On Torsion-Free Semigroups generated by Mealy Automata

Thibault Godin, Ines Klimann and Matthieu Picantin
Seminaire Doctorant LIAFA-PPS, 11 Mars 2015

ANR JCJC 12 JS02 012 01

Torsion-Free Semigroups generated by Mealy Automaton

How to generate
Semigroups?

Torsion-Free

Semigroups
generated by
Mealy Automaton

How to generate Semigroups?

Why generate Semigroups with Mealy Automata?

Torsion-Free Semigroups generated by Mealy Automaton

How to generate Semigroups?

Why generate Semigroups with Mealy Automata?

Torsion-Free Semigroups generated by Mealy Automaton

What can we say about such Semigroups?

SemiGroups

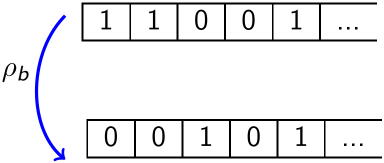
SemiGroups

$(\{g_1, g_2, ...\},\ *)$

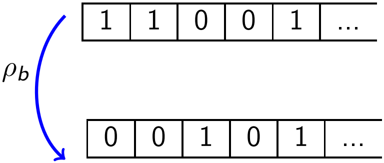
SemiGroups

$$(\{g_1, g_2, ...\},\ *) \qquad (\mathbb{N}, +), \qquad (\textit{words}, .)$$

SemiGroups



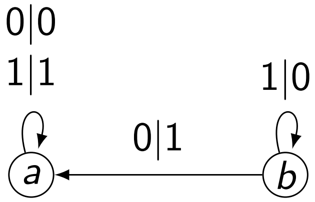
SemiGroups



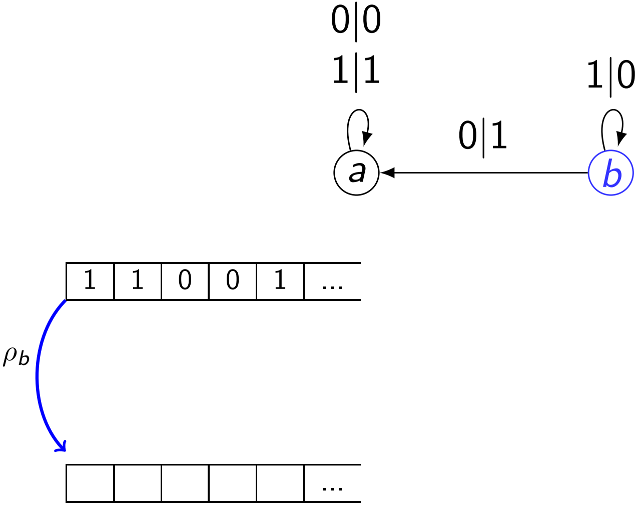
$$(\{\rho_a, \rho_b, ...\}, \circ)$$

How

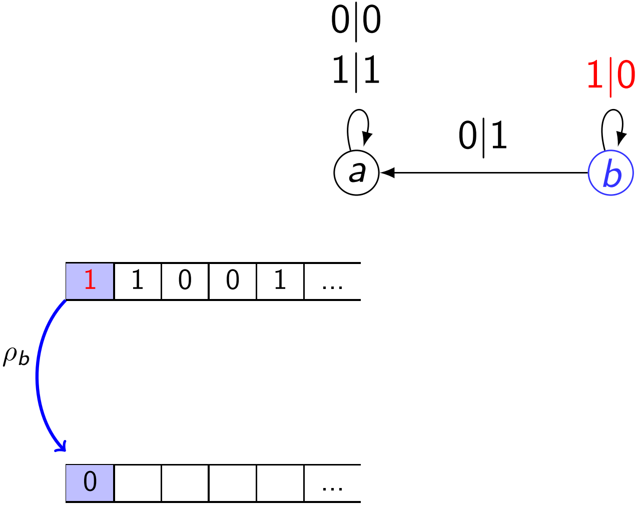
Mealy automaton



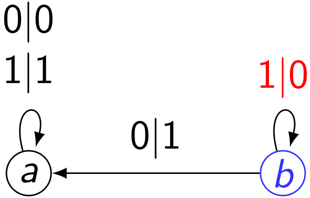
Mealy automaton

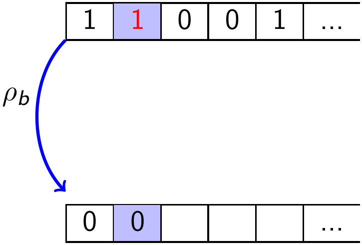


Mealy automaton

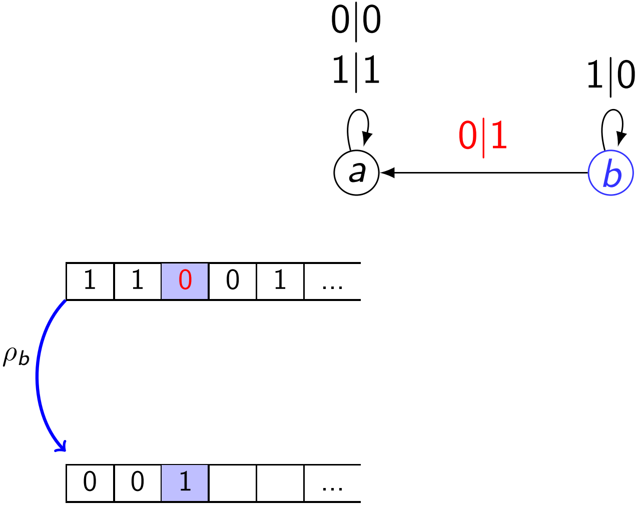


Mealy automaton

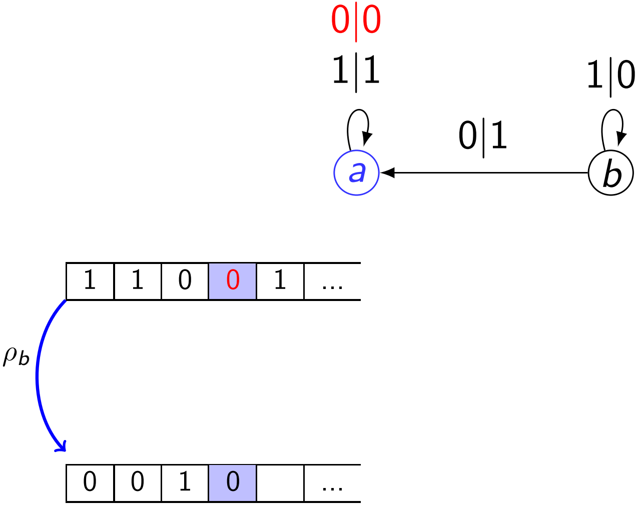




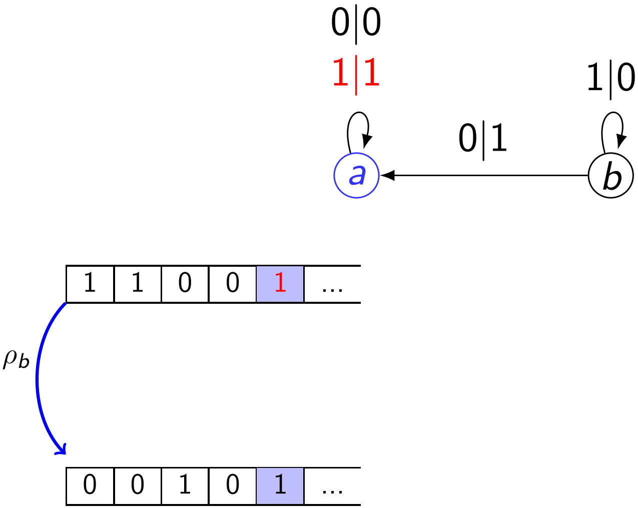
Mealy automaton



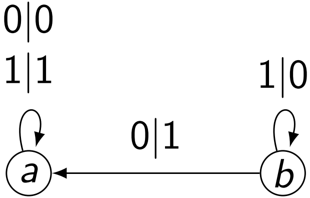
Mealy automaton

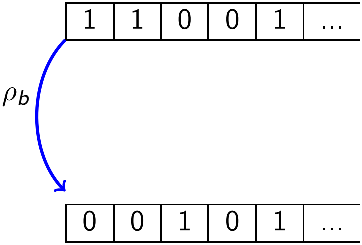


Mealy automaton



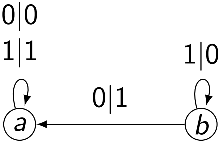
Mealy automaton



