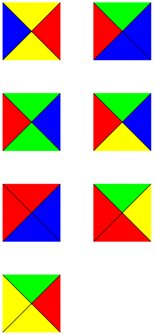
Mealy Automata, Singular Points and Wang tilings

Daniele D'Angeli, Thibault Godin, Ines Klimann, Matthieu Picantin, and Emanuele Rodaro

Tilings in Oléron

ANR JCJC 12 JS02 012 01

Wang tilings and transducers



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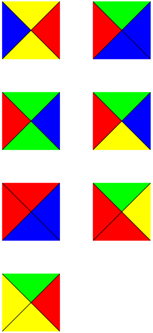

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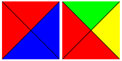

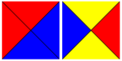

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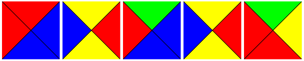

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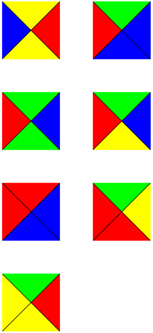

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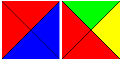

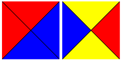

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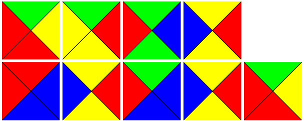

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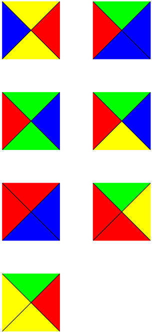

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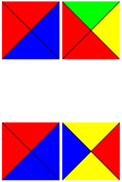

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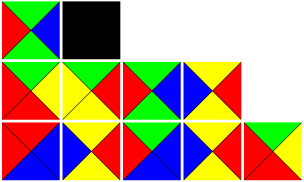

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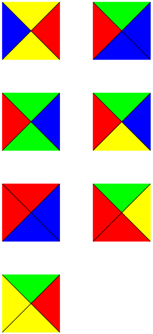

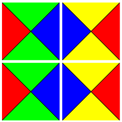

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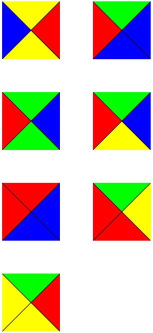

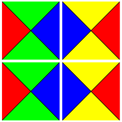

[Berger 1964]

The Domino Problem is undecidable.

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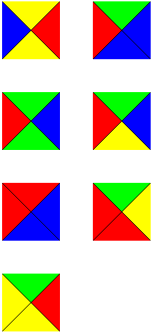

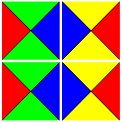

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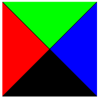
Key property: aperiodicity

Wang tiling and transducers



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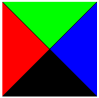
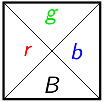


$\Leftrightarrow$



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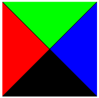

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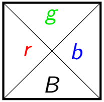

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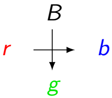

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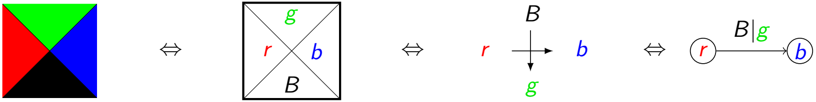

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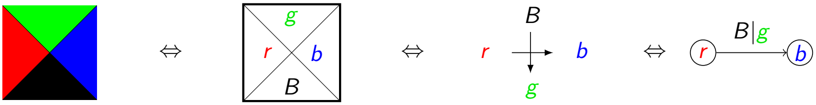

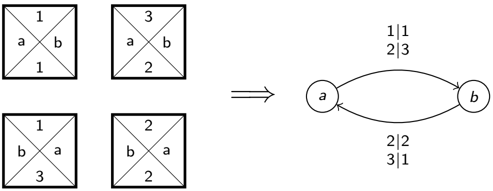

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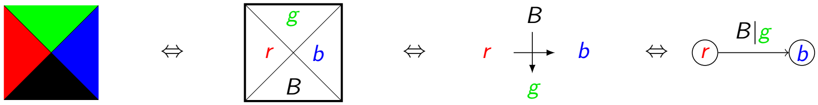
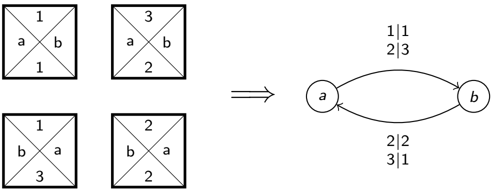
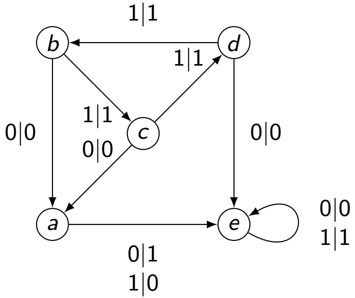




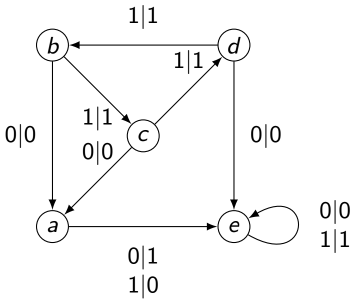
- line of dominoes $\iff$ run in the transducer
- multiple lines $\iff$ transducer composition

