Provability of Functionnal Reactive Programming type system

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Functionnal Reactive programming

History

Eliott and Hudak introduced it in 1997 with Functionnal Reactive animation
Functionnal Reactive programming

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Program paradigm concerned by propagating a reactive input (such as stream) to ensure properties or modify values over time.
Elliott and Hudak introduced it in 1997 with Functionnal Reactive animation. Program paradigm concerned by propagating a reactive input (such as stream) to ensure properties or modify values over time. Exemple: Spreadsheet, graphical interface, web app.
Linear Temporal Logic

Connectives:

- $\circ A$
- $\Diamond A$
- $\square A$
- $A U B$
Linear Temporal Logic

Connectives:

- $\bigcirc A$
- $\lozenge A$
- $\Box A$
- $A \mathbin{\lor} B$

Pnueli used temporal logic to reason on reactive programs in 1977.
Linear Temporal Logic

Connectives:

\(\circ A\)  \(\lozenge A\)  \(\square A\)  \(A \cup B\)

Pnueli used temporal logic to reason on reactive programs in 1977

Yuse and Igarashi use temporal logic to encode multi-level generating code extensions with persistent code
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Functionnal Reactive Programming typed by a Linear Temporal Logic system

We will name this system FRP (for Fair Reactive Programming)
Functionnal Reactive Programming typed by a Linear Temporal Logic system

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Types of FRP

\[
\mathcal{F} ::= \text{Var} \mid \mathcal{F} \lor \mathcal{F} \mid \mathcal{F} \land \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F} \mid \bigcirc \mathcal{F} \mid \mu X. \mathcal{F} \mid \nu X. \mathcal{F} \mid 1
\]
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Types of FRP

\[ \mathcal{F} ::= \text{Var} \mid \mathcal{F} \lor \mathcal{F} \mid \mathcal{F} \land \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F} \mid \bigcirc \mathcal{F} \mid \mu X. \mathcal{F} \mid \nu X. \mathcal{F} \mid 1 \]

\[ \square A ::= \nu X. A \land \bigcirc X \]

\[ \Diamond A ::= \mu X. A \lor \bigcirc X \]

\[ A \cup B ::= \mu X. (B \lor (A \land \bigcirc X)) \]
Temporal terms and derivation rules

\[ \bullet t : \bigcirc A \quad \text{let} \quad \bullet x = t \ \text{in} \quad t_2 : C \]
Temporal terms and derivation rules

\[ \bullet t : \bigcirc A \quad \text{let} \quad \bullet x = t \text{ in } t_2 : C \]

Sequents

\[ \Theta; \Gamma \vdash A \]
Temporal terms and derivation rules

\[ \begin{align*}
  \bullet t & : \bigcirc A \\
  \text{let } \bullet x = t & \text{ in } t_2 : C
\end{align*} \]

Sequents

\[ \Theta; \Gamma \vdash A \]

Typing rules

\[ \frac{\Theta; \Gamma \vdash t : A}{\Theta; \Gamma \vdash \bullet t : \bigcirc A} \quad (\bigcirc i) \]

\[ \frac{\Theta; \Gamma \vdash t_1 : \bigcirc A}{\Theta; \Gamma \vdash \text{let } \bullet x = t_1 \text{ in } t_2 : B} \quad (\bigcirc e) \]
Causality

Causality

\[ f(s_1, \ldots, s_n, s_{n+1}, \ldots) = f(s_1, \ldots, s_n, s'_{n+1}, \ldots) \] at time \( n \)
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\[ f(s_1, \ldots, s_n, s_{n+1}, \ldots) = f(s_1, \ldots, s_n, s'_{n+1}, \ldots) \] at time \( n \)

\[ \circ A \rightarrow A \]

should no be provable:

\[ \text{predictor1}(x : \circ A) := \text{let } \bullet x' = x \text{ in } x' \]
Causality

\[ f(s_1, \ldots, s_n, s_{n+1}, \ldots) = f(s_1, \ldots, s_n, s'_{n+1}, \ldots) \text{ at time } n \]

\[ \bigcirc A \rightarrow A \]

should no be provable:

\[
\text{predictor1}(x : \bigcirc A) := \text{let } \bullet x' = x \text{ in } x'
\]

\[ \bigcirc (A \lor B) \rightarrow \bigcirc A \lor \bigcirc B \]

should not be provable:

\[
\text{predictor2 } (x : \bigcirc (A \lor B)) = \text{let } \bullet x' = x \text{ in case } x' \text{ of }
\begin{align*}
| \text{inl } a & \rightarrow \text{inl } (\bullet a) \\
| \text{inr } b & \rightarrow \text{inr } (\bullet b)
\end{align*}
\]
Other rejected formula