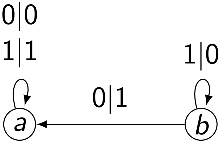


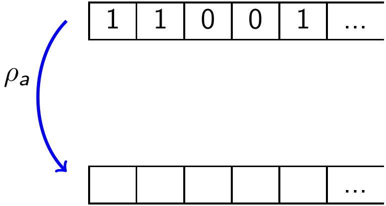
$\rho_b : \Sigma^\omega \to \Sigma^\omega$

How to Generate Semigroups

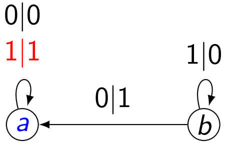


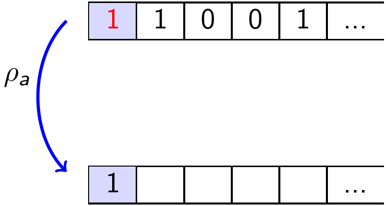
How to Generate Semigroups



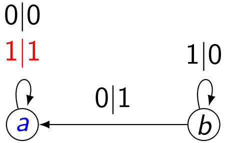


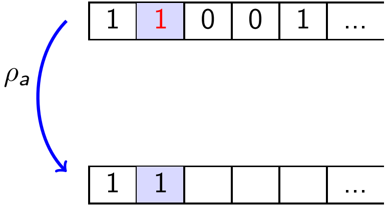
How to Generate Semigroups



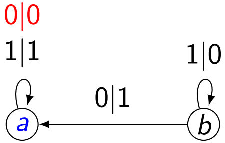


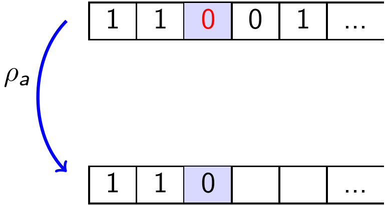
How to Generate Semigroups



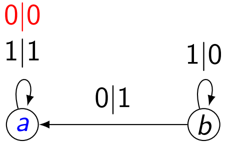


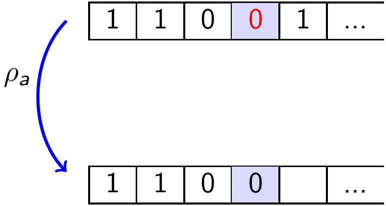
How to Generate Semigroups



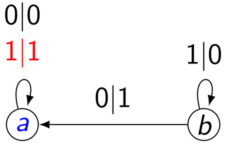


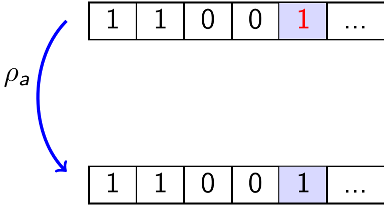
How to Generate Semigroups



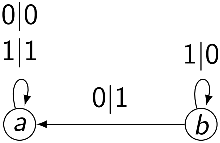


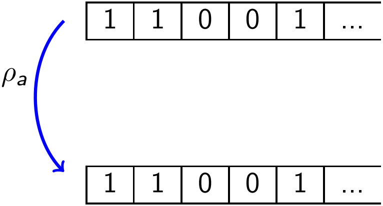
How to Generate Semigroups





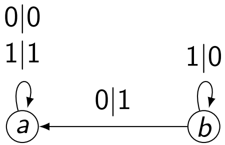
How to Generate Semigroups

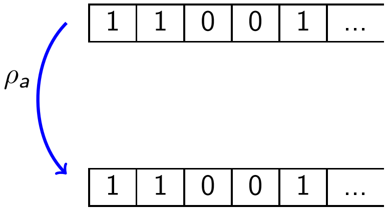




$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$

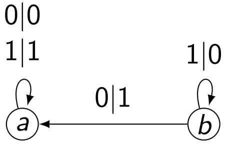
How to Generate Semigroups

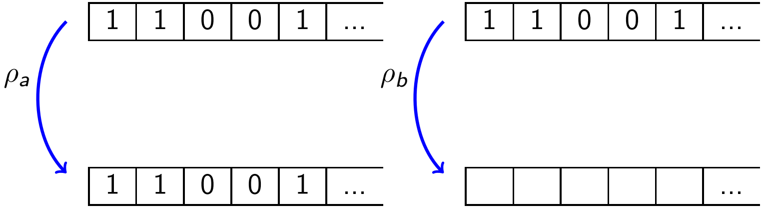




$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

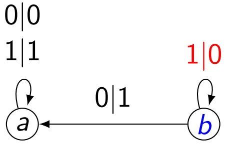
How to Generate Semigroups

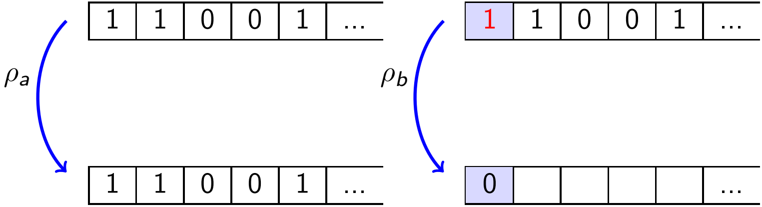




$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

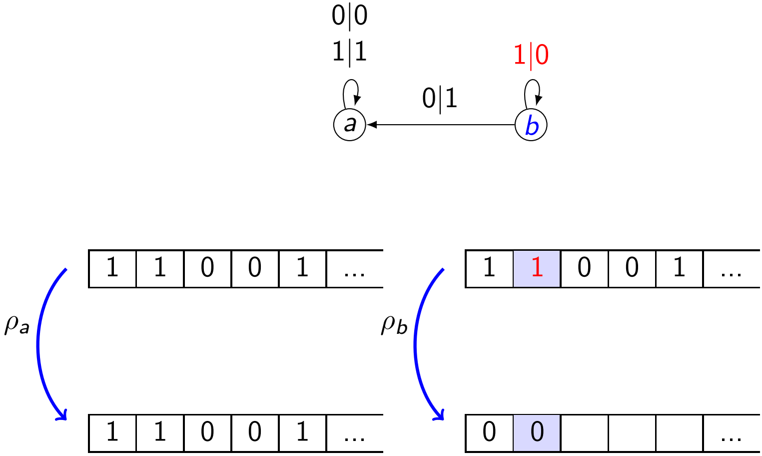
How to Generate Semigroups





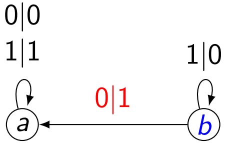
$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