Transducer



Transducer

Run



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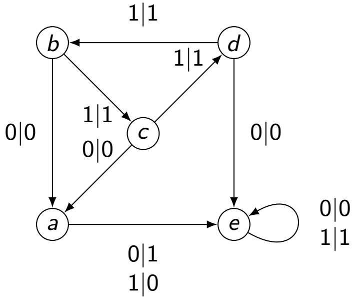

Run

$\rho_q : \Sigma^\omega \to \Sigma^\omega$

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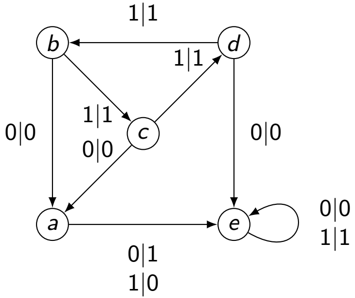

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	1	1	1	0	1
b					

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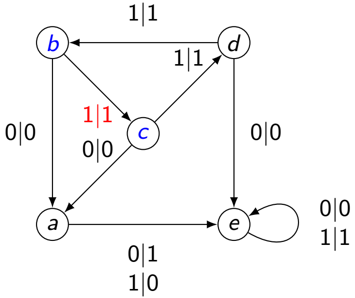

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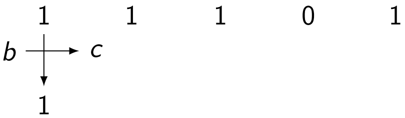

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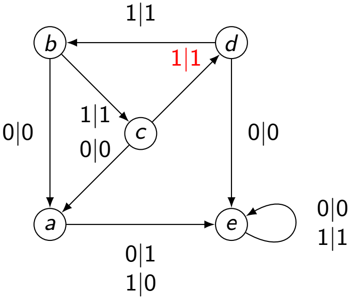

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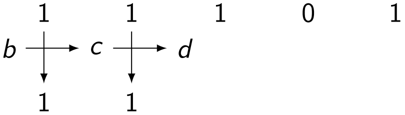

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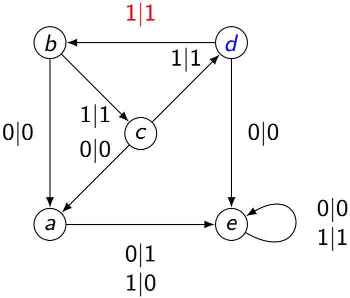

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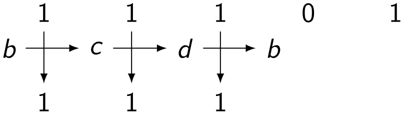

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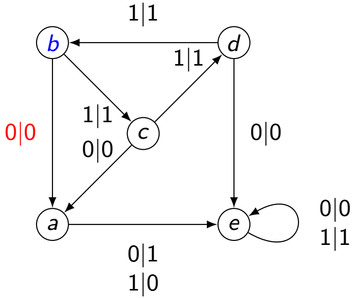

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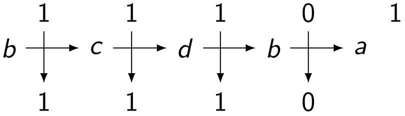

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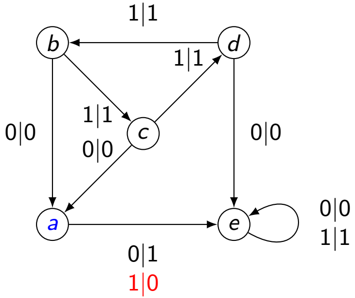

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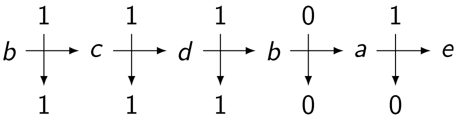

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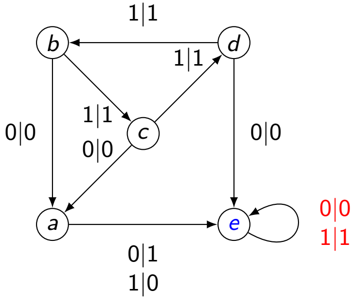

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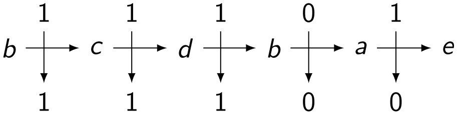

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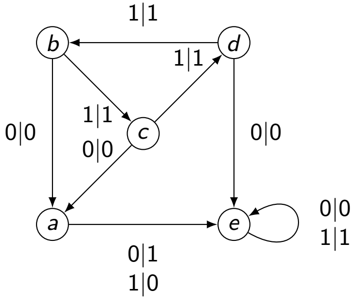

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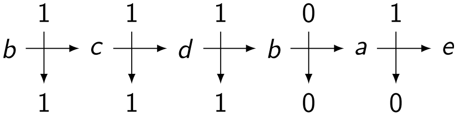
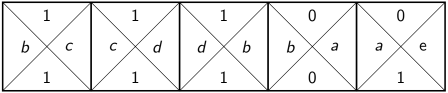
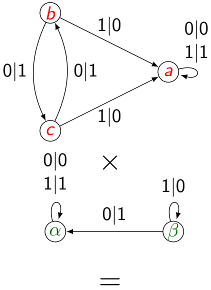