$A \rightarrow \odot A$

would not break causality, but we refute it anyway. (accepted in Krishnaswami and Benton 2011 but rejected in Krishnaswami 2013 paper for managing space)

import $(x : A) = \bullet x$
Some example of implementation

Type of Temporal Streams on $A : \square A$ (namely $\nu X. A \land \Diamond X$)
Some example of implementation

Type of Temporal Streams on \( A : \square A \) (namely \( \nu X. A \land \bigcirc X \))

\[
\text{coit app } f \ a : \square (A \to B) \to \square A \to \square B := \\
\text{let } ha, hf = \text{hd } a, \text{hd } f \text{ in} \\
\text{let } \bullet ta, \bullet tf = \text{tl } a, \text{tl } f \text{ in } (hf \ ha, \bullet \text{app } tf \ ta)
\]
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Linear Temporal Logic from Kojima and Igarashi (LJ\(\bigcirc\)) - 2011

No \(\lozenge\), \(\Box\) and fixpoint, only \(\bigcirc\) to deal with time
No ◇, □ and fixpoint, only ◯ to deal with time

Sequents are made of formulas indexed with a natural number
Sequents are made of formulas indexed with a natural number

\[
\frac{\Gamma^\vec{m}, A^{n+1} \vdash B^m}{\Gamma^\vec{m}, (\Diamond A)^n \vdash B^m} \quad \Diamond I
\]

\[
\frac{\Gamma^\vec{m} \vdash A^{n+1}}{\Gamma^\vec{m} \vdash (\Diamond A)^n} \quad \Diamond R
\]
Thanks to the side condition, 

\[ \#(A \lor B) \rightarrow (\#A \lor \#B) \]

is not provable in the system.
Causality

\[ \Gamma \vec{m}, A^n \vdash C^m \quad \Gamma \vec{m}, B^n \vdash C^m \quad n \leq m \]
\[ \Gamma \vec{m}, (A \lor B)^n \vdash C^m \quad \lor_I \]

Thanks to the side condition, \((\bigcirc (A \lor B)) \rightarrow (\bigcirc A \lor \bigcirc B)\) is not provable in the system.
Same sequents than LJ°.
Same sequents than $LJ^\circ$.

Additional side conditions:

\[
\begin{align*}
\Gamma^\vec{m}, A^n & \vdash C^m & \Gamma^\vec{m}, B^n & \vdash C^m & n \leq m, \vec{m} & \quad \vee_l \\
\Gamma^\vec{m}, (A \lor B)^n & \vdash C^m
\end{align*}
\]

\[
\begin{align*}
\Gamma^\vec{m}, A^n & \vdash B^n & n \leq \vec{m} & \quad \rightarrow_r \\
\Gamma^\vec{m} & \vdash (A \rightarrow B)^n
\end{align*}
\]
Same sequents than LJ°.

Additional side conditions:

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$\mu$ and $\nu$-free FRP is provably equivalent to FRP°.
Same sequents than $\text{LJ}^\bigcirc$.

Additional side conditions:

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\begin{align*}
\Gamma \vec{m}, A^n &\vdash C^m & 
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\end{align*}
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\Gamma \vec{m}, A^n &\vdash B^n & n \leq \vec{m} & \rightarrow_r \\
\Gamma \vec{m} &\vdash (A \rightarrow B)^n
\end{align*}
\]

$\mu$ and $\nu$-free FRP is provably equivalent to $\text{FRP}^\bigcirc$.