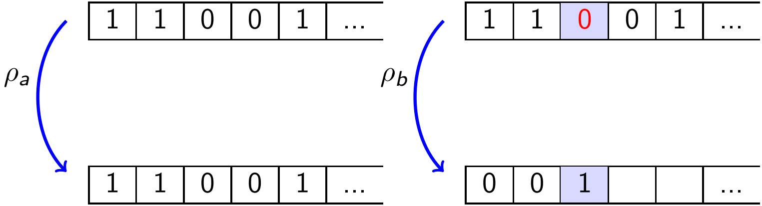
How to Generate Semigroups



$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

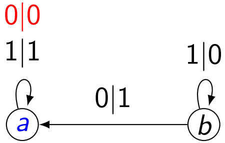
How to Generate Semigroups

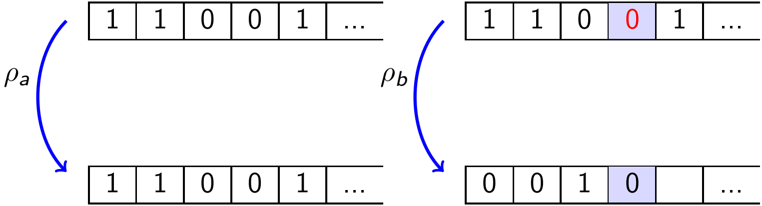




$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

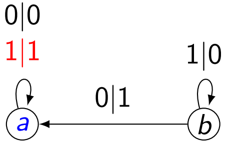
How to Generate Semigroups

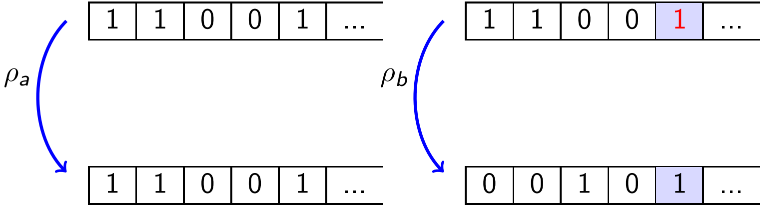




$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

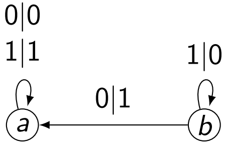
How to Generate Semigroups

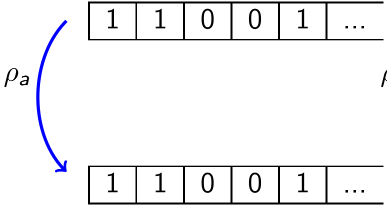


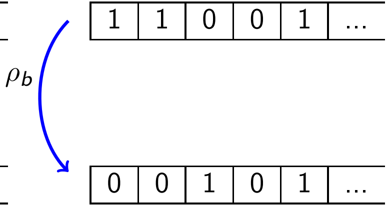


$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

How to Generate Semigroups



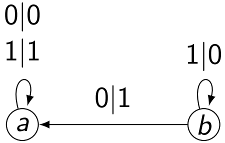


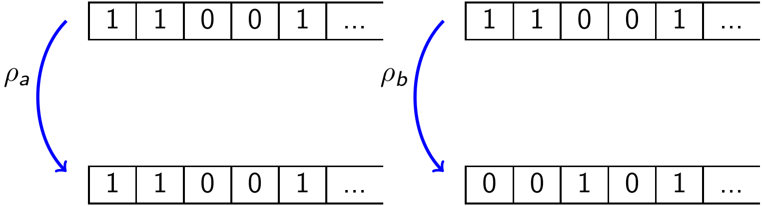


$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

$$\rho_b : \Sigma^\omega \to \Sigma^\omega$$

How to Generate Semigroups

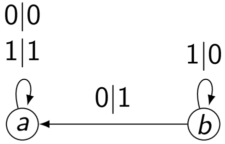




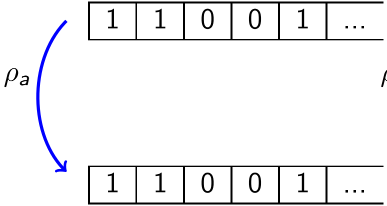
$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

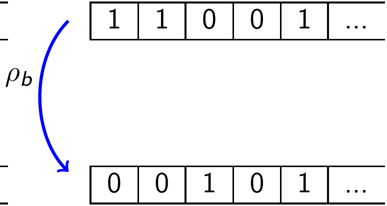
$$\rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u + 1$$

How to Generate Semigroups



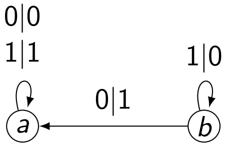
Claim: this transducer generates $\mathbb{N}$



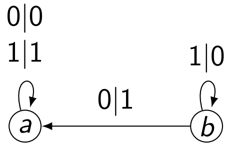


$$\rho_a : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u$$

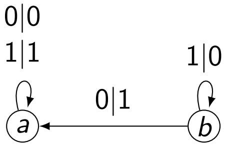
$$\rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u + 1$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u+1$$



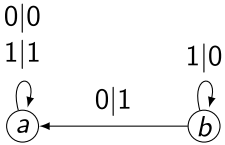
$$\rho_a : \Sigma^\omega \to \Sigma^\omega \quad u \mapsto u \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega \quad u \mapsto u+1 \qquad \rho_a = 1_{\Sigma^\omega \to \Sigma^\omega}$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u + 1$$

$$\rho_a \approx 0$$

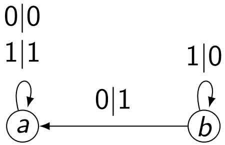
$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u+1$$

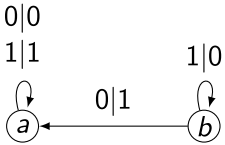
$$\rho_a \approx 0$$

$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$
$$\rho_b \circ \rho_b = u \mapsto u + 2$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \quad u \mapsto u \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega \quad u \mapsto u+1 \qquad \rho_a \approx 0$$

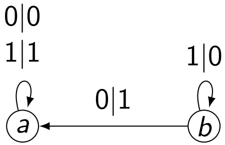
$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$
$$\rho_b \circ \rho_b = u \mapsto u+2$$
$$\rho_b^m \circ \rho_b^n = u \mapsto u+(m+n)$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u+1$$

$$\rho_a \approx 0$$
$$\rho_b \approx 1$$

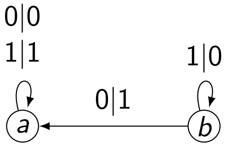
$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$
$$\rho_b \circ \rho_b = u \mapsto u+2$$
$$\rho_b^m \circ \rho_b^n = u \mapsto u+(m+n)$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \quad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u+1$$

$$\rho_a \approx 0$$
$$\rho_b \approx 1$$
$$\circ \approx +$$

$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$
$$\rho_b \circ \rho_b = u \mapsto u+2$$
$$\rho_b^m \circ \rho_b^n = u \mapsto u+(m+n)$$



$$\rho_a : \Sigma^\omega \to \Sigma^\omega \qquad \rho_b : \Sigma^\omega \to \Sigma^\omega$$
$$u \mapsto u \qquad\qquad u \mapsto u+1$$

$$\rho_a \circ \rho = \rho \circ \rho_a = \rho$$
$$\rho_b \circ \rho_b = u \mapsto u+2$$
$$\rho_b^m \circ \rho_b^n = u \mapsto u+(m+n)$$

$$\rho_a \approx 0$$
$$\rho_b \approx 1$$
$$\circ \approx +$$
$$(\langle \rho_a, \rho_b \rangle, \circ) \simeq (\mathbb{N}, +)$$