Composition/Product of Transducers



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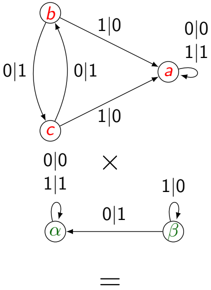

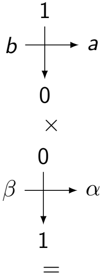

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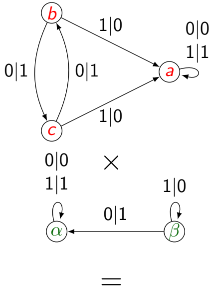

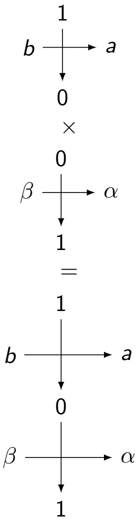

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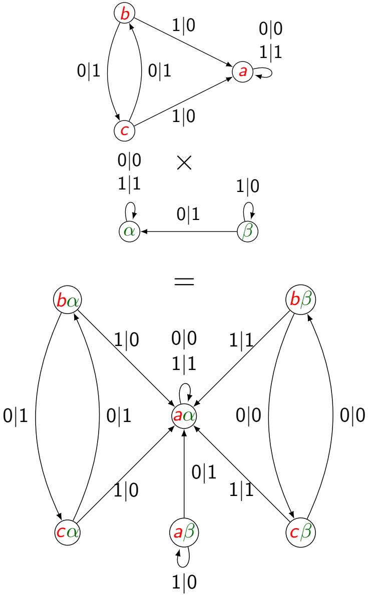

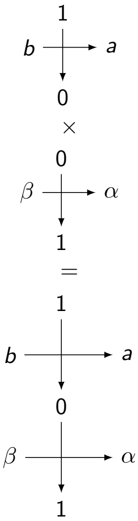

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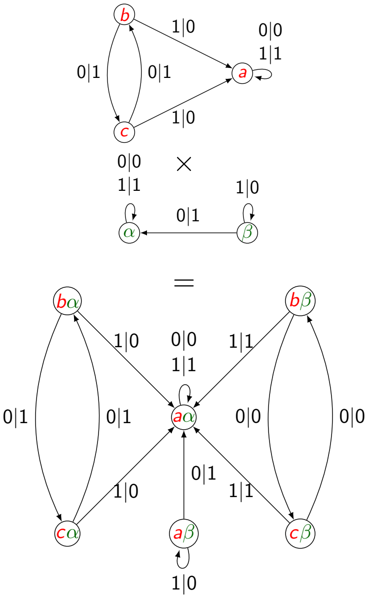

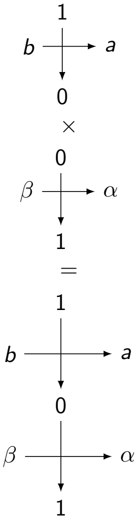

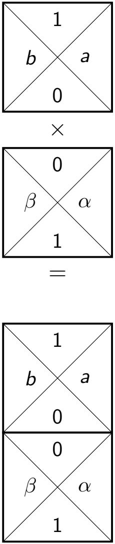

Torsion

Torsion element

$x \in G$ is torsion (finite order) if $\exists n \geq 1, x^n = e$

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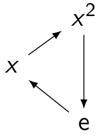

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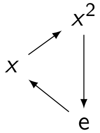
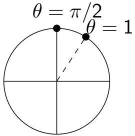


$x^3 = e$, x is torsion

- $\mathbb{Z}/n\mathbb{Z}$ is torsion (all its elements have finite order)
- $\mathbb{Z}$ is torsion-free (0 is the only element of finite order)
- On the circle $\mathbb{R}/2\pi\mathbb{Z}$; $\pi/2$ has finite order but 1 has infinite order



The Burnside problem

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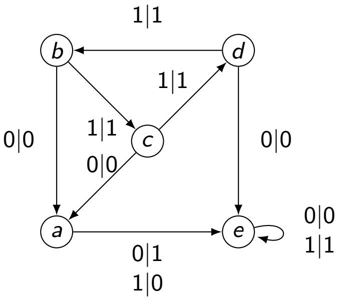
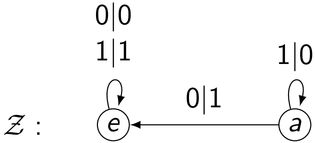


The Grigorchuk automaton $\mathcal{G}$

How to Generate Groups



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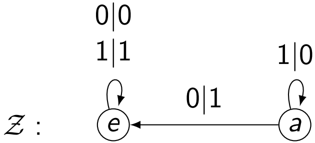

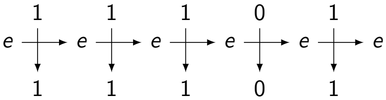

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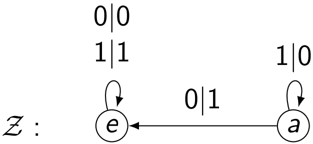
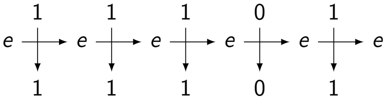
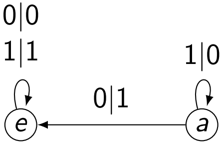


