Université
Paris Cité

INDEXED AND FIBERED PRESENTATIONS OF PRESHEAVES IN TYPE THEORY

And their equivalence via the Grothendieck construction

Moana Laurent JUBERT

Advisor :

Hugo HERBELIN

M2 de Mathématiques fondamentales

Septembre 2022

Abstract

There are two "canonical" ways to define presheaves in type theory: with an *indexed* presentation, or with a *fibered* one. Both presentations should be of course equivalent, and the exact nature of this equivalence appears to be related to a well-known result of category theory, the *Grothendieck construction*. This master's thesis aims at presenting all the basic tools necessary to understand this equivalence in the special case of direct categories.

Contents

Introduction

The initial goal of my intership was to investigate the Grothendieck construction in type theory. Hugo Herbelin have been working for quite some time on the various ways to implement presheaves in type theory, most notably semi-simplicial types [4]. He presented to me the observation that there are at least two "canonical" ways to define presheaves: with an *indexed presentation*, or with a *fibered* one. The correspondence between both appears to be tightly related with the Grothendieck construction.

As I come from more "classical" areas of mathematics I was unfamiliar with type theory, and in the end I have spent a considerable amount of my time trying to find a similar distinction on the presentation of presheaves directly at the level of category theory — in the hopes of constructing a clearer picture in my mind. Even though I did not succeed, I have learned a lot on the Grothendieck construction and type theory over the course of my three months at the Institut de recherche en informatique fondamentale (IRIF). In the end, this internship took the turn of making general conjectures about the indexed and fibered presentations of presheaves.

This text is divided in four parts of roughly the same size. I first define presheaves, and make a series of general statements about them. I then move on to introducing the Grothendieck construction in category theory, in its simplest versions. Next, I explain what the indexed and fibered presentations of presheaves are in type theory, in the case of direct categories. I finally present some unfinished work on a variant of the indexed presentation that I call "Yoneda-esque" for its "family resemblance" with the Yoneda lemma, and formulate a number of open questions.

Although it is barely mentioned in this work, *homotopy type theory* (HoTT) had a special place in my considerations. HoTT is a recent paradigm shift in type theory, with the observation that the equality type $x =_A y$ can be thought as the type of "all paths from x to y". Higher up, $p =_{x=_A y} p'$ can be thought as the type of "all continuous deformations of the path p into the path p'", and so on. In short, types such as A are interpreted as being ∞-*groupoids* (see [11] for more). As a consequence, what would be interesting is how the higher structure would interact with the different presentations and their correspondence, and how what we already know in higher category theory could help us understand what would be a reasonable construction of presheaves in HoTT.

Acknowledgements

I would like to thank my advisor Hugo Herbelin for his kind support, and for guiding me during my three months as an intern. I would also like to thank Daniel de Rauglaudre with whom I have had the pleasure of being his last "co-bureau", as he went into retirement at the end of my stay.

1 Presheaves

Presheaves are a key ingredient of category theory. It is important to get them right, as I will make heavy use of presheaves thereafter.

I assume the reader is already familiar with the most basic notions of category theory (e.g. definition, functors, limits and colimits, ...), otherwise a good introduction written by Emily Riehl is available at [13].

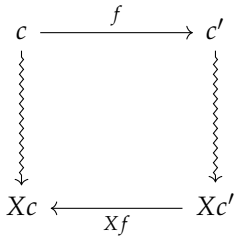
1.1 Definition

DEFINITION 1. A *presheaf* over some category $\mathcal{C}$ is the data of a functor X from the opposite category $\mathcal{C}^{\mathsf{op}}$ to the category of (small) sets Set:

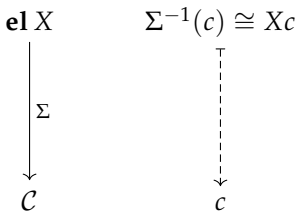
$$X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

It is standard to write Xc instead of $X(c)$ when applying the presheaf X to some object $c \in \mathcal{C}$. One can think of X as a "generalized set", where each element is labeled with an object of $\mathcal{C}$. In this case, Xc is the set of elements labeled with c.

As X is also a functor, given a morphism $f : c \longrightarrow c'$ of $\mathcal{C}$, the corresponding map Xf sends elements labeled with c' to elements labeled with c:



Thus if presheaves are indeed generalized sets, they additionally reflect the structure of the base category $\mathcal{C}$ in a sense that will be soon made precise. A different, albeit completely equivalent point of view is to think of $\mathcal{C}$ as the base space of some projection Σ such that the *fibers* are exactly the labeled elements:



This, again, will be made precise in a moment. The domain $\mathbf{el}\ X$ is called the *category of elements of* X, and every pair $(\mathbf{el}\ X, \Sigma)$ is uniquely associated to a presheaf X. This bijective correspondence is already a special case of the *Grothendieck construction*, as we will see in the next section.

The collection of presheaves over a category $\mathcal{C}$ forms a well-behaved category usually noted $\mathsf{PSh}(\mathcal{C})$, or sometimes $\widehat{\mathcal{C}}$. Morphisms are the natural transformations between presheaves, and it is both complete and cocomplete. A mildly nontrivial fact is that when $\mathcal{C}$ is locally small, $\mathsf{PSh}(\mathcal{C})$ contains a copy of the original category $\mathcal{C}$ through the *Yoneda embedding*, which we will now see.

1.2 Yoneda lemma

Let us assume that $\mathcal{C}$ is locally small, that is, for any two objects $c, c' \in \mathcal{C}$ the collection $\mathcal{C}(c, c')$ of morphisms from c to c' is a (small) set. This gives us a well-defined hom functor into the category of sets:

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$$

It acts *contravariantly* on the left as pre-composition is contravariant, while post-composition is covariant. Another — perhaps obnoxious — way to formulate this is to say that $\mathcal{C}$ is a Set-*enriched category*. Now given any object $c \in \mathcal{C}$, we have:

$$\mathcal{C}(-, c) : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

i.e. to every object of a locally small category, we can associate a canonical presheaf. This is the definition of the Yoneda embedding:

DEFINITION 2. The map $Y : \mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{C})$ defined as:

$$Y(c) \stackrel{\text{def}}{=} \mathcal{C}(-, c)$$

is a well-defined functor, called the *Yoneda embedding*.

