F^\perp}\,\mathsf{ax}}{\vdash F \otimes F, F^\perp, F^\perp}\,\otimes}{\vdash F, F^\perp, F^\perp}\,\nu}{\vdash F, F^\perp \parr F^\perp}\,\parr
 \qquad
 \dfrac{\overset{\pi}{\vdash F} \qquad \overset{\pi}{\vdash F}}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\mathsf{mcut}
}{\vdash F \otimes F}\,\otimes}{\vdash F}\,\nu
$$

# Multi-cut in action

Taking $F := \nu X.X \otimes X$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{mcut}
\BinaryInfC{$\vdash F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F, F^{\perp}, F^{\perp}$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F, F^{\perp}, F^{\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash F, F^{\perp} \parr F^{\perp}$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\AxiomC{$\pi$}
\noLine
\UnaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{mcut}
\BinaryInfC{$\vdash F$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash F \otimes F$}
\RightLabel{$\nu$}
\UnaryInfC{$\vdash F$}
\end{prooftree}
$$

### Fairness

An (mcut)-reduction sequence is fair, if each reduction that can be made is made at some point.

# Cut-elimination theorem

**Cut-elimination (Baelde et al. 2016)**

Each fair reduction sequence converges to a cut-free valid proof.