Why

Growth

Cayley Graph: $\Gamma(G, S)$

$$g \xrightarrow{s} h \qquad g.s = h,\ g, h \in G,\ s \in S$$

Growth

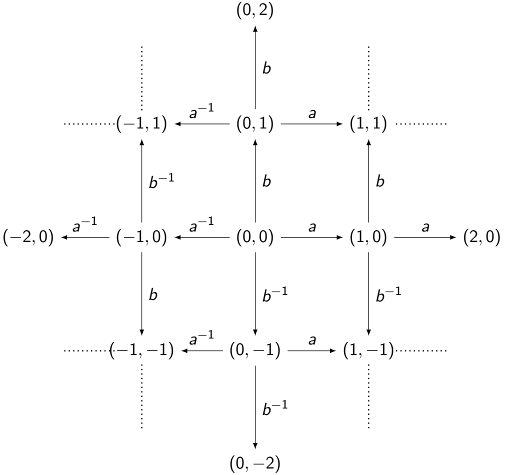
Cayley Graph: $\Gamma(G,S)$ $ex: \mathbb{Z}^2,\ \{a=(0,1), b=(1,0)\}$

$g \xrightarrow{s} h \qquad g.s = h,\ g,h \in G,\ s \in S$

Growth

Cayley Graph: $\Gamma(G, S)$ $ex : \mathbb{Z}^2, \ \{a = (0,1), b = (1,0)\}$

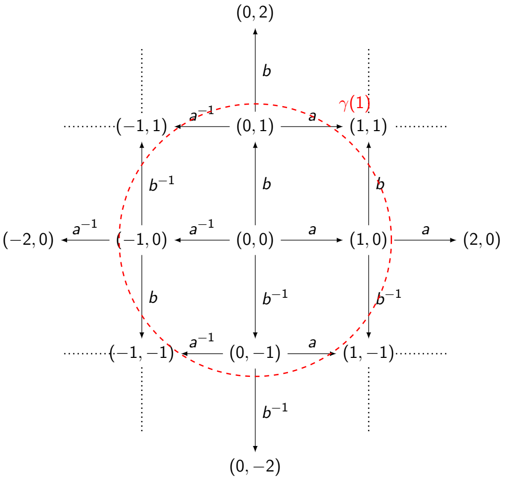
$g \xrightarrow{s} h \qquad g.s = h, \ g, h \in G, \ s \in S$



Growth

Cayley Graph: $\Gamma(G, S)$ $ex : \mathbb{Z}^2,\ \{a = (0,1), b = (1,0)\}$

$g \xrightarrow{s} h$ $\quad g.s = h,\ g, h \in G,\ s \in S$



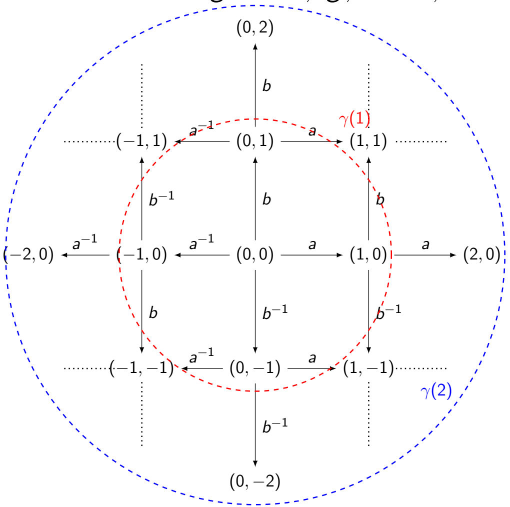
$\gamma(0) = 1$

$\gamma(1) = 5$

Growth

Cayley Graph: $\Gamma(G,S)$ $ex : \mathbb{Z}^2,\ \{a=(0,1), b=(1,0)\}$

$g \xrightarrow{s} h$ $g.s = h,\ g,h \in G,\ s \in S$

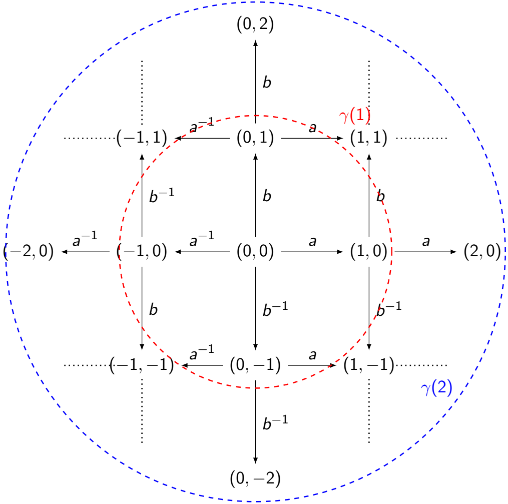


$\gamma(0) = 1$
$\gamma(1) = 5$
$\gamma(2) = 13$

Growth

Cayley Graph: $\Gamma(G, S)$ $ex : \mathbb{Z}^2,\ \{a = (0,1), b = (1,0)\}$

$g \xrightarrow{s} h$ $\quad g.s = h,\ g, h \in G,\ s \in S$



$\gamma(0) = 1$

$\gamma(1) = 5$

$\gamma(2) = 13$

$\vdots$

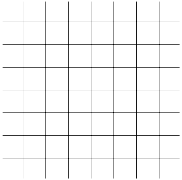
$\gamma(n) = 2n^2 - 2n + 1$

Milnor's Problem

- growth bounded: finite groups

Milnor's Problem

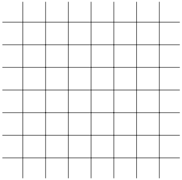
- growth bounded: finite groups
- polynomial growth: $\mathbb{Z}^d$, Abelian groups



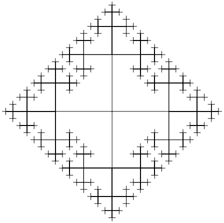
$\Gamma(\mathbb{Z}^2)$

Milnor's Problem

- growth bounded: finite groups
- polynomial growth: $\mathbb{Z}^d$, Abelian groups
- exponential growth: $\mathbb{F}_d$



$\Gamma(\mathbb{Z}^2)$



$\Gamma(\mathbb{F}_2)$

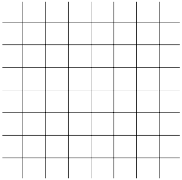
Milnor's Problem

- growth bounded: finite groups
- polynomial growth: $\mathbb{Z}^d$, Abelian groups

Milnor's Problem (1968):

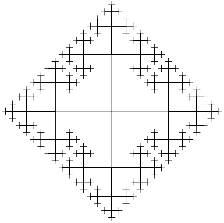
Do groups with growth growth between polynomial and exponential exist?

- exponential growth: $\mathbb{F}_d$



$\Gamma(\mathbb{Z}^2)$

$\Gamma(?)$



$\Gamma(\mathbb{F}_2)$

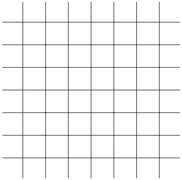
Milnor's Problem

- growth bounded: finite groups
- polynomial growth: $\mathbb{Z}^d$, Abelian groups

Milnor's Problem (1968):

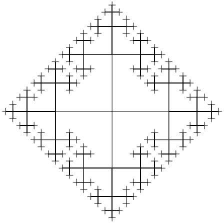
Do groups with growth growth between polynomial and exponential exist?
1983 (Grigorchuk) Yes, automaton-generated example

- exponential growth: $\mathbb{F}_d$



$\Gamma(\mathbb{Z}^2)$

$\Gamma(?)$



$\Gamma(\mathbb{F}_2)$

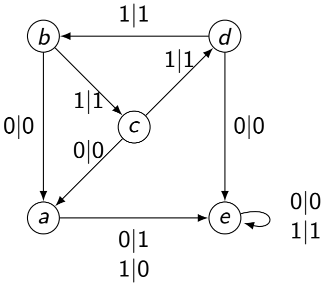
Milnor's Problem

- growth bounded: finite groups
- polynomial growth: $\mathbb{Z}^d$, Abelian groups

Milnor's Problem (1968):

Do groups with growth growth between polynomial and exponential exist?
1983 (Grigorchuk) Yes, automaton-generated example