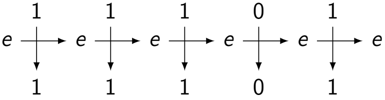


$$\rho_e : \Sigma^\omega \to \Sigma^\omega$$

How to Generate Groups

$\mathcal{Z}$:





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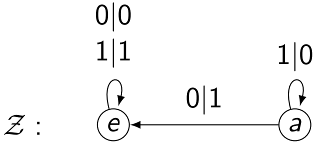
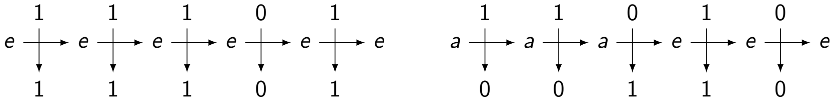
How to Generate Groups





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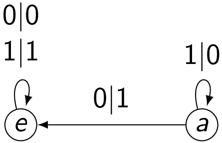
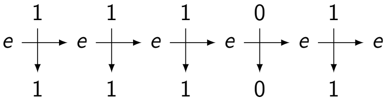
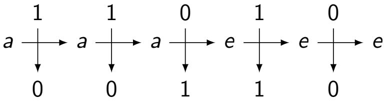
How to Generate Groups

$\mathcal{Z}$:







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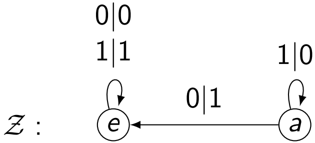

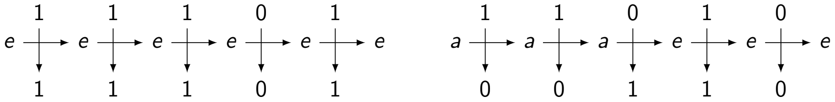

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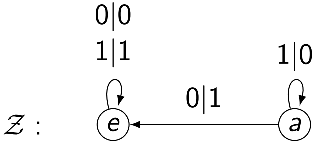

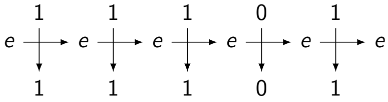

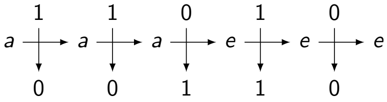

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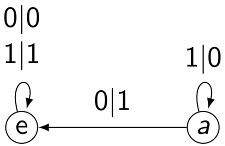
$\langle \mathcal{A} \rangle = (\langle \rho_{\mathrm{e}}, \rho_a \rangle, \circ)$

Claim: the transducer $\mathcal{Z}$ generates $\mathbb{Z}$



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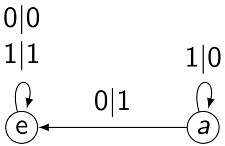

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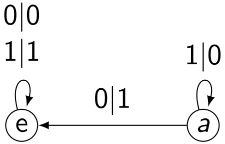

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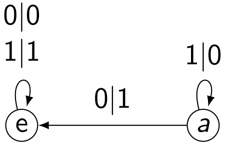

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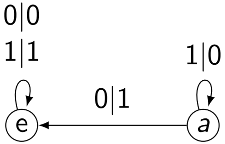

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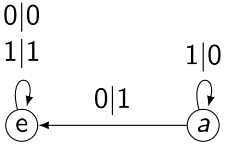

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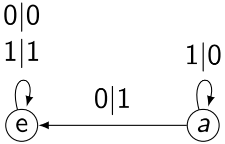

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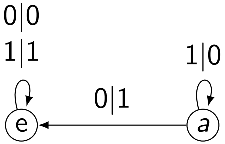

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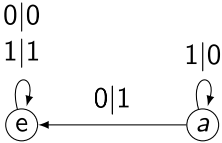

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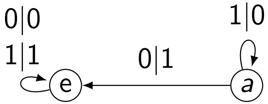
group element $\Leftarrow$ state in a power $\Longleftrightarrow$ word of states

Action on a regular rooted tree



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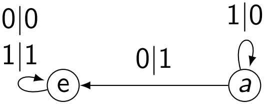

Set of word $\Sigma^* \simeq$ regular rooted tree T

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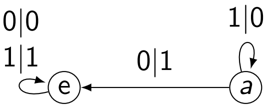

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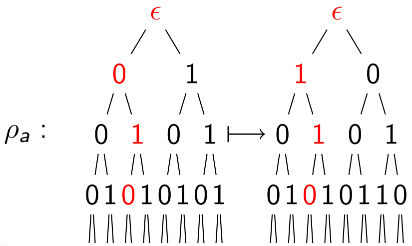

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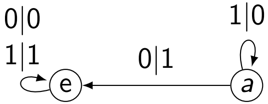

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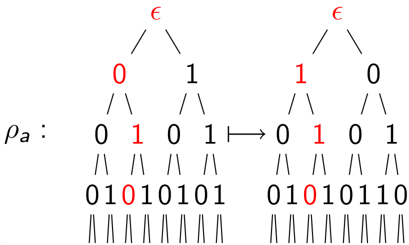

