Cut-elimination for modal μ-calculus: exploring the expressivity of μMALL

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Formulas and derivation rules of $\mu\mathsf{LK}^\infty_\Box$

$$\phi, \psi ::= \phi \vee \psi \mid \phi \wedge \psi \mid \phi \to \psi \mid T \mid F \mid X \in \mathcal{V} \mid \Box\phi \mid \Diamond\phi \mid \mu X.\phi \mid \nu X.\phi.$$

Derivation rules are those of LK together with two rules for the modalities:

$$\frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}\,\Diamond \qquad \frac{\Gamma \vdash A, \Delta}{\Box\Gamma \vdash \Box A, \Diamond\Delta}\,\Box$$

and four rules for fixpoints:

$$\frac{\Gamma, A[X := \mu X.A] \vdash \Delta}{\Gamma, \mu X.A \vdash \Delta}\,\mu_l \qquad \frac{\Gamma \vdash A[X := \mu X.A], \Delta}{\Gamma \vdash \mu X.A, \Delta}\,\mu_r$$

$$\frac{\Gamma, A[X := \nu X.A] \vdash \Delta}{\Gamma, \nu X.A \vdash \Delta}\,\nu_l \qquad \frac{\Gamma \vdash A[X := \nu X.A], \Delta}{\Gamma \vdash \nu X.A, \Delta}\,\nu_r$$

Some example of infinite proofs

$$\mathsf{Nat} := \mu X.\top \vee X$$

Inhabitant of natural number type

$$\pi_0 := \dfrac{\dfrac{\dfrac{}{\vdash \top}\,\top}{\vdash \top \vee \mathsf{Nat}}\,\vee_1}{\vdash \mathsf{Nat}}\,\mu \qquad \pi_{n+1} := \dfrac{\dfrac{\overset{\pi_n}{\vdash \mathsf{Nat}}}{\vdash \top \vee \mathsf{Nat}}\,\vee_2}{\vdash \mathsf{Nat}}\,\mu$$

$\pi_{\text{successeur}} :=$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\top \vdash \top}\,\mathsf{ax}}{\top \vdash \top \vee \mathsf{Nat}}\,\vee^r_1}{\top \vdash \mathsf{Nat}}\,\mu_r}{\top \vdash \top \vee \mathsf{Nat}}\,\vee^r_2}{\top \vdash \mathsf{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\mathsf{Nat} \vdash \mathsf{Nat}}{\mathsf{Nat} \vdash \top \vee \mathsf{Nat}}\,\vee^r_2}{\mathsf{Nat} \vdash \mathsf{Nat}}\,\mu_r}{\dfrac{\top \vee \mathsf{Nat} \vdash \mathsf{Nat}}{\mathsf{Nat} \vdash \mathsf{Nat}}\,\mu_l}\,\vee_l$$

$\pi_{\text{double}} :=$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\top \vdash \top}\,\mathsf{ax}}{\top \vdash \top \vee \mathsf{Nat}}\,\vee^r_1}{\top \vdash \mathsf{Nat}}\,\mu_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Nat} \vdash \mathsf{Nat}}{\mathsf{Nat} \vdash \top \vee \mathsf{Nat}}\,\vee^r_2}{\mathsf{Nat} \vdash \mathsf{Nat}}\,\mu_r}{\mathsf{Nat} \vdash \top \vee \mathsf{Nat}}\,\vee^r_2}{\mathsf{Nat} \vdash \mathsf{Nat}}\,\mu_r}{\dfrac{\top \vee \mathsf{Nat} \vdash \mathsf{Nat}}{\mathsf{Nat} \vdash \mathsf{Nat}}\,\mu_l}\,\vee_l$$

Example with modalities

Taking $F := \nu X.\Diamond X$:

$$\dfrac{\dfrac{\vdash F^\perp, F}{\vdash \Box F^\perp, \Diamond F}\ \Box_{\mathsf{p}}}{\vdash F^\perp, F}\ \mu, \nu$$

Cut-elimination for $\mu LK^{\infty}_{\Box}$

Several results:

Mints & Studer '12, Brünnler & Studer '12

Cut-elimination for fragment modal μ-calculus with ω-rule.

Niwiński & Walukiewicz '96

Cut-free non-wellfounded system with a completeness theorem.

Afshari & Leigh '16

Cut-free cyclic proof system without cut-elimination procedure.

Kloibhofer '23

Incompleteness of Afshari & Leigh's system.