- exponential growth: $\mathbb{F}_d$



Torsion

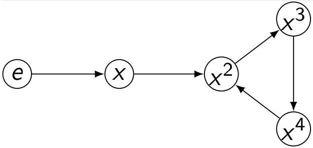
Torsion Element

$x \in S$ is torsion (finite order) if $\exists n \neq p \geq 1, x^n = x^p$

Torsion

Torsion Element

$x \in S$ is torsion (finite order) if $\exists n \neq p \geq 1, x^n = x^p$

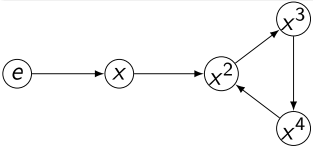


$x^2 = x^5$, x is torsion

Torsion

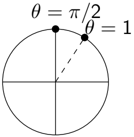
Torsion Element

$x \in S$ is torsion (finite order) if $\exists n \neq p \geq 1, x^n = x^p$



$x^2 = x^5$, x is torsion

- $\mathbb{Z}/n\mathbb{Z}$ is torsion (all its elements have finite order)
- $\mathbb{N}$ is torsion-free (0 is the only element of finite order)
- On the circle $\mathbb{R}/2\pi\mathbb{Z}$; $\pi/2$ has finite order but 1 has infinite order



Burnside problem

Burnside Problem (1902):

Can a finitely generated group be torsion and infinite?

Burnside problem

Burnside Problem (1902):

Can a finitely generated group be torsion and infinite?
1964 (Golod and Shafarevich) Yes!

Burnside problem

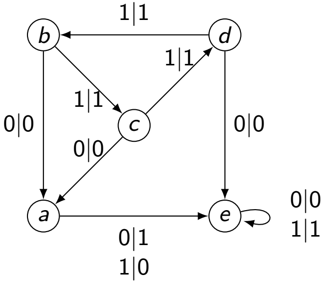
Burnside Problem (1902):

Can a finitely generated group be torsion and infinite?
1964 (Golod and Shafarevich) Yes!
1980 (Grigorchuk) an automaton-generated example

Burnside problem

Burnside Problem (1902):

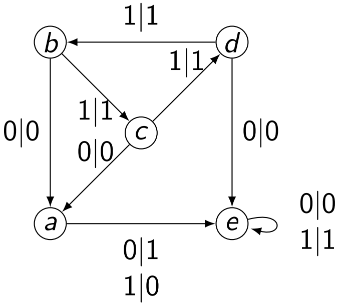
Can a finitely generated group be torsion and infinite?
1964 (Golod and Shafarevich) Yes!
1980 (Grigorchuk) an automaton-generated example



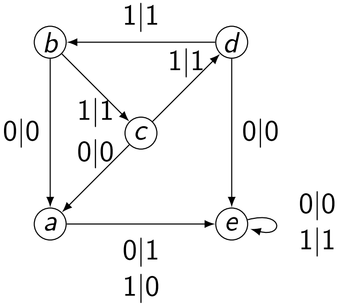
Grigorchuk automaton

What

About Grigorchuk automaton



About Grigorchuk automaton



Action of a state on the letters:

$\rho_a : 0 \mapsto 1 \mapsto 0$

$\rho_b : 0 \mapsto 0; \quad 1 \mapsto 1$

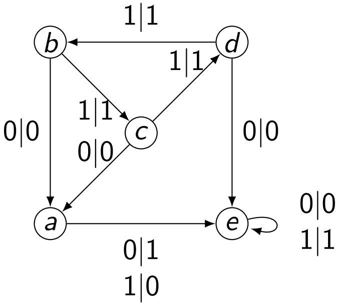
$\rho_c : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_d : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_e : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rightarrow$permutations

About Grigorchuk automaton



Action of a state on the letters:

$\rho_a : 0 \mapsto 1 \mapsto 0$

$\rho_b : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_c : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_d : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_e : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rightarrow$permutations

Action of a letter on the states:

$\delta_0 : a, d, e \mapsto e; \quad b, c \mapsto a$

$\rightarrow$not a permutation

Structure properties

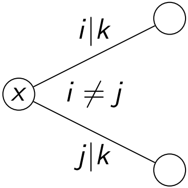
Invertibility:
Each state induces a permutation of the letters

Reversibility:
Each input letter induces a permutation of the stateset

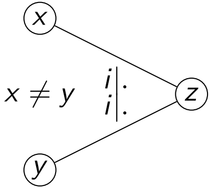
Coreversibility:
Each output letter induces a permutations of the stateset

Structure properties

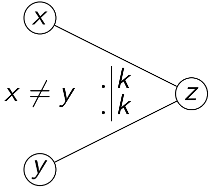
Invertibility:
Each state induces a permutation of the letters



Reversibility:
Each input letter induces a permutation of the stateset

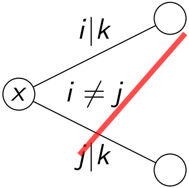


Coreversibility:
Each output letter induces a permutations of the stateset

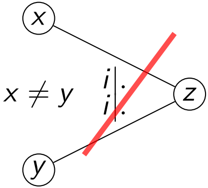


Structure properties

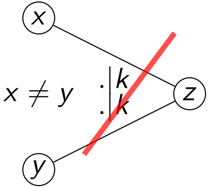
Invertibility:
Each state induces a permutation of the letters



Reversibility:
Each input letter induces a permutation of the stateset



Coreversibility:
Each output letter induces a permutations of the stateset



Fact

Every known automaton generating an infinite Burnside group happen to be non-reversible

Question

Can an (invertible) reversible automaton generate an infinite Burnside group?

Theorem 1 [Klimann 2013]

A 2-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

Theorem 1 [Klimann 2013]

A 2-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

Theorem 2 [Klimann, Picantin and Savchuk 2014]

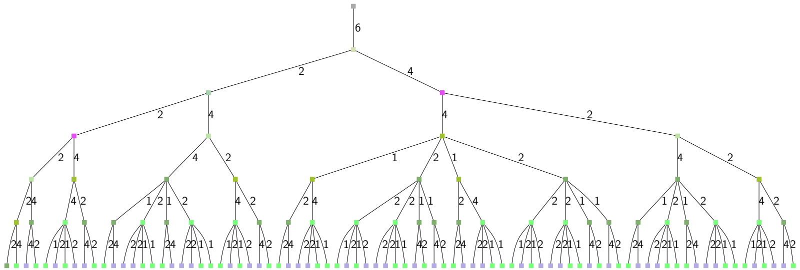
A connected 3-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

Theorem 1 [Klimann 2013]

A 2-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

Theorem 2 [Klimann, Picantin and Savchuk 2014]

A connected 3-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group



Theorem 1 [Klimann 2013]

A 2-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

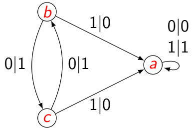
Theorem 2 [Klimann, Picantin and Savchuk 2014]

A connected 3-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

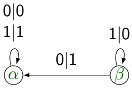
Theorem [G., Klimann and Picantin 2015]

An invertible-reversible Mealy automaton without coreversible component cannot generate an infinite Burnside group