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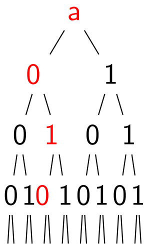

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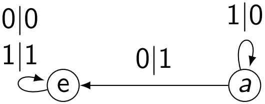

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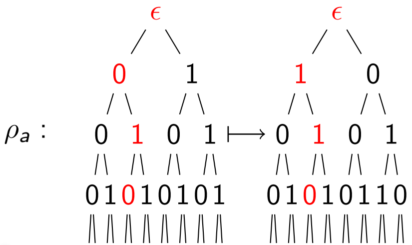

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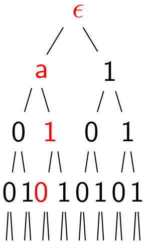

$\rho_a(0) = 1,\ \delta_0(a) = \mathrm{e}$

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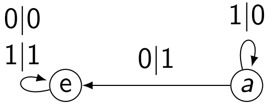

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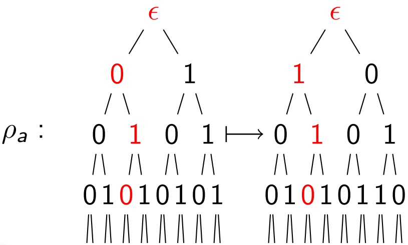

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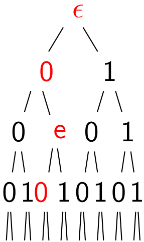
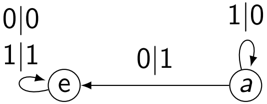


$$\rho_{\mathrm{e}}(1) = 1,\ \delta_1(\mathrm{e}) = \mathrm{e}$$

Action on a regular rooted tree



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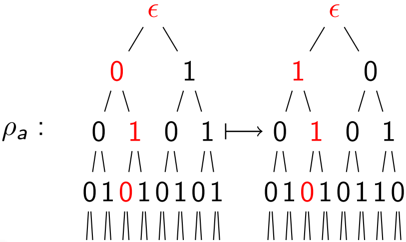

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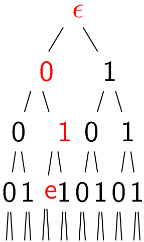

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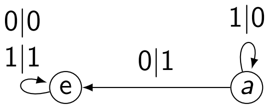

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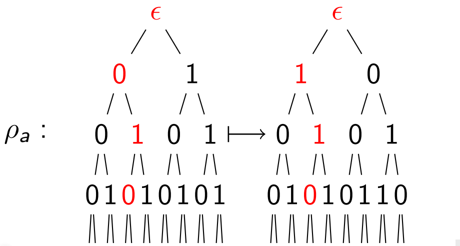

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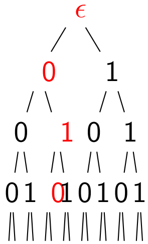

Stabilisers

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$\mathsf{Stab}_{\langle \mathcal{A} \rangle}(\xi) = \{g \in \langle \mathcal{A} \rangle \mid \rho_g(\xi) = \xi\}$

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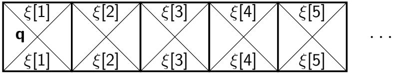

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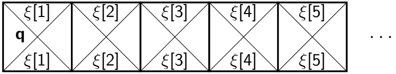
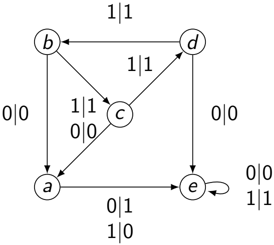


Example



e, $b, c, d \in \mathrm{Stab}_{\langle \mathcal{G} \rangle}(1^\omega)$
studied by Y. Vorobets

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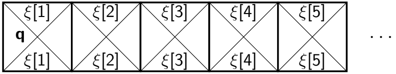

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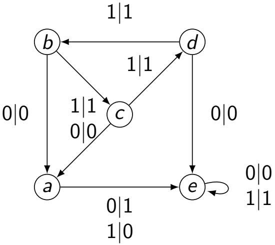

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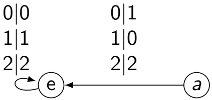
Interesting elements



2^ω is stabilised by a

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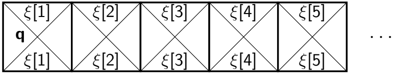

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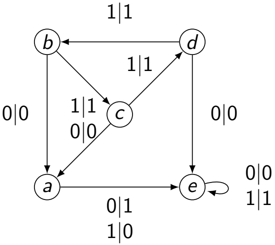

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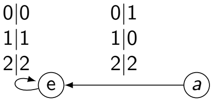

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Avoid ending in e

Singular points

$$St : \xi \mapsto \mathsf{Stab}_{\langle \mathcal{A} \rangle}(\xi)$$

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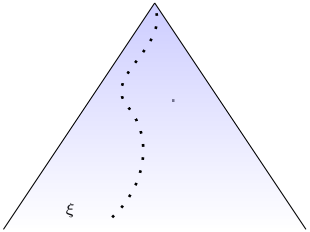
Neighbourhood



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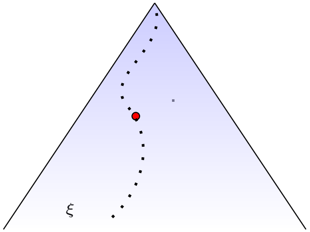