Formulas and derivation rules of μMALL$^\infty$

$$\phi, \psi ::= \phi \parr \psi \mid \phi \otimes \psi \mid \phi \,\&\, \psi \mid \phi \oplus \psi \mid \bot \mid 1 \mid \top \mid 0 \mid X \in \mathcal{V} \mid \mu X.\phi \mid \nu X.\phi.$$

$$\dfrac{}{\vdash A, A^\perp}\,\mathsf{ax} \qquad \dfrac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\mathsf{cut}$$

$$\dfrac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \otimes B, \Delta_1, \Delta_2}\,\otimes \qquad \dfrac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,\parr$$

$$\dfrac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,\oplus^1 \qquad \dfrac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,\oplus^2 \qquad \dfrac{\vdash A_1, \Gamma \qquad \vdash A_2, \Gamma}{\vdash A_1 \,\&\, A_2, \Gamma}\,\&$$

$$\dfrac{\vdash A[X := \mu X.A], \Gamma}{\vdash \mu X.A, \Gamma}\,\mu \qquad \dfrac{\vdash A[X := \nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\,\nu$$

Multi-cut rule

Multi-cut

In order to prove the cut-elimination theorem, we use the multi-cut rule:

$$\dfrac{\vdash \Gamma_1, \Delta_1 \qquad \dots \qquad \vdash \Gamma_n, \Delta_n}{\vdash \Gamma_1, \dots, \Gamma_n}\ \text{mcut}$$

with $\Delta_1, \dots, \Delta_n$ being the *cut-formulas* and satisfying an acyclicity and connexity condition (relative to the cut relation).

$$\dfrac{\mathcal{C} \qquad \dfrac{\dfrac{\pi_1}{\vdash C, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash C^\perp, \Gamma_2}}{\vdash \Gamma_1, \Gamma_2}\ \text{cut}}{\vdash \Delta}\ \text{mcut} \rightsquigarrow \dfrac{\mathcal{C} \qquad \dfrac{\pi_1}{\vdash C, \Gamma_1} \qquad \dfrac{\pi_2}{\vdash C^\perp, \Gamma_2}}{\vdash \Delta}\ \text{mcut}$$

Multi-cut reduction rule

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash A,\Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash B,\Gamma_2\end{array}}{\vdash A\otimes B,\Gamma_1,\Gamma_2}\otimes \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdash A^\perp,B^\perp,\Gamma_3\end{array}}{\vdash A^\perp \parr B^\perp,\Gamma_3}\parr}{\Gamma_1,\Gamma_2,\Gamma_3}\,\text{cut} \;\leadsto\; \dfrac{\begin{array}{c}\pi_1\\ \vdash A,\Gamma_1\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash B,\Gamma_2\end{array} \qquad \begin{array}{c}\pi_3\\ \vdash B^\perp,A^\perp,\Gamma_3\end{array}}{\vdash A^\perp,\Gamma_2,\Gamma_3}\,\text{cut}}{\vdash \Gamma_1,\Gamma_2,\Gamma_3}\,\text{cut}$$

$$\dfrac{\mathcal{C} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdash A,\Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash B,\Gamma_2\end{array}}{\vdash A\otimes B,\Gamma_1,\Gamma_2}\otimes \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdash A^\perp,B^\perp,\Gamma_3\end{array}}{\vdash A^\perp \parr B^\perp,\Gamma_3}\parr}{\vdash \Delta}\,\text{mcut}$$

$$\leadsto\; \dfrac{\mathcal{C} \qquad \begin{array}{c}\pi_1\\ \vdash A,\Gamma_1\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash B,\Gamma_2\end{array} \qquad \begin{array}{c}\pi_3\\ \vdash A^\perp,B^\perp,\Gamma_3\end{array}}{\vdash \Delta}\,\text{mcut}$$

Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X . X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Multi-cut in action

Taking $F := \nu X.X \otimes X$

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Cut-elimination theorems

Cut-elimination of μALL$^\infty$ (Fortier & Santocanale 2013)

The cut-rule is admissible for Additive Linear Logic with fixpoints.

Cut-elimination of μMALL$^\infty$ (Baelde et al. 2016)

Each fair multi-cut reduction sequences of μMALL$^\infty$ are converging to a μMALL$^\infty$-cut-free proof.

Cut-elimination of μLL$^\infty$ (Saurin 2023)

Each fair multi-cut reduction sequences of μLL$^\infty$ are converging to a μLL$^\infty$-cut-free proof.