Again, it is standard to write Yc, or even $\widehat{c}$ instead of $Y(c)$. If a presheaf X is (naturally) isomorphic to some $\widehat{c}$ it is said to be *representable*, and we say that X is *represented* by c. The name comes from the fact that it is enough to know how to "probe" the object c — i.e. to know $\mathcal{C}(-, c)$ — in order to know X. In some sense, c then "encodes" everything there is to know about the shape of X.

Of course, most presheaves are not representable. Yet, their structure is completely determined by their relation to the representable ones. This is the

statement of the so-called "Yoneda lemma":

LEMMA 1 (Yoneda). $\mathrm{Hom}(\widehat{c}, X) \cong Xc$

Proof. We only have to unfold the definition of a natural transformation from $\widehat{c}$ to X. The data of a natural transformation $\gamma \in \mathrm{Hom}(\widehat{c}, X)$ is the data of a family of morphisms $\gamma_d : \widehat{c}(d) \longrightarrow Xd$ in Set such that for any arrow $f : d \longrightarrow d'$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{c}(d') & \xrightarrow{\gamma_{d'}} & Xd' \\ {\scriptstyle \widehat{c}(f)}\downarrow & & \downarrow{\scriptstyle Xf} \\ \widehat{c}(d) & \xrightarrow[\gamma_d]{} & Xd \end{array}$$

The identity $1_c \in \mathcal{C}(c, c)$ is sent to an element of Xc through γ_c. I now claim that γ as a whole is completely determined by $\gamma_c(1_c)$. For any arrow $f : d \longrightarrow c \in \widehat{c}(d)$, we have:

$$\begin{array}{ccc} \mathcal{C}(c, c) & \xrightarrow{\gamma_c} & Xc \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle Xf} \\ \mathcal{C}(d, c) & \xrightarrow[\gamma_d]{} & Xd \end{array}$$

where f^* is defined as the pre-composition by f. We then conclude by observing that $\gamma_d(f) = Xf(\gamma_c(1_c))$. ∎

By analogy, one could argue that a natural transformation γ from $\widehat{c}$ to X is some sort of "linear map" with $\gamma_c(1_c)$ as its coefficient. We will see it again when discussing the "Yoneda trick" in type theory. For now, what the lemma states is that you only need to look at the "incoming arrows" of a presheaf to know everything about it:

$$\widehat{c} \longrightarrow X$$

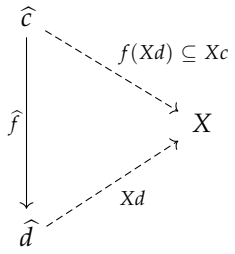
With the "generalized sets" point of view, it amounts to picking exactly one element labeled with c from X. This is not unlike the usual intuition in Set, where looking at the elements of a set A, or at the arrows $* \longrightarrow A$ from any

singleton to A, is exactly the same thing. In fact, it is quite easy to show that the category Set is isomorphic to $\mathsf{PSh}(*)$.

I claimed earlier that $\mathsf{PSh}(\mathcal{C})$ contains a copy of the original category $\mathcal{C}$. This is an immediate consequence of the Yoneda lemma:

LEMMA 2. The Yoneda embedding is fully faithful.

Indeed, for any two objects $c, d \in \mathcal{C}$ we have that $\mathrm{Hom}(\widehat{c}, \widehat{d}) \cong \mathcal{C}(c, d)$ by the Yoneda lemma. Thus Y truly is an *embedding* of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$. As presheaves are completely determined by the arrows "coming out of" $Y(\mathcal{C})$ into them, the reader should now easily see how the structure of $\mathcal{C}$ and the "internal" structure of a presheaf are tightly related:



One could make the observation that, since $\mathsf{PSh}(\mathcal{C})$ is cocomplete, a family of arrows from $\widehat{c}$ to X is the same as a single arrow from the coproduct $\coprod \widehat{c}$ to X:

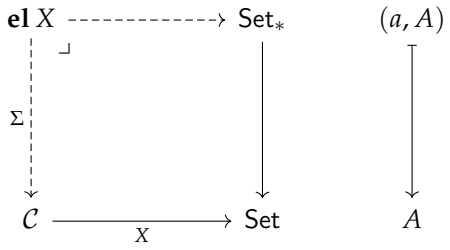
More generally, if we were to "break X down" into its arrows $\widehat{c} \longrightarrow X$ for every object $c \in \mathcal{C}$ instead of a fixed one, we would get a canonical arrow from a special colimit into X. The shape of this colimit is the aforementioned *category of elements of* X, and as I said before, knowing about X is the same as knowing about its elements, so it should come as no surprise that X is actually isomorphic to this colimit.

1.3 Density theorem

DEFINITION 3. The *category of elements of* X, noted $\mathbf{el}\, X$, is the data of objects (c, x) for every $x \in Xc$, together with morphisms $f : (c, x) \longrightarrow (c', y)$ for every $f : c \longrightarrow c'$ of $\mathcal{C}$ such that $Xf(y) = x$.

$\mathbf{el}\, X$ contains all the elements of X paired with their appropriate label, hence the name "category of elements". There is a canonical projection $\Sigma : \mathbf{el}\, X \longrightarrow \mathcal{C}$ such that $\Sigma(c, x) \stackrel{\text{def}}{=} c$. Then we indeed have that $\Sigma^{-1}(c) \cong Xc$ by definition.