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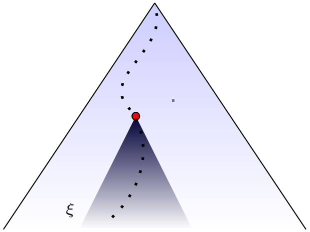

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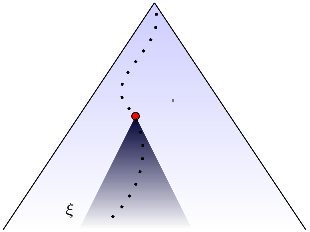

Definition

ξ singular $\Leftrightarrow$ St is not continuous in ξ

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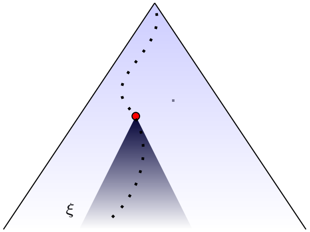

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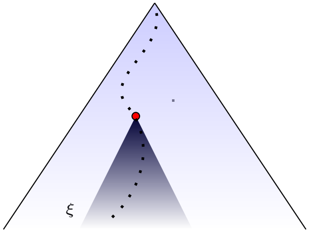
ξ not singular $\Longleftrightarrow$
$\forall g \in \mathrm{Stab}_{\langle \mathcal{A} \rangle}(\xi)$, $\exists n$, $\delta_{\xi[:n]}(g) = \mathrm{e}$

Singular points

$$St : \xi \mapsto \mathrm{Stab}_{\langle \mathcal{A} \rangle}(\xi)$$

Neighbourhood



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The set of singular points has measure 0.

Characterising singular points

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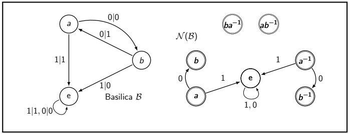
Expressing Sing as a language:

(fractal) contracting automata



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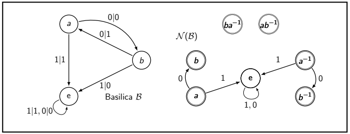

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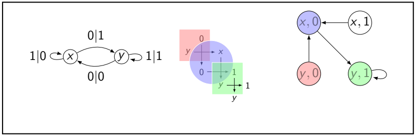

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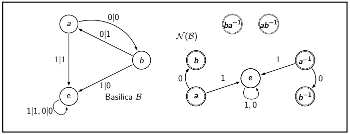

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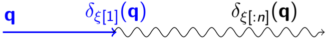

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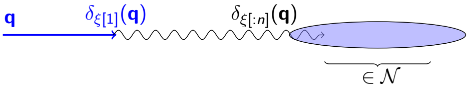

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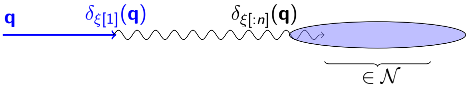

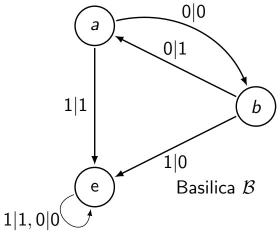

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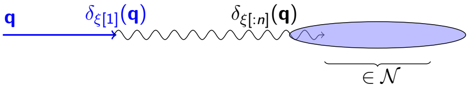

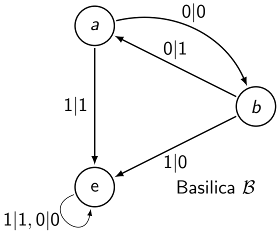

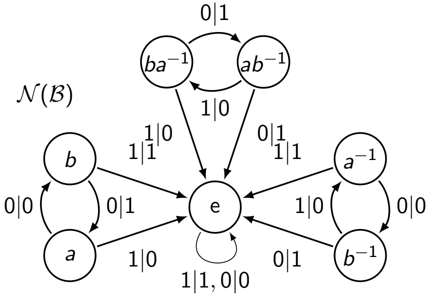

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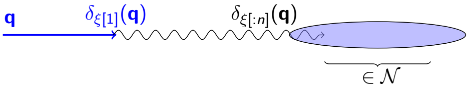

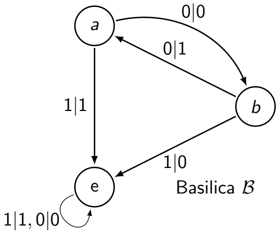

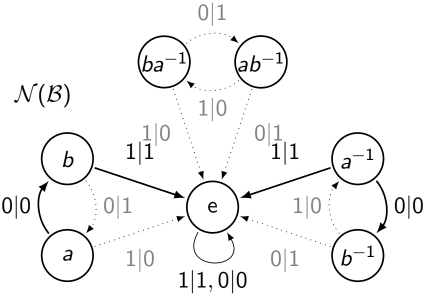

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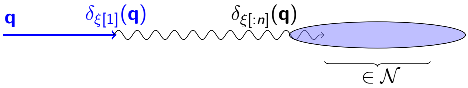