$\text{FRP} \not\vdash (\bigcirc A \rightarrow \bigcirc B) \rightarrow \bigcirc (A \rightarrow B)$
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Kripke models

**IM-frame**

Quadruplet \((W, \leq, R, \models)\) such that:

1. \(W\) is a non-empty set,
2. \(\leq\) is a partial order on \(W\),
3. \(R\) is a binary relation on \(W\),
4. \(\models\) is a relation between elements of \(W\) and propositional variables.

Interpretation of formulas

\(w \models A \rightarrow B \iff (\forall w' \geq w, (w' \models A) \Rightarrow (w' \models B))\)

\(w \models \#A \iff (\forall v, w R v \Rightarrow v \models A)\)
Kripke models

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**Interpretation of formulas**

\[
\begin{align*}
w \models A \rightarrow B & \iff (\forall w' \geq w, (w' \models A) \Rightarrow (w' \models B)) \\
w \models \bigcirc A & \iff (\forall v, w R v \Rightarrow v \models A)
\end{align*}
\]
Correctness and Completeness

Two axioms:

axiom 1:
\[(\leq, R, \leq) = R\]

axiom 2:
\[\forall w, v, w R v \rightarrow (\exists w', w \leq w' \text{ and } \forall u, (w' R u) \iff (v \leq u))\]
Correctness and Completeness

Two axioms:

axiom 1:

$$(\leq, R, \leq) = R$$

axiom 2:

$$\forall w, v, w R v \rightarrow (\exists w', w \leq w' \text{ and } \forall u, (w' R u) \iff (v \leq u))$$

Correctness and Completeness for LJ$\Box$ (Kojima and Igarashi - 2011)

IM-frame together with axiom 1 and 2 are correct and complete relatively to LJ$\Box$. 
Correctness and Completeness

Two axioms:

axiom 1:

\[ (\leq, R, \leq) = R \]

axiom 2:

\[ \forall w, v, w R v \rightarrow (\exists w', w \leq w' \text{ and } \forall u, (w' R u) \iff (v \leq u)) \]

Correctness and Completeness for LJ\(\Box\) (Kojima and Igarashi - 2011)

IM-frame together with axiom 1 and 2 are correct and complete relatively to LJ\(\Box\).

Correctness and Completeness for FRP\(\Box\)

IM-frame satisfying axiom 1 are correct and complete relatively to FRP\(\Box\).
Kripke models

\[ R = \text{Id} \]
Kripke models

R = Id

Intuitionistic logic
Kripke models

Intuitionistic logic

$R = \text{Id}$

$\leq = \text{Id}$
Kripke models

\[ R = \text{Id} \]

\[ \leq = \text{Id} \]

Intuitionistic logic

Modal Logic K
Kripke models

Intuitionistic logic

Modal Logic K

$R = \text{Id}$

$\leq = \text{Id}$
Kripke models

\[ R = Id \]

Intuitionistic logic

\[ \leq = Id \]

Modal Logic K

Classical Logic

Intuitionistic logic

Classical Logic

Modal Logic K
Kripke models

Intuitionistic logic

Modal Logic K

Classical Logic

\[ R = \text{Id} \quad \leq \text{Id} \]

axiom 1
Kripke models

Intuitionistic logic

Classical Logic

Modal Logic K

FRP

R = Id

≤ = Id

axiom 1

Intuitionistic logic
Kripke models

Intuitionistic logic

Modal Logic K

Classical Logic

axiom 1

axiom 2

R = Id

<= Id

FRP
Kripke models

- Intuitionistic logic
  - axiom 1
  - FRP
  - $\leq = \text{Id}$
- Modal Logic K
  - axiom 2
  - LJ
  - Classical Logic
  - $R = \text{Id}$
Kripke models

- **Intuitionistic logic**
  - R = Id
  - ≤ = Id

- **Modal Logic K**
  - Classical Logic

- **axiom 1**
  - LJ

- **axiom 2**
  - FRP
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Future works

Extending our results to full FRP (with fixpoints)
Future works

Extending our results to full FRP (with fixpoints)

Consider a classical setting for FRP ($\lambda\mu - calculus$ from Parigot)