The definition of $\mathbf{el}\, X$ may prompt us to look for a universal property. If Set_* is the category of pointed sets (i.e. pairs (a, A) with $a \in A$), then $\mathbf{el}\, X$ can be seen as a subcategory of the product $\mathcal{C} \times \mathsf{Set}_*$ with objects $(c, (x, Xc))$. This is exactly the data of a *pullback square* along X (with the appropriate variance) and the evident projection from Set_* to Set:



A remarkable fact of Σ is that because X is a presheaf, for any $y \in Xc'$ and any morphism $f : c \longrightarrow c'$ of $\mathcal{C}$, the arrow $(Xf(y), c) \longrightarrow (y, c')$ is the *unique* lift of f into $\mathbf{el}\, X$. Maps such as Σ with this property are called *discrete right fibrations*. As I hinted earlier, this is a first illustration of the Grothendieck construction: *any discrete right fibration is equivalent to the data of a presheaf*.

But for now, we are still only looking at presheaves. The category of elements tells us exactly how the pieces fit together:

THEOREM 1 (Density). If $\mathcal{C}$ is also small, then X is isomorphic to the colimit of the projection Σ composed with the Yoneda embedding:

$$\varinjlim Y\Sigma \cong X$$

This is usually called the "density theorem". One way to understand the name is to think of the image of $\mathcal{C}$ through the Yoneda embedding as "being dense", since any presheaf is actually a colimit of representables. The size

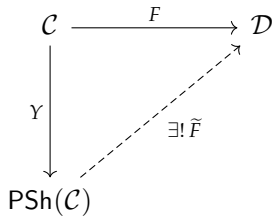
condition on $\mathcal{C}$ is necessary to make sure that the colimit is small. An idea of proof is the following: glue as many disjoint copies of $\widehat{c}$'s as there are in each Xc, then identify what should be equal under the action of the Xf's. This is trivial to state using *coends*, but it would require us to make an important détour in the realm of abstract nonsense. In any case, an introduction to coends is available in [12] and a complete proof of the previous theorem using *Kan extensions* instead is available in [13, p. 212].

The somehow "converse" statement is also true: $\mathsf{PSh}(\mathcal{C})$ is the "most general" category containing a copy of $\mathcal{C}$ and having all (small) colimits, i.e.:

THEOREM 2. $\mathsf{PSh}(\mathcal{C})$ is the *free (small) cocompletion* of $\mathcal{C}$, that is for any functor:

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

into a *cocomplete* category $\mathcal{D}$, there exists an extension:



which is *unique* up to isomorphism.

The proof is more technical, and again can be found in [13, p. 213] as a remark. What is more interesting is that this theorem provides a completely different characterization of presheaves by a universal property, which might be worth exploring in type theory.

1.4 Semi-simplicial sets

I have presented some of the general properties of presheaves so far. I will now focus on the specifics of *semi-simplicial sets* (also sometimes called Δ-*sets* in the literature).

Simplicial sets (without the semi-) encode the ways to glue n-simplices together along their faces in order to build arbitrarily complex shapes. They are "rich" enough to entirely capture the "algebraic" data of (nice) topological spaces as well as the higher structure of categories, up to equivalence. Topological

and homotopical aspects of simplicial sets are extensively covered in [10] or [3], while their use as a model for $(\infty, 1)$-categories is covered in [9].

Semi-simplicial sets provide a more "rigid" (thus less powerful) approach to simplicial sets, with the benefit of being easier to define and manipulate. As we are eventually interested in constructing semi-simplicial *types*, different (albeit equivalent) definitions will yield different constructions in the end.

1.4.1 Equivalent definitions

NOTATION. We write $[n]$ for the nonempty finite set $\{0, \dots, n\}$.

DEFINITION 4. The *semi-simplex category* Δ_+ is defined as having all the sets $[n]$ as objects, and all the *strictly increasing* maps between them as morphisms. A *semi-simplicial set* is then a presheaf $X : \Delta_+^{\mathsf{op}} \longrightarrow \mathsf{Set}$.

Notice that all the arrows can only go from an object $[n]$ to another $[m]$ with $n < m$, except for the identity morphisms. Δ_+ is an example of what is called a *direct category*, and we will soon study them in more details.

The category of semi-simplicial sets is noted SSSet, and is not to be confused with the category of simplicial sets, similarly noted sSet. The main difference between the two is that we allow ourselves to look at *nonstrictly* increasing maps as well in the definition of the latter. Now, for the main result:

THEOREM 3. The following characterizations of SSSet are all equivalent:

i) A semi-simplicial set is a presheaf $X : \Delta_+^{\mathsf{op}} \longrightarrow \mathsf{Set}$.

ii) A semi-simplicial set is the data of a family $(X_n)_{n\in\mathbb{N}}$ of sets together with maps $d_i : X_n \longrightarrow X_{n-1}$ such that $0 \leqslant i \leqslant n$ called *face maps*, which satisfy the relation:

$$d_i d_j = d_{j-1} d_i \qquad \text{when } i < j$$

iii) Strictly increasing maps $[n] \longrightarrow [m]$ are in bijection with the subsets of $[m]$ of cardinal $n+1$, thus giving a different definition of Δ_+.

iv) A semi-simplicial set is the data of a discrete right fibration:

$$\Sigma : \mathcal{E} \longrightarrow \Delta_+$$

v) SSSet is the free (small) cocompletion of Δ_+.

Point ii) is actually the standard approach to defining semi-simplicial sets. The category Δ_+ is generated by *coface maps* $d^i : [n] \longrightarrow [n+1]$ such that d^i is the only strictly increasing map which "misses" i in its image. Unsurprisingly, the coface maps also satisfy the dual relation:

$$d^j d^i = d^i d^{j-1} \qquad \text{when } i < j$$

What is interesting is that it gives us some kind of description of Δ_+ by means of *generators* and *relations*, as indeed the following (easy) lemma holds:

LEMMA 3. Any strictly increasing map $[n] \longrightarrow [m]$ can be canonically written as a composition of d^i's.

Points i) and iii) are essentially different sets of generators and relations for the category Δ_+. One might then be tempted to say that Δ_+ is "presentable" just as the "presentation" of a group encapsulates the corresponding notion in group theory, but unfortunately the term is already used by category theorists for a distinct (although similar) concept. Instead, a category theorist would rather say that Δ_+ is a "2-*polygraph*" (or a "2-*computad*" if they are Australian). I will present the full definition of polygraphs in a later section.