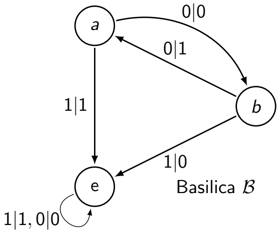

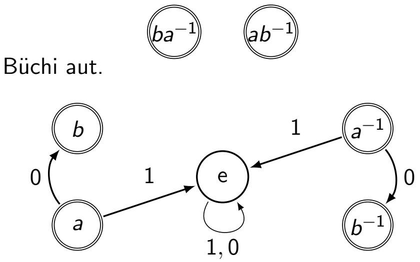

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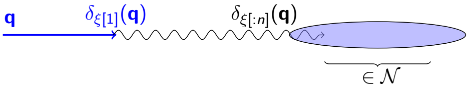
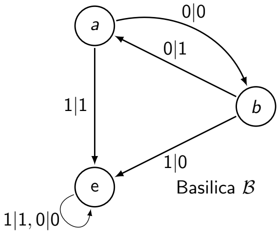
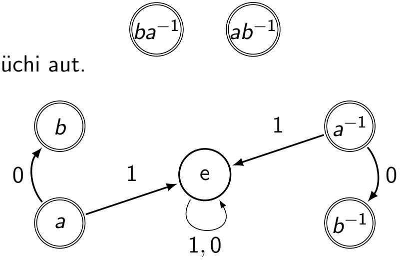




Basilica $\mathcal{B}$

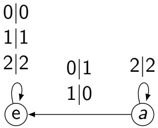
Büchi aut.



Lemma

$\mathrm{Sing}(\mathcal{B}) = \emptyset$

Fractal



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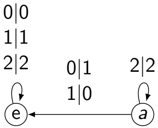

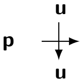

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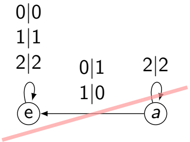

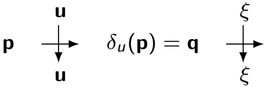

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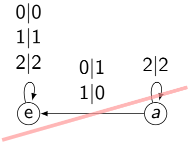

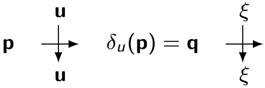

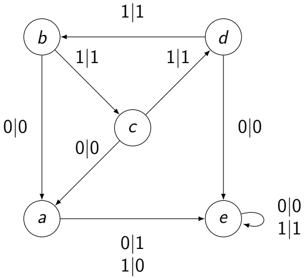

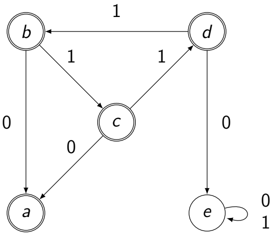

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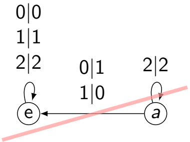
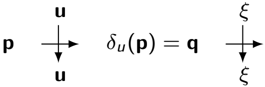
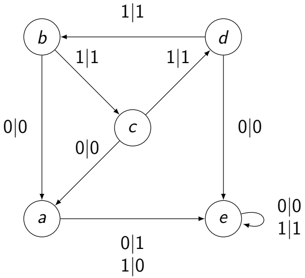
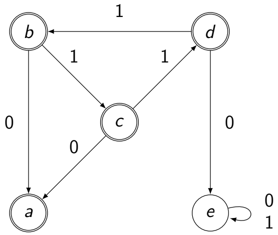




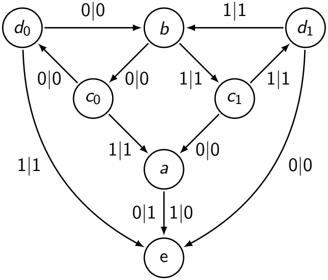




Proposition [Vorobets, AGKPR]

$\mathsf{Sing}(\mathcal{G}) = (0 + 1 + 2)^* 1^\omega$

Singular points



Automaton $\mathcal{W}$

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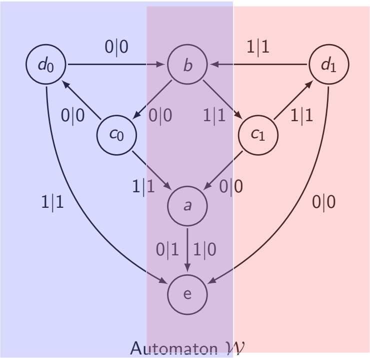

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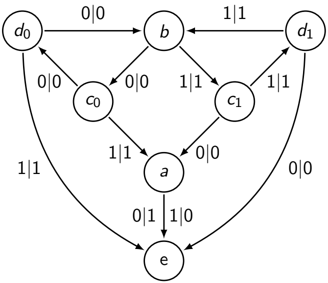

Automaton $\mathcal{W}$

- Contracting

nucleus of size 1027

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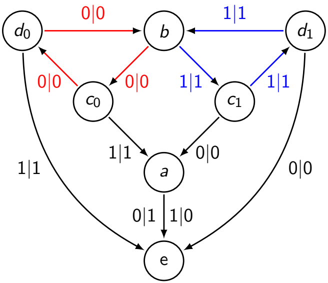

Automaton $\mathcal{W}$

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Corollary

$(000 + 111)^\omega \subset \mathrm{Sing}(\mathcal{W})$

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Consider specific stabilisers, via commuting pairs:

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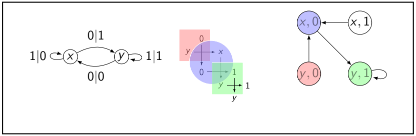

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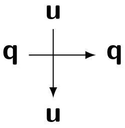

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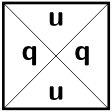

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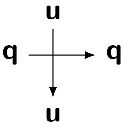

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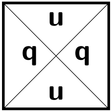

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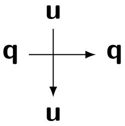

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$(\mathbf{q}, \mathbf{u})$ commuting pair

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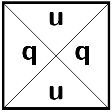

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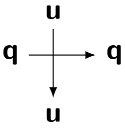

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Commuting pair and helix graph

How to find commuting pairs?