Goal

We want to prove cut-elimination in a syntactic way

$$\begin{tikzcd}
\mu LK^{\infty}_{\Box} \arrow[r, "(-)^{\bullet}"] & \mu LL^{\infty}_{\Box} \arrow[d, "(-)^{\circ}"] \\
\mu MALL^{\infty} & \mu LL^{\infty} \arrow[l, "\text{TABLEAUX '23}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\mu LK^{\infty}_{\Box} \arrow[rr, "(-)^{\bullet}"] & & \mu LL^{\infty}_{\Box} \arrow[d, "(-)^{\bullet}"] \\
\mu MALL^{\infty} & & \mu LL^{\infty} \arrow[ll, "\text{TABLEAUX '23}"']
\end{tikzcd}
$$

Exponentials

We add exponentials to $\mu MALL^\infty$:

$$?A \quad \text{and} \quad !A$$

As well as the corresponding rules:

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,\mathsf{w} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{c} \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{d} \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

This new system is called μLL^∞

Cut-elimination steps of μLL$^\infty$

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1}}{\vdash ?A, \Gamma_1}\,c \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2,}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2}\,\mathsf{cut}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash ?A, ?A, \Gamma_1} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash ?A, \Gamma_1, ?\Gamma_2}\,\mathsf{cut} \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2, ?\Gamma_2}\,\mathsf{cut}}{\vdash \Gamma_1, ?\Gamma_2}\,c^{\#\Gamma_2}
$$

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash ?A, \Gamma_1}\,w \qquad \dfrac{\overset{\pi_2}{\vdash A^\perp, ?\Gamma_2}}{\vdash !A^\perp, ?\Gamma_2}\,!}{\vdash \Gamma_1, ?\Gamma_2}\,\mathsf{cut}
\quad\rightsquigarrow\quad
\dfrac{\overset{\pi_1}{\vdash \Gamma_1}}{\vdash \Gamma_1, ?\Gamma_2}\,w^{\#\Gamma_2}
$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\ \mathsf{w} \qquad \dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ \mathsf{c} \qquad \dfrac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ \mathsf{d} \qquad \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of weakening

$$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\ \mathsf{w} \rightsquigarrow \dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma^\bullet}{\vdash \bot, \Gamma^\bullet}\ \bot}{\vdash \bot \oplus ((?A)^\bullet \parr (?A)^\bullet), \Gamma^\bullet}\ \oplus^1}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet \parr (?A)^\bullet)), \Gamma^\bullet}\ \oplus^2}{\vdash \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)), \Gamma^\bullet}\ \mu$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,w \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,c \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,d \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of contraction

$$\frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,c \rightsquigarrow \frac{\dfrac{\dfrac{\dfrac{\vdash (?A)^\bullet, (?A)^\bullet, \Gamma^\bullet}{\vdash (?A)^\bullet \parr (?A)^\bullet, \Gamma^\bullet}\,\parr}{\vdash \bot \oplus ((?A)^\bullet \parr (?A)^\bullet), \Gamma^\bullet}\,\oplus^2}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet \parr (?A)^\bullet)), \Gamma^\bullet}\,\oplus^2}{\vdash \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)), \Gamma^\bullet}\,\mu$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,\mathsf{w} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{c} \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{d} \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of dereliction

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{d} \rightsquigarrow \frac{\dfrac{\vdash A^\bullet, \Gamma^\bullet}{\vdash A^\bullet \oplus (\bot \oplus ((?A)^\bullet \parr (?A)^\bullet)), \Gamma^\bullet}\,\oplus^1}{\vdash \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)), \Gamma^\bullet}\,\mu$$

Translation of μLL^∞ in $\mu MALL^\infty$

$$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\,\mathsf{w} \qquad \dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{c} \qquad \dfrac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,\mathsf{d} \qquad \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,!$$

Translation of formulas

$$(?A)^\bullet = \mu X.A^\bullet \oplus (\bot \oplus (X \parr X)) \qquad (!A)^\bullet = \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X))$$

Translation of promotion

$$\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \rightsquigarrow \dfrac{\dfrac{\vdash A^\bullet, (?\Gamma)^\bullet \qquad \dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1, (?\Gamma)^\bullet}\,\mathsf{w}^\bullet \qquad \dfrac{\dfrac{\vdash (?A)^\bullet, (?\Gamma)^\bullet \qquad \vdash (?A)^\bullet, (?\Gamma)^\bullet}{\vdash (?A)^\bullet \otimes (?A)^\bullet, (?\Gamma)^\bullet, (?\Gamma)^\bullet}\,\otimes}{\vdash (?A)^\bullet \otimes (?A)^\bullet, (?\Gamma)^\bullet}\,\mathsf{c}^\bullet}{\vdash A^\bullet \,\&\, (1 \,\&\, ((?A)^\bullet \otimes (?A)^\bullet)), (?\Gamma)^\bullet}\,\& \times 2}{\vdash \nu X.A^\bullet \,\&\, (1 \,\&\, (X \otimes X)), (?\Gamma)^\bullet}\,\nu$$