Product of automata

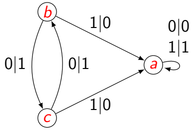


×

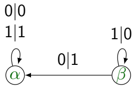


=

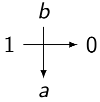
Product of automata



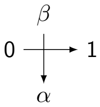
×



=

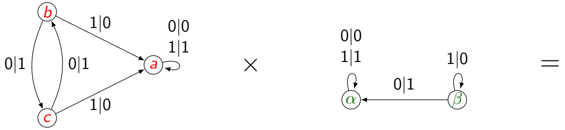


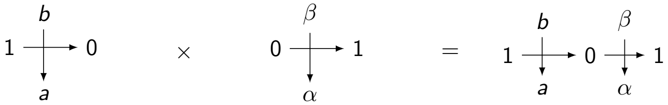
×



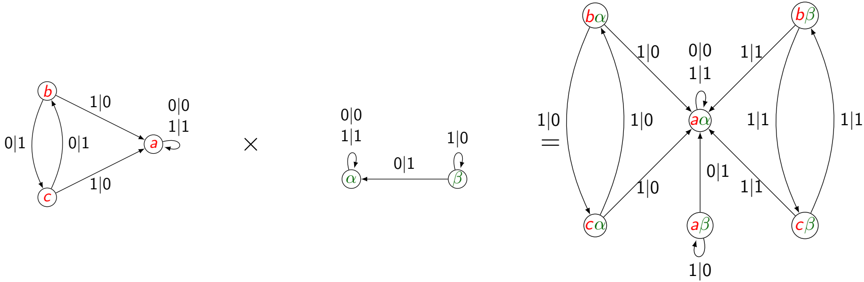
=

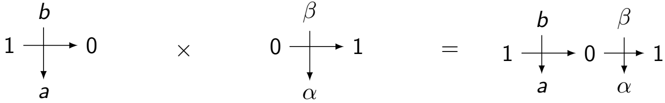
Product of automata





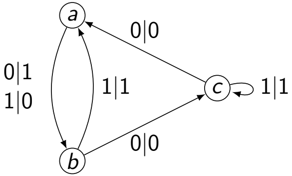
Product of automata





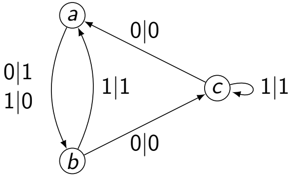
Orbit Tree

$\mathcal{A}$

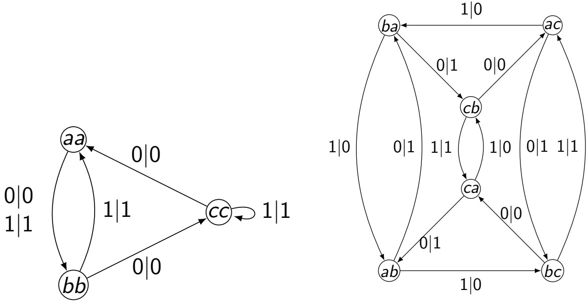


Orbit Tree

$\mathcal{A}$



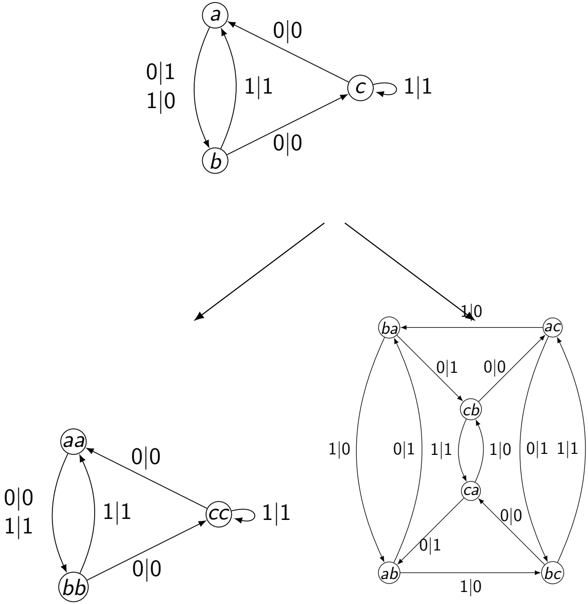
$\mathcal{A}^2$



Orbit Tree

$\mathcal{A}$

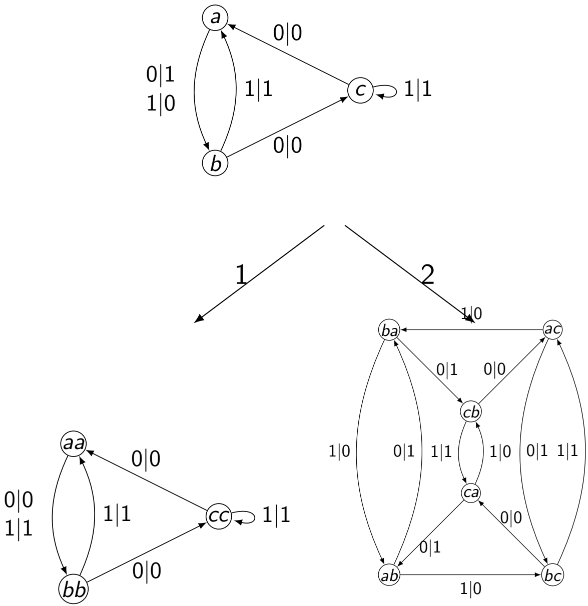
$\mathcal{A}^2$



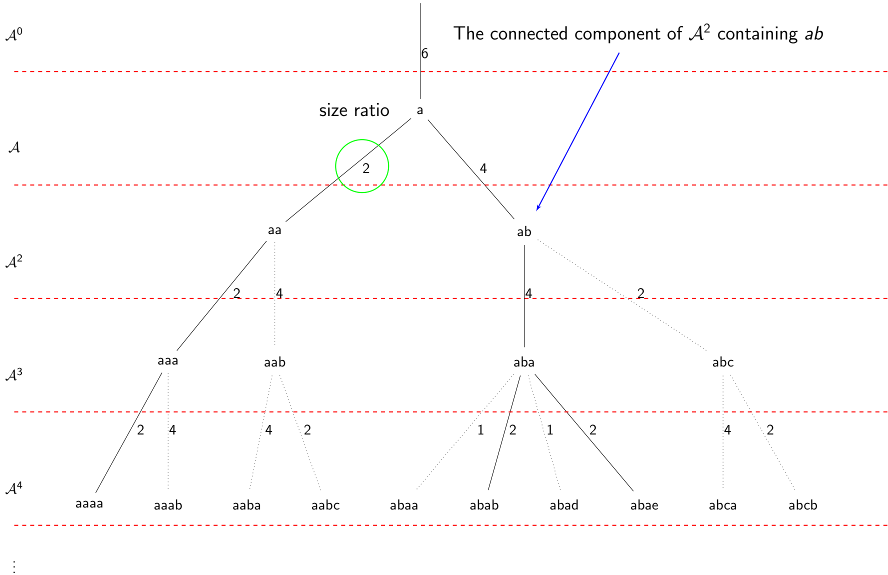
Orbit Tree

$\mathcal{A}$

$\mathcal{A}^2$



Orbit Tree



$\mathcal{A}^0$
$\mathcal{A}$
$\mathcal{A}^2$
$\mathcal{A}^3$
$\mathcal{A}^4$
⋮
6
The connected component of $\mathcal{A}^2$ containing *ab*
size ratio
a
2
4
aa
ab
2
4
4
2
aaa
aab
aba
abc
2
4
4
2
1
2
1
2
4
2
aaaa
aaab
aaba
aabc
abaa
abab
abad
abae
abca
abcb