Commuting pair and helix graph

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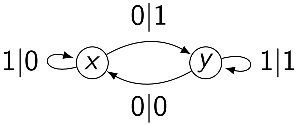

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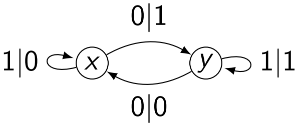

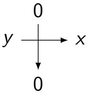

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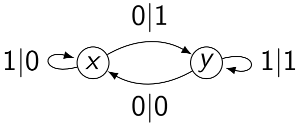

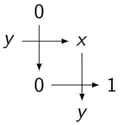

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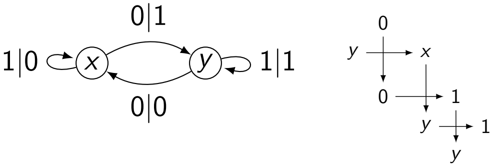

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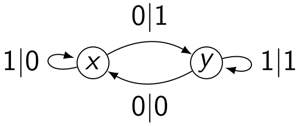

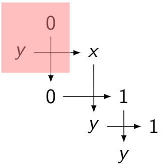

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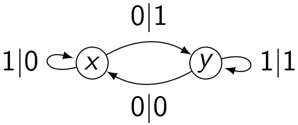

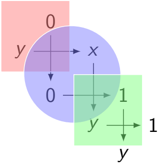

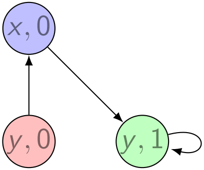

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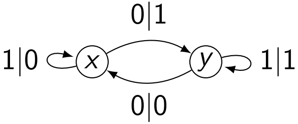
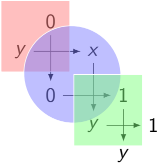
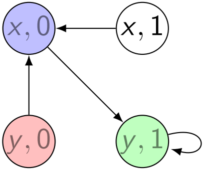
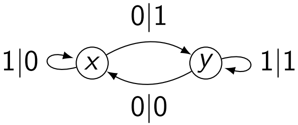


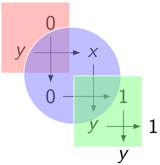


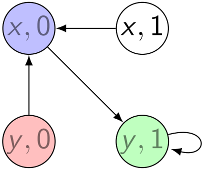


Commuting pair and helix graph

How to find commuting pairs?







Lemma

$\exists\, (\mathbf{q}, \mathbf{u})$ commuting pair

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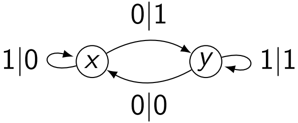

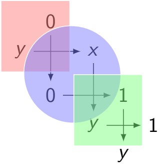

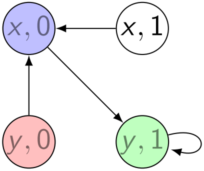

Lemma

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Lemma

Tileset associated with Mealy automaton
$\Longrightarrow$ periodic tiling

Restricted tilesets and undecidability

$(\mathrm{e}, \mathbf{u})$ is always a commuting pair but $\mathbf{u}^{\omega}$ is not always singular

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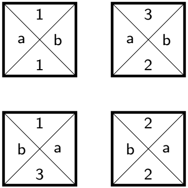

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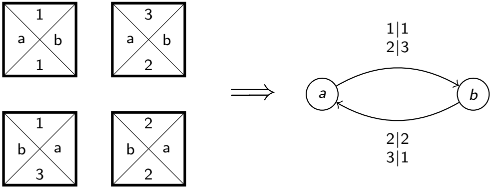

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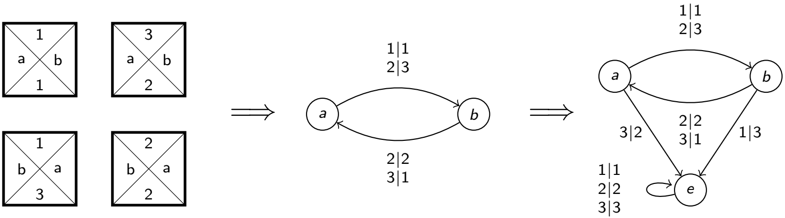


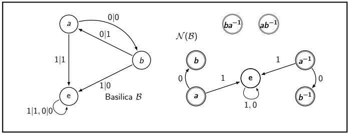
Characterising singular points

Theorem

The set of singular points has measure 0.

Expressing Sing as a language:

(fractal) contracting automata



Consider specific stabilisers, via commuting pairs:

(bi)reversible automata