Translation of μLK$^\infty$ in μLL$^\infty$ and cut-elimination for μLK$^\infty$

$$
\begin{aligned}
(\mu X.A)^\bullet &:= !\mu X.?A^\bullet & (\nu X.A)^\bullet &:= !\nu X.?A^\bullet \\
(A_1 \vee A_2)^\bullet &:= !(?A_1{}^\bullet \oplus ?A_2{}^\bullet) & (A_1 \wedge A_2)^\bullet &:= !(?A_1{}^\bullet \,\&\, ?A_2{}^\bullet) \\
F^\bullet &:= !0 & \top^\bullet &:= !\top \\
X^\bullet &:= !X & a^\bullet &:= !a \\
(A_1 \to A_2)^\bullet &:= !(?A_1{}^\bullet \multimap ?A_2{}^\bullet) & (\Gamma \vdash \Delta)^\bullet &:= \Gamma^\bullet \vdash ?\Delta^\bullet
\end{aligned}
$$

$$
\begin{tikzcd}
\mu LK^\infty & & \mu LL^\infty \\
\pi = \pi_0 \arrow[rr] \arrow[d, "*"] & & \pi^\bullet = \theta = \theta_0 \arrow[d, "*"] \\
U(\theta_n) \arrow[d, "\infty"] & & \theta_n \arrow[ll] \arrow[d, "\infty"] \\
U(\theta_\infty) & & \theta_\infty \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mu LK^\infty_\Box \arrow[rr, "(-)^\bullet"] & & \mu LL^\infty_\Box \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & & \mu LL^\infty \arrow[ll]
\end{tikzcd}
$$

Naïve extension of the translation and issue with it

Let's consider the two-sided system μLL$^\infty$ together with the two modal rules:

$$\frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}\ \Diamond \qquad \frac{\Gamma \vdash A, \Delta}{\Box\Gamma \vdash \Box A, \Diamond\Delta}\ \Box$$

We extend the translation, to get a translation from μLK$^\infty_\Box$ to μLL$^\infty_\Box$:

$$(\Box A)^\bullet := !\,\Box\, A^\bullet \qquad\qquad (\Diamond A)^\bullet := !\Diamond A^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

We start with the sequent:

$$\vdash ?!\square A^\bullet, ?!\Diamond B^\bullet$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \lozenge B}\ \square$$

We apply a dereliction a promotion:

$$\frac{\vdash ?!\,\square\, A^\bullet, \lozenge B^\bullet}{\vdash ?!\,\square\, A^\bullet, ?!\lozenge B^\bullet}\ \mathrm{d}, !$$

Problem

Promotion rule

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\;!$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \Box A, \Diamond B}\;\Box$$

We apply a dereliction again:

$$\dfrac{\dfrac{\vdash !\Box A^\bullet, \Diamond B^\bullet}{\vdash ?!\Box A^\bullet, \Diamond B^\bullet}\;\mathrm{d}}{\vdash ?!\Box A^\bullet, ?!\Diamond B^\bullet}\;\mathrm{d}, !$$

And we are blocked.

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \Box A, \Diamond B}\ \Box$$

We apply a dereliction again:

$$\frac{\dfrac{\vdash !\Box A^\bullet, \Diamond B^\bullet}{\vdash ?!\Box A^\bullet, \Diamond B^\bullet}\ \mathrm{d}}{\vdash ?!\Box A^\bullet, ?!\Diamond B^\bullet}\ \mathrm{d}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\,!$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\,\square$$

We apply a dereliction again:

$$\dfrac{\dfrac{\dfrac{\vdash \square A^\bullet, \Diamond B^\bullet}{\vdash !\square A^\bullet, \Diamond B^\bullet}\,!}{\vdash ?!\square A^\bullet, \Diamond B^\bullet}\,\mathrm{d}}{\vdash ?!\square A^\bullet, ?!\Diamond B^\bullet}\,\mathrm{d}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