The general idea is this: if we have distinct but equivalent definitions of a category using generators and relations, then from a constructive point of

view we should see distinct but equivalent constructions of presheaves. These distinctions "vanish" in classical mathematics, while they are unavoidable in type theory. The primary goal of this thesis is precisely to better understand these unavoidable distinctions.

Before ending this first section, I will now go back to point ii) and elaborate on the "geometric" aspect of face maps.

1.4.2 Geometric intuition

NOTATION. As $\widehat{[n]}$ is pretty much unreadable, we write Δ^n_+ for the representable functors of SSSet instead.

DEFINITION 5. Δ^n_+ is called the (*standard*) *n*-(*semi-*)*simplex*.

Simplices are "generalized triangles in higher dimensions". Their direct counterpart in topology are the sets:

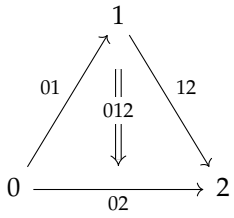
$$\{\,(x_1,\ldots,x_n) \in \left(\mathbb{R}^+\right)^n \mid x_1 + \cdots + x_n \leqslant 1\,\}$$

Any map $\Delta^n_+ \longrightarrow X$ can really be thought as picking a single n-simplex from the space X. Simplices themselves are made of smaller simplices. If we write $\alpha_0\alpha_1\ldots\alpha_n$ the strictly injective map from $[n]$ to some $[m]$ such that:

$$\alpha_0 < \alpha_1 < \cdots < \alpha_n \in [m]$$

Then we can easily see that the "filled" triangle Δ^2_+ is made up of three vertices, three segments , and one surface:

$$\begin{aligned}
(\Delta^2_+)_0 &= \{0,1,2\}\\
(\Delta^2_+)_1 &= \{01,12,02\}\\
(\Delta^2_+)_2 &= \{012\}\\
(\Delta^2_+)_3 &= \varnothing\\
(\Delta^2_+)_4 &= \varnothing\\
&\vdots
\end{aligned}$$



The face maps d_i really select the *faces* of the simplex, as for example:

$$d_2(012) = 01 \qquad d_1(12) = 1 \qquad d_1(01) = 0$$

i.e. vertices, segments, etc. Semi-simplicial sets are then made of simplices which are glued along common faces indicated by the face maps.

REMARK. One way of building a 2-sphere is to contract all the edges of a triangle to a point, like a rubber balloon that you would seal at its end.

This is impossible to describe with semi-simplicial sets as the gluing can only occur on faces *of matching dimension*: we cannot contract a segment to a single point.

Simplicial sets (without the semi-), by contrast, allow for this sort of "compact" constructions at the cost of being even more difficult to use.

2 Grothendieck construction

The Grothendieck construction relates functors $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathcal{U}$ into a sufficiently nice category $\mathcal{U}$ (e.g. Set or Cat) with "well-behaved" projections. We already have seen an example of it in the previous section, with the correspondence:

$$X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set} \qquad \simeq \qquad \Sigma : \mathbf{el}\, X \longrightarrow \mathcal{C}$$

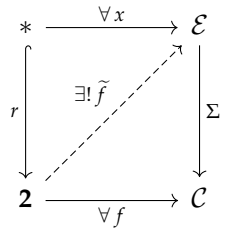
It is essentially equivalent to either look for the list of objects of a given label, or given an object look its label up. If we look at presheaves $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ instead, the correspondence in fact still holds — provided the fibers of the projection now additionally has the structure of a category.

2.1 Discrete version

Let $\mathcal{E}$ and $\mathcal{C}$ be small categories.

DEFINITION 6. A *discrete right fibration* (over $\mathcal{C}$) is a functor $\Sigma : \mathcal{E} \longrightarrow \mathcal{C}$ such that for any $x \in \mathcal{E}$ and any arrow $f : c \longrightarrow \Sigma(x)$ in $\mathcal{C}$, there exists a *unique* lift $\widetilde{f} : y \longrightarrow x$ in $\mathcal{E}$ such that $\Sigma(\widetilde{f}) = f$.

If we write **2** for the category with two objects and a single nontrivial arrow between them and $r : * \hookrightarrow \mathbf{2}$ for the canonical inclusion on the right, then the definition of a discrete right fibration can be succintly formulated with a diagram:



Remember that Cat is the category of all *small* categories:

NOTATION. I will use the nonstandard notation $\mathrm{Discr}\,\mathcal{C}$ for the full subcategory of $\mathsf{Cat}/\mathcal{C}$ (the slice category over $\mathcal{C}$) of discrete right fibrations.

THEOREM 4. The categories $\mathsf{PSh}(\mathcal{C})$ and $\mathrm{Discr}\,\mathcal{C}$ are *equivalent*.

The proof is very standard: we define a functor $F : \mathrm{Discr}\,\mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{C})$, and we show that it is essentially surjective and fully faithful. Let us consider:

$$F : \mathrm{Discr}\,\mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{C})$$
$$\Sigma : \mathcal{E} \longrightarrow \mathcal{C} \longmapsto \Sigma^{-1}$$

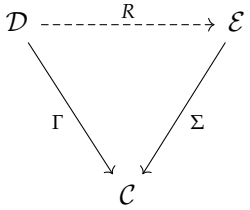
It is easy to check that F is well-defined.

On the one hand, given a presheaf $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ we already know that its associated $\Sigma : \mathbf{el}\, X \longrightarrow \mathcal{C}$ is a discrete right fibration. Moreover, we have that:

$$\Sigma^{-1}(c) = \{\, (x, c) \mid x \in Xc \,\} \cong Xc$$

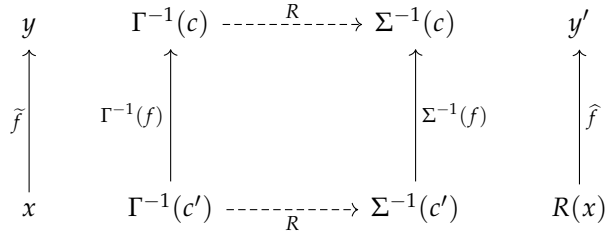
and similarly $\Sigma^{-1}(f) \simeq Xf$ for any arrow f of $\mathcal{C}$. As a result $F(\Sigma) \simeq X$, i.e. F is essentially surjective.