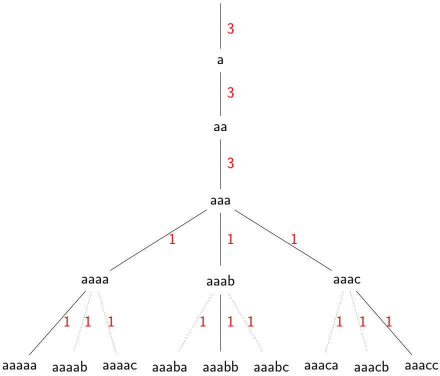
Boundedness

Prop

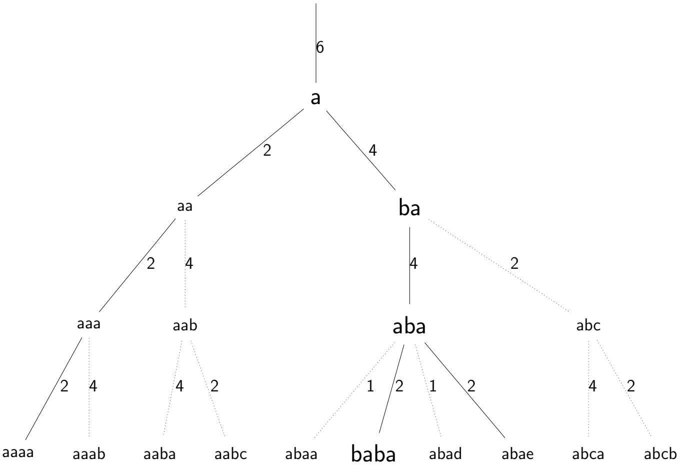
$\langle \mathcal{A} \rangle_+$ is finite iff the sizes of the connected components of $(\mathcal{A}^n)_n$ are bounded

Prop

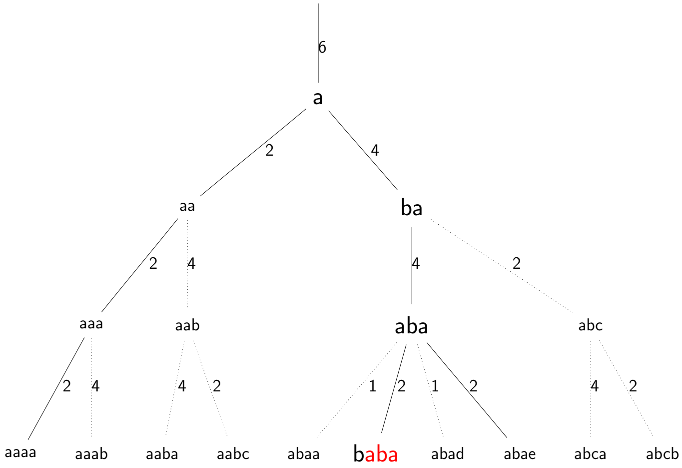
ρ_u has finite order iff the sizes of the connected components of $(\mathcal{A}^n)_n$ containing u^n are bounded



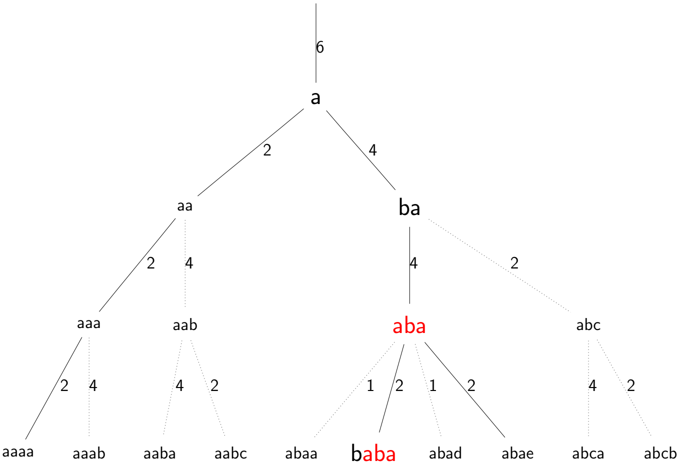
1-liftable paths



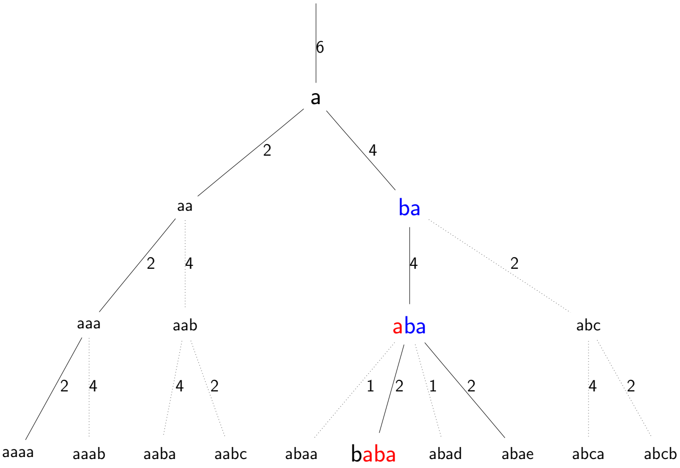
1-liftable paths



1-liftable paths

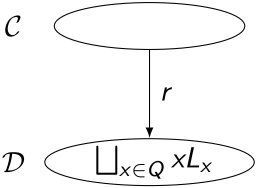


1-liftable paths



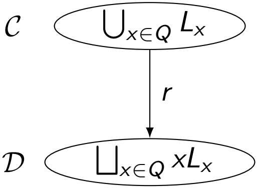
Torsion Elements

if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$



Torsion Elements

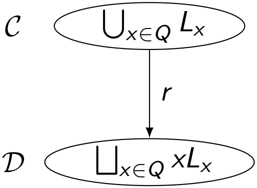
if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$

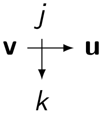


Torsion Elements

if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$

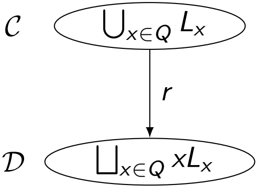
Invertibility



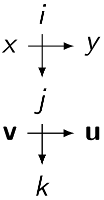


Torsion Elements

if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$

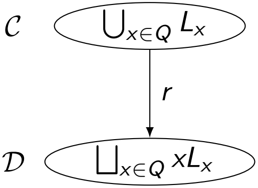


Invertibility

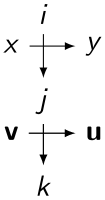


Torsion Elements

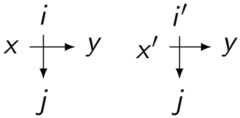
if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$



Invertibility

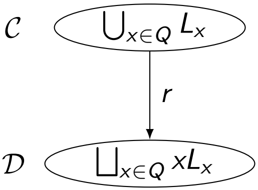


non Bireversibility

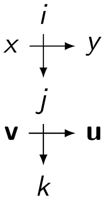


Torsion Elements

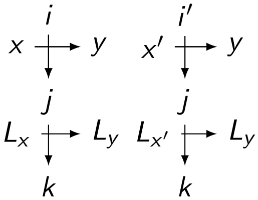
if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$



Invertibility

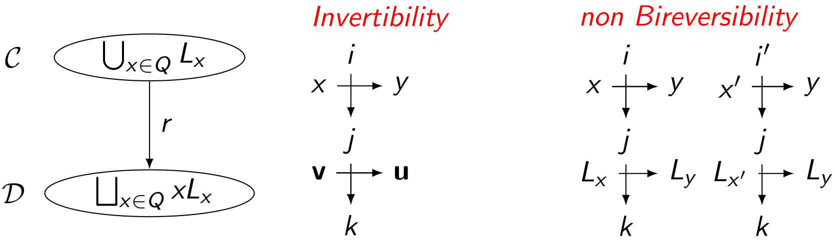


non Bireversibility



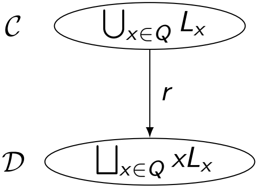
Torsion Elements

if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$



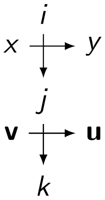
Torsion Elements

if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$

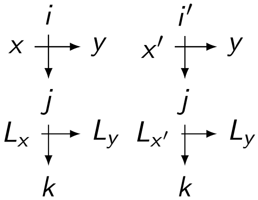


Hence $L_x = L_{x'}$

Invertibility

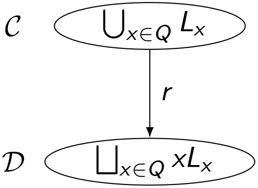


non Bireversibility

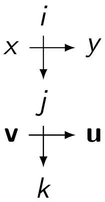


Torsion Elements

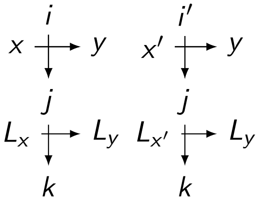
if $\mathbf{u} \in \mathcal{C}$ then $\exists \mathbf{v} \in \mathcal{C}$ s.t. $x\mathbf{v} \in \mathcal{D}$