We want to translate:

$$\frac{\vdash A, B}{\vdash \square A, \Diamond B}\ \square$$

We apply a dereliction again:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash A^\bullet, B^\bullet}{\vdash \square A^\bullet, \Diamond B^\bullet}\ \square}{\vdash !\square A^\bullet, \Diamond B^\bullet}\ !}{\vdash ?!\square A^\bullet, \Diamond B^\bullet}\ \mathrm{d}}{\vdash ?!\square A^\bullet, ?!\Diamond B^\bullet}\ \mathrm{d}, !$$

Solution

Promotion rule

$$\frac{\vdash A, ?\Gamma, \Diamond\Delta}{\vdash !A, ?\Gamma, \Diamond\Delta}\ !$$

Consequences

$$\frac{\vdash \Delta}{\vdash \Diamond A, \Delta}\ w \qquad \frac{\vdash \Diamond A, \Diamond A, \Delta}{\vdash \Diamond A, \Delta}\ c$$

$$
\begin{tikzcd}
\mu LK^\infty_\square \arrow[r, "(-)^\bullet"] & \mu LL^\infty_\square \arrow[d, "(-)^\circ"] \\
\mu MALL^\infty & \mu LL^\infty \arrow[l]
\end{tikzcd}
$$

Translation from $\mu LL^\infty_\square$ to μLL^∞

Translation from $\mu LL^\infty_\square$ to μLL^∞ and cut-elimination for $\mu LL^\infty_\square$

We define:

$$(\Diamond A)^\circ \to ?A^\circ \text{ and } (\square A)^\circ \to !A^\circ.$$

We easily get weakening and contractions of $\Diamond$ with weakening and contraction of ?.
For the modality rule, we have:

$$\dfrac{\vdash A, \Gamma}{\vdash \square A, \Diamond\Gamma}\ \square \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\ \mathrm{d}}{\vdash !A^\circ, ?\Gamma^\circ}\ !$$

Translation from $\mu LL^\infty_\square$ to μLL^∞

Translation from $\mu LL^\infty_\square$ to μLL^∞ and cut-elimination for $\mu LL^\infty_\square$

We define:

$$(\lozenge A)^\circ \to ?A^\circ \text{ and } (\square A)^\circ \to !A^\circ.$$

We easily get weakening and contractions of $\lozenge$ with weakening and contraction of $?$.
For the modality rule, we have:

$$\dfrac{\vdash A, \Gamma}{\vdash \square A, \lozenge \Gamma}\,\square \rightsquigarrow \dfrac{\dfrac{\vdash A^\circ, \Gamma^\circ}{\vdash A^\circ, ?\Gamma^\circ}\,\mathsf{d}}{\vdash !A^\circ, ?\Gamma^\circ}\,!$$

Cut-elimination theorems

Using $(-)^\circ$, we obtain cut-elimination for $\mu LL^\infty_\square$.

Using $(-)^\bullet$, we get a proof of $\mu LL^\infty_\square$, from which we can eliminate cuts, we then can come back to $\mu LK^\infty_\square$ and get a cut-free proof of $\mu LL^\infty_\square$.

Sub-exponentials

B. & Laurent TLLA '20

The previous work actually works with a sub-exponential system inspired from the work of Nigam & Miller '09. With a promotion rule on signed exponentials:

$$\frac{\vdash A, ?_{e_1}A_1, \ldots, ?_{e_n}A_n \qquad e \leq_g e_i}{\vdash !_e A, ?_{e_1}A_1, \ldots, ?_{e_n}A_n}\ ! \qquad \frac{\vdash A, A_1, \ldots, A_n \qquad e \leq_f e_i}{\vdash !_e A, ?_{e_1}A_1, \ldots, ?_{e_n}A_n}\ !_f$$

and structural rules authorized only on some signed exponentials:

$$\frac{\vdash \Gamma \qquad e \in \mathcal{W}}{\vdash ?_e A, \Gamma}\ w \qquad \frac{\vdash ?_e A, ?_e A, \Gamma \qquad e \in \mathcal{C}}{\vdash ?_e A, \Gamma}\ c \qquad \frac{\vdash A, \Gamma \qquad e \in \mathcal{D}}{\vdash ?_e A, \Gamma}\ d$$

Modal logic is an instance of this sub-exponential system:

With two signatures e and e', with $! := !_e$ and $\square = !_{e'}$.

$e' \leq_g e, \quad e \leq_g e, \quad e' \leq_f e', \quad e \not\leq_f e', \quad e' \not\leq_g e, e'$

$e, e' \in \mathcal{W},\ e, e' \in \mathcal{C}$ and $e \in \mathcal{D}$