On the other hand, suppose that we have a morphism $R : \mathcal{D} \longrightarrow \mathcal{E}$ in Cat as well as two discrete right fibrations $\Gamma : \mathcal{D} \longrightarrow \mathcal{C}$ and $\Sigma : \mathcal{E} \longrightarrow \mathcal{C}$. We want to know what necessary and sufficient conditons we need on R to be a morphism in $\mathrm{Discr}\,\mathcal{C}$, i.e. such that the following diagram commutes:



For any $x \in \mathcal{D}$ and $f : c \longrightarrow c'$ of $\mathcal{C}$ such that $\Gamma(x) = c'$, there is a unique lift $\widetilde{f} : y \longrightarrow x$ in $\mathcal{D}$ such that $\Gamma(\widetilde{f}) = f$. Similarly, given $R(x) \in \mathcal{E}$ there is a unique lift $\widehat{f} : y' \longrightarrow R(x)$ in $\mathcal{E}$ such that $\Sigma(\widehat{f}) = f$.

Clearly, what we need is R to send $\widetilde{f}$ to $\widehat{f}$ for any choice of f. This is the data of a commutative square:



which exactly describes a natural transformation between Γ^{-1} and Σ^{-1}, and thus it shows that F is fully faithful. ∎

2.2 Categorical version

A more general statement of the Grothendieck construction relates "categorical" presheaves with a special sort of fibrations, called *Grothendieck fibrations*:

$$X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat} \qquad \simeq \qquad \Sigma : \int X \longrightarrow \mathcal{C}$$

The precise theorem below is adapted from [7, p. 264].

DEFINITION 7. The *Grothendieck category* of a presheaf $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$, noted $\int X$, is defined as:

- Objects are pairs (c, x) for every object $x \in Xc$ of the small category Xc
- Morphisms are pairs $(f, \alpha) : (c, x) \longrightarrow (c', y)$ of arrows:

$$\begin{aligned} f &: c \longrightarrow c' && \text{of } \mathcal{C} \\ \alpha &: x \longrightarrow Xf(y) && \text{of } Xc \end{aligned}$$

  such that, given $(g, \beta) : (c', y) \longrightarrow (c'', z)$ we have:

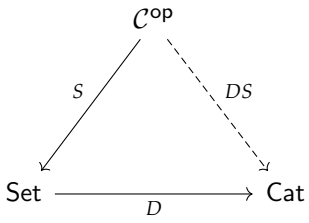
$$x \xrightarrow{\alpha} Xf(y) \xrightarrow{Xf(\beta)} Xf(Xg(z)) = X(gf)(z)$$

  i.e. composition is defined as $(g, \beta) \circ (f, \alpha) = (gf, Xf(\beta)\alpha)$.

The definition is very similar to the one of categories of elements with presheaves over Set. Just like then, we have a canonical projection $\Sigma : \int X \longrightarrow \mathcal{C}$ defined such that $\Sigma(c, x) \stackrel{\text{def}}{=} c$. Even more so, there is an obvious functor:

$$D : \mathsf{Set} \longrightarrow \mathsf{Cat}$$

which turns a set A into the discrete category DA with A as its set of objects. Now, if we have $S : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ we can push it forward into Cat:



Then indeed we have that $\mathbf{el}\, S \cong \int DS$, i.e. the two notions do coincide.

REMARK. In the last step of the definition of composition in $\int X$, we have:

$$Xf(Xg(z)) = X(gf)(z)$$

because X is a "strict" contravariant functor.

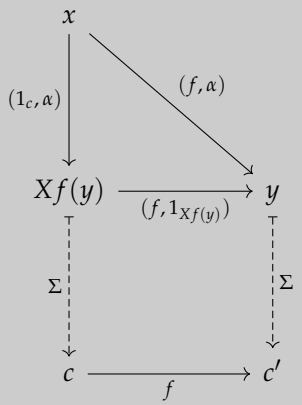
One could ask $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ to be merely a *pseudofunctor* instead, that is a functor which preserves composition only up to isomorphism. Then the last step would become:

$$Xf(Xg(z)) \xrightarrow[\sim]{\theta_{f,g}} X(gf)(z)$$

where $\theta_{f,g}$ is the corresponding isomorphism. Everything that I will present next also applies to pseudofunctors, at the cost of being extra careful with the compositions involved. The fully "pseudofunctorial" approach is the one done in [7].

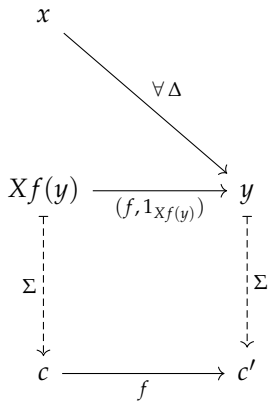
Again, just like with categories of elements, the projection $\Sigma : \int X \longrightarrow \mathcal{C}$ has special lifting properties. But first, let us have a closer look to the structure of morphisms in $\int X$:

DEFINITION 8. A morphism $(f, \alpha) : (c, x) \longrightarrow (c', y)$ can be decomposed the following way:

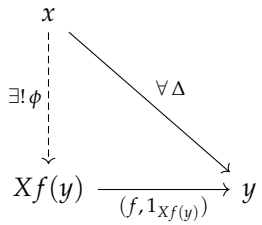


- Morphisms with shape $(1_c, \alpha) : (c, x) \longrightarrow (c, Xf(y))$ lie entirely within the fiber $\Sigma^{-1} \cong Xc$ and are said to be *vertical*
- Morphisms with shape $(f, 1_{Xf(y)}) : (c, Xf(y)) \longrightarrow (c', y)$ are said to be *horizontal*

The factorization $(f, \alpha) = (f, 1_{Xf(y)}) \circ (1_c, \alpha)$ is clearly unique. Now, for any diagram such as below:

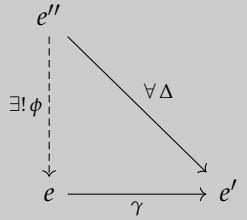


where both $x, Xf(y) \in Xc$, and $\Sigma(\Delta) = f$. By definition of $\int X$, there must be a *unique* vertical arrow ϕ which completes the diagram:



In that case $(f, 1_{Xf(y)}) : (c, Xf(y)) \longrightarrow (c', y)$ can be thought as the "canonical lift" of $f : c \longrightarrow c'$ into $\int X$ at point y, since any morphism $\Delta : x \longrightarrow y$ over f is uniquely an extension of $(f, 1_{Xf(y)})$. This is the special lifting property we are looking for:

DEFINITION 9. Let $\Sigma : \mathcal{E} \longrightarrow \mathcal{C}$ be a functor. A morphism $\gamma : e \longrightarrow e'$ of $\mathcal{E}$ is a *cartesian lift* of a morphism $f : c \longrightarrow c'$ of $\mathcal{C}$ if $\Sigma(\gamma) = f$, and for any other morphism $\Delta : e'' \longrightarrow e'$ with the same codomain as γ and such that $\Sigma(\Delta) = f$, there exists a *unique* $\phi : e'' \longrightarrow e$ such that $\Sigma(\phi) = 1_c$ and $\gamma\phi = \Delta$:



This is exactly the same diagram as before, except for a general category E. Cartesian lifts such as γ are the direct analogue of horizontal morphisms, while the factor ϕ is a vertical morphism. Now, there is no reason for a cartesian lift to be unique a priori — but it is indeed the case:

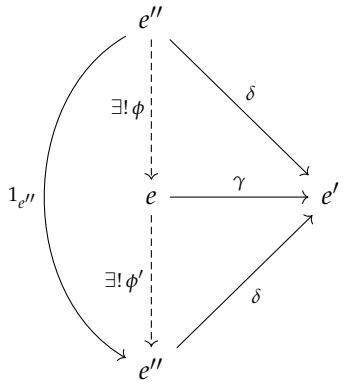
LEMMA 4. A cartesian lift, if it exists, is unique up to (unique) isomorphism.

Proof. This is a matter of straightforward verification of a universal property.

Given a functor $\Sigma : \mathcal{E} \longrightarrow \mathcal{C}$ and two possible cartesian lifts:

$$\gamma : e \longrightarrow e' \qquad \text{and} \qquad \delta : e'' \longrightarrow e'$$

of a morphism $f : c \longrightarrow c'$ of $\mathcal{C}$, we must have:

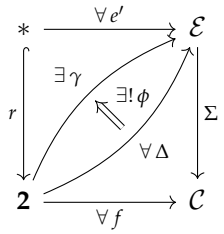


and similarly in the other direction by interverting γ and δ. This shows that the arrow ϕ in the diagram must be an isomorphism. ∎

DEFINITION 10. A functor $\Sigma : \mathcal{E} \longrightarrow \mathcal{C}$ is called a *Grothendieck fibration* if every arrow $f : c \longrightarrow c'$ of $\mathcal{C}$ has a cartesian lift.

NOTATION. The full subcategory of $\mathsf{Cat}/\mathcal{C}$ of Grothendieck fibrations is noted $\mathrm{Fib}\,\mathcal{C}$.

Grothendieck fibrations are the "correct" notion of fibrations for categorical presheaves. Just as with discrete right fibrations, the lifting property can be succintly formulated with a diagram:



REMARK. A nice (and important) property of Grothendieck fibrations is that cartesian lifts are stable under composition, up to isomorphism.

If $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ were a pseudofunctor instead, we would need to relax the definition of a cartesian lift in order to keep track of the isomorphisms. As a result we would also need to make an explicit choice of lifts, and include this additional data in our definition of fibrations — losing the stability under composition in the process. This is done in [7] with the notion of *cleavage* and *cloven fibrations*.

Unsurprisingly, the projection $\Sigma : \int X \longrightarrow \mathcal{C}$ is a Grothendieck fibration. We now know all the ingredients for the main result:

THEOREM 5. The categories $[\mathcal{C}^{\mathsf{op}}, \mathsf{Cat}]$ and $\mathrm{Fib}\,\mathcal{C}$ are equivalent.

The structure of the proof is essentially the same as in the discrete case. ■

2.3 To infinity and beyond

An even more general statement of the Grothendieck construction exists within the setting of $(\infty, 1)$-categories, and is covered in [9, p. 168].

For the rest of this thesis, we will focus on the Grothendieck construction in type theory:

$$X : A \longrightarrow \mathsf{Type} \qquad \simeq \qquad \pi_1 : \sum_{a\,:\,A} X\,a \longrightarrow A$$

which relates dependent types with Σ-types. We will now see how with the definition of presheaves in type theory.

3 Presheaves in type theory

From this point I will adopt a more informal style, as I will only be presenting intuitions for the most part in the absence of concrete results. I assume the reader has a basic knowledge of Martin-Löf type theory (Π-types and Σ-types, identity types, ...) and homotopy type theory (HoTT). Any introduction should suffice, although [11] covers both extensively.

A question raised during the Univalent foundations special year [8] asked whether it would be possible to construct semi-simplicial sets as dependently-typed families of sets, i.e. something along the lines of:

$$
\begin{aligned}
&X_0 : \mathsf{Type} \\
&X_1 : X_0 \longrightarrow X_0 \longrightarrow \mathsf{Type} \\
&X_2 : \prod_{a\ b\ c\ :\ X_0} X_1(a,b) \longrightarrow X_1(b,c) \longrightarrow X_1(a,c) \longrightarrow \mathsf{Type} \\
&\quad \vdots
\end{aligned}
$$

where X_0 would be the type of points, $X_1(a,b)$ would be the type of segments with endpoints a and b, and so on. One notable issue arising in the setting of HoTT is the need of *higher coherence* witnesses as the identity types have then unbounded *homotopy level*. The investigations pertaining to the construction of semi-simplicial types and the higher coherence issues are covered for example in [4] or [1].