Invertibility

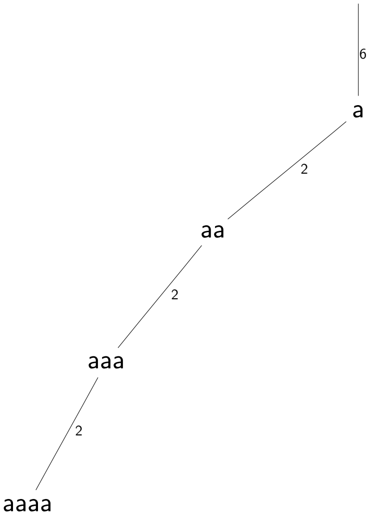


non Bireversibility



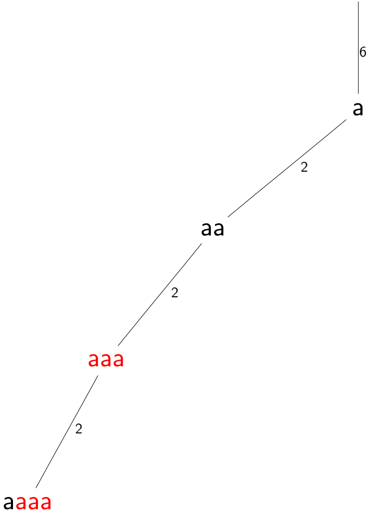
Hence $L_x = L_{x'} \;\Rightarrow |\mathcal{D}| > |\mathcal{C}|$

1-liftable paths exist



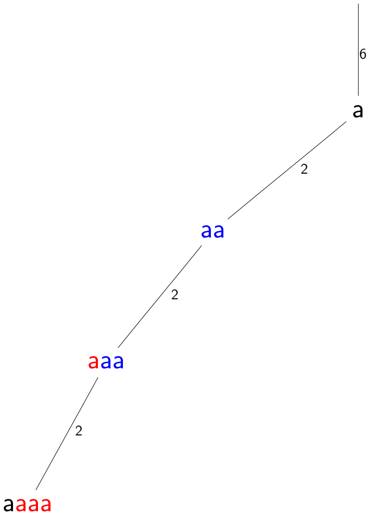
$(a^n)_n$ is on a 1-self-liftable path

1-liftable paths exist



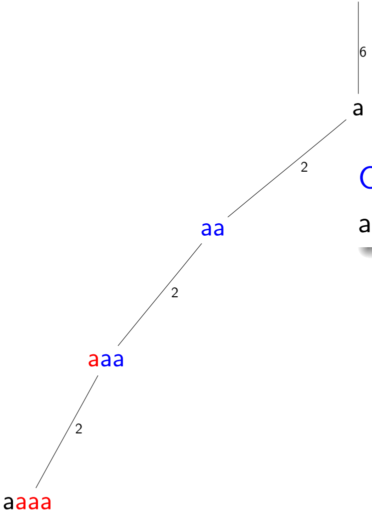
$(a^n)_n$ is on a 1-self-liftable path

1-liftable paths exist



$(a^n)_n$ is on a 1-self-liftable path

1-liftable paths exist

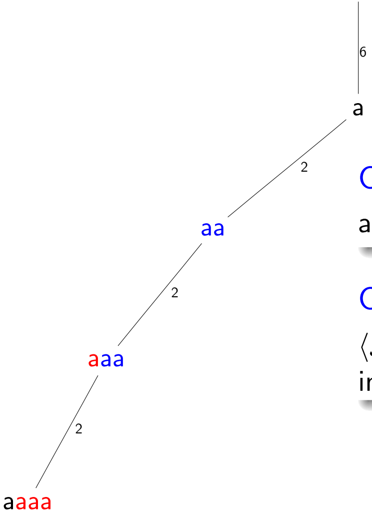


Corollary

any state ρ_a has infinite order

$(a^n)_n$ is on a 1-self-liftable path

1-liftable paths exist



Corollary

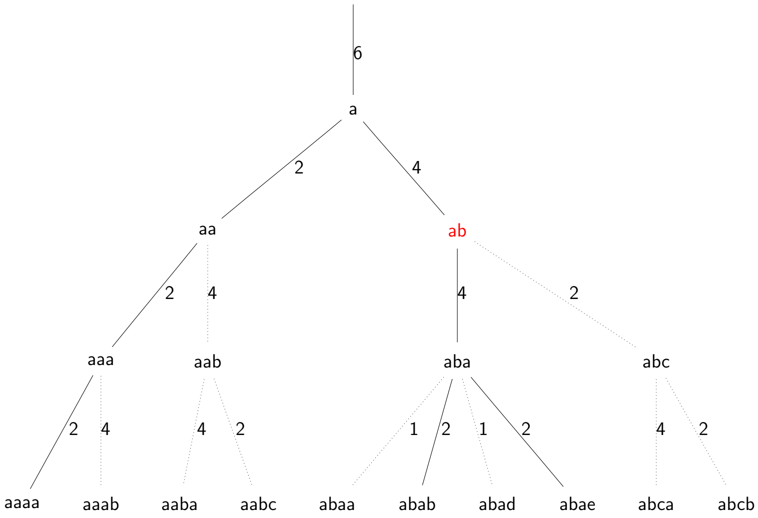
any state ρ_a has infinite order

Corollary

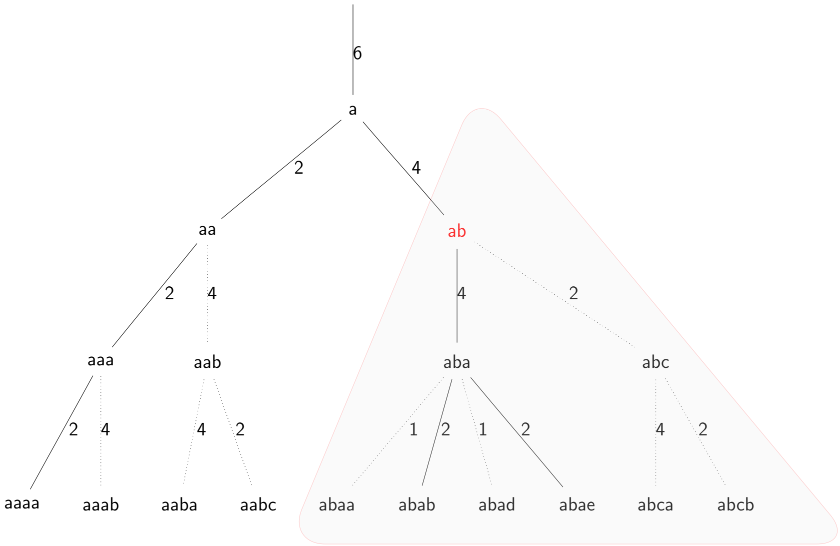
$\langle \mathcal{A} \rangle$ cannot be Burnside infinite.

$(a^n)_n$ is on a 1-self-liftable path

Every element has infinite order



Every element has infinite order



Theorem [G., Klimann and Picantin 2015]

An invertible-reversible Mealy automaton without coreversible component generates a torsion-free semigroup.

- Extend the result to coreversible automata
- Decide the finiteness for coreversible automata
- Find other structural properties