3.1 Two possible presentations

There is one "obvious" way of defining semi-simplicial types. We know from our previous section on semi-simplicial sets that we only need the data of a family of sets $(X_n)_{n\in\mathbb{N}}$ and an associated family of face maps $d_i : X_n \longrightarrow X_{n-1}$ which satisfy the relation:

$$d_i d_j = d_{j-1} d_i \qquad \text{when } i < j$$

That would informally translate in type theory into a family of types:

$$X : \mathsf{nat} \longrightarrow \mathsf{Type}$$

together with a family of face maps and relations:

$$
\begin{aligned}
d &: \prod_{i\ n\ :\ \mathsf{nat}} i \leqslant n \longrightarrow X\ n \longrightarrow X\ (n-1) \\
\alpha &: \prod_{i\ j\ n\ :\ \mathsf{nat}} i < j \longrightarrow d_i d_j = d_{j-1} d_i
\end{aligned}
$$

It would not be that simple in HoTT as, in general, we need additional data to ensure that the α's are compatible together higher up in a "coherent" way. One workaround is to *truncate*, i.e. replace Type with hSet (that is, types with homotopy level 2).

In any case (truncated or not), we now have two possible approaches to defining semi-simplicial types:

- The *indexed presentation* mentioned earlier with dependent types, where the face maps are mere projections and thus implicit
- The *fibered presentation* I just showed, where the face maps and their relations are explicitly part of the data

Now as both approaches supposedly produce the same semi-simplicial types in the end, there must exist an isomorphism that turns indexed presentations into fibered ones. Consider a much simpler case where, instead of presheaves over Δ_+, we look at presheaves over $\mathbf{2}$. An element of $\mathsf{PSh}(\mathbf{2})$ is the data of two sets X_0, X_1 together with an application $f : X_1 \longrightarrow X_0$. In type theory, this would give:

$$X_0 : \mathsf{Type}$$
$$X_1 : X_0 \longrightarrow \mathsf{Type}$$

for the indexed presentation, and:

$$Y_0, Y_1 : \mathsf{Type}$$
$$f : Y_1 \longrightarrow Y_0$$

for the fibered one. Mapping X_0 to Y_0 is direct, but what about X_1 and Y_1? Clearly, the only sensible choice would be for Y_1 to be the "elements" of X_1:

$$Y_1 \overset{\text{def}}{=} \sum_{a\,:\,X_0} X_1\ a$$

Then f would be naturally defined as the projection $\pi_1 : \sum X_1\ a \longrightarrow X_0$. This is exactly the discrete Grothendieck construction!

Of course, we restricted ourselves to the case where the base category is $\mathbf{2}$ for illustrative purposes. In general, we would have many more dependencies and coherence equations to keep track of in order to build an isomorphism. Even worse, in the case of simplicial sets we have seen that there are multiple ways to describe the base category Δ_+. Making these distinctions more explicit is the subject of the next subsections.

3.2 Reedy categories

Let us be a little bit more rigorous on what we need to give a precise formulation of the indexed and fibered presentations.

It seems unavoidable that in order to turn presheaves $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ into a type-theoretic concept, $\mathcal{C}$ must have special properties which makes it at least definable in type theory. At the very least, both its objects and hom sets should be definable as types (and, as a consequence, at most countable). Now, in order to build a dependently-typed family of sets, we need an *inductive* structure on the objects of $\mathcal{C}$, i.e. there cannot be any kind of "circular dependencies". This is where the notion of *Reedy category* comes in:

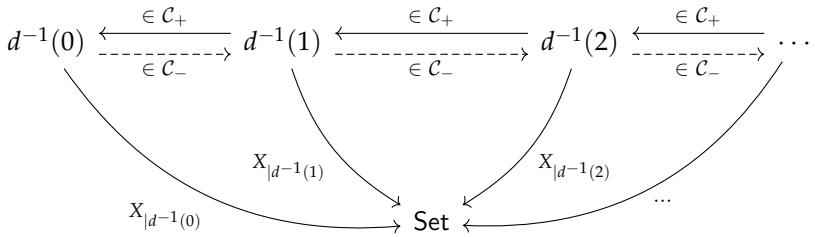
DEFINITION 11. A *Reedy category* is a category $\mathcal{C}$ such that:

- Every object $c \in \mathcal{C}$ has an associated *degree* $d(c) \in \omega$, where ω is the first countable ordinal
- There exist two subclasses of arrows of $\mathcal{C}$ noted $\mathcal{C}_+$ and $\mathcal{C}_-$ such that:
  i) Morphisms $c \longrightarrow c'$ of $\mathcal{C}_+$ raise degrees, i.e. $d(c) < d(c')$
  ii) Morphisms $c \longrightarrow c'$ of $\mathcal{C}_-$ lower degrees, i.e. $d(c) > d(c')$
  iii) Every morphism of $\mathcal{C}$ factors uniquely as the composition of a morphism in $\mathcal{C}_-$ followed by a morphism in $\mathcal{C}_+$

One could also think of $\mathcal{C}_+$ and $\mathcal{C}_-$ as *wide subcategories* of $\mathcal{C}$.

Many classical examples of finite or countable categories are actually Reedy categories. In particular, the categories Δ (simplex category) and Δ_+ (semi-simplex category) are Reedy categories. Even more so, Δ_+ really is the wide subcategory of Δ of degree-raising morphisms!

The fact that every object of a Reedy category has a well-ordered degree opens up the possibility of an indexed presentation of presheaves. Assuming we have the "degree functor" $d : \mathcal{C} \longrightarrow \omega$ and a presheaf $X : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$, we can decompose X the following way:



where the $X_{|d^{-1}(i)}$'s are the presheaf X restricted to the domain $d^{-1}(i)$. Notice how the arrows of $\mathcal{C}_+$ and $\mathcal{C}_-$ obviously change directions in $\mathcal{C}^{\mathsf{op}}$.

By extension, I say that the "degree" of a set Xc is the degree of its corresponding object c. Now, the arrows of $\mathcal{C}_+$ then indicate a *dependency* of some set Xd on sets $Xc_1, Xc_2, \ldots$ of lower degree. An arrow $f_k : c_k \longrightarrow d$ of $\mathcal{C}_+$ is associated with an application $Xf_k : Xd \longrightarrow Xc_k$ which, for any $x \in Xd$, tells me that the element x "projects to", or is "made up of" $f_k(x) \in Xc_k$. From an indexed point of view, this would give in type theory:

$$\begin{array}{l} \vdots \\ Xc_1, Xc_2, \ldots : \ldots \\ \qquad Xd : Xc_1 \longrightarrow Xc_2 \longrightarrow \cdots \longrightarrow \mathsf{Type} \\ \vdots \end{array}$$

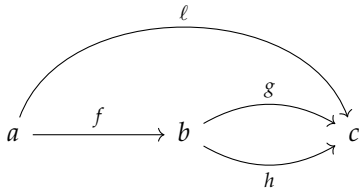
Then $x : Xd\ f_1(x)\ f_2(x)\ \cdots$

On the other hand, the arrows of $\mathcal{C}_-$ could be thought as "elevating a copy" of an element from a low degree set into a higher degree set. This is much more difficult to express properly in type theory, but an unpublished note by Hugo Herbelin [5] makes a proposition in that direction. As such, I will only focus on categories for which $\mathcal{C}_-$ is empty, i.e. what are called *direct categories*:

DEFINITION 12. A *direct category* is a Reedy category for which there are no degree-lowering morphisms.

Using the same notations as before, if $\mathcal{C}$ is a Reedy category then it is a direct category if "$\mathcal{C} = \mathcal{C}_+$". Δ_+ is the most immediate example of a direct category, and as a consequence if we have a general theory of presheaves over direct categories in type theory, then in particular we will be able to use it to construct semi-simplicial types.

I have hinted before that the description of a category in terms of generators and relations would play a key role. Let me illustrate this with an elementary example of a direct category that I will call $\mathcal{M}$:



The category $\mathcal{M}$ has three objects a, b, c and four nontrivial arrows f, g, h, ℓ such that $gf = \ell = hf$. It is easy to see that $\mathcal{M}$ is indeed a direct category. Now, I could drop ℓ in my definition of $\mathcal{M}$ as it is "generated" by gf or hf. At the very least, the fibered presentation of a presheaf $X : \mathcal{M}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ in type theory would change as the generators and relations change:

$$\begin{aligned} Xa, Xb, Xc &: \mathsf{Type} \\ Xf &: Xb \longrightarrow Xa \\ Xg &: Xc \longrightarrow Xb \\ Xh &: Xc \longrightarrow Xb \end{aligned}$$

Then I either add the data of $X\ell : Xc \longrightarrow Xa$ and $\alpha : XfXg = X\ell = XfXh$, or I only add the data of $\alpha : XfXg = XfXh$. This is perfectly equivalent, but not strictly equal. From the indexed point of view on the other hand, we would get:

$$\begin{aligned} Xa &: \mathsf{Type} \\ Xb &: Xa \longrightarrow \mathsf{Type} \\ Xc &: \prod_{i\,:\,Xa} Xb\ i \longrightarrow Xb\ i \longrightarrow \mathsf{Type} \end{aligned}$$

The relations have been "flattened" into definitional equalities of projections, and the generators have been completely eliminated. Indeed, if we consider:

$$\begin{aligned} i &: Xa \\ j, k &: Xb\ i \\ x &: Xc\ i\ j\ k \end{aligned}$$

Then somehow we could say that:

$$g^*x = j \qquad h^*x = k \qquad \ell^*x = f^*g^*x = f^*h^*x = i$$

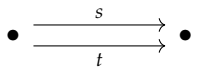
provided we interpret f^*, g^*, h^*, ℓ^* as being the obvious projections. What happens is that the Π-type "eats" all of the complexity when we explicitly know how to compute the relations of our category. But, let us say that, instead of a description as explicit as with $\mathcal{M}$ we are given an abstract set of generating

morphisms G, and an abstract set of relations R identifying some compositions of morphisms of G. As definitional equality is *stricter*, it would not be possible for us to turn the *propositional* equalities of R into definitional ones in general, inside of our Π-type (at least, not this way). In any case, we need to make the notion of generators and relations in the setting of category theory more explicit.

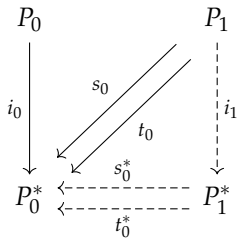
3.3 Polygraphs

Given an arbitrary graph, it should be quite easy to imagine how to construct the "free category" over this graph, by formally adding all the finite paths as arrows. To be more precise, the process is the following: take a set of 0-*generators* P_0 that will serve as objects of our category, and a set of 1-*generators* P_1 which connect objects to one another, effectively describing arrows. This additionally requires two maps $s_0, t_0 : P_1 \longrightarrow P_0$ usually called *source* and *target*, such that given an arrow $f \in P_1$ then its "domain" and "codomain" are $s_0(f)$ and $t_0(f)$.

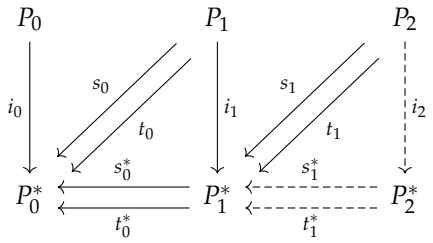
It is actually a classical result of abstract nonsense that directed graphs are exactly the data of a presheaf over the category:



Now, the pieces I introduced would fit in a (slightly more complete) diagram:



The bottom row contains the structures freely generated by the data on the top row. One could think of P_0^* as the "0-cells" of the "free 0-category" (i.e. with only objects) generated by P_0. Then we really have $P_0^* = P_0$, witnessed by the inclusion i_0. We can thus ask for the maps s_0, t_0 to land in P_0^* instead as it is strictly the same, just like on the diagram. P_1^* would be, in turn, the "1-cells" of the "free (1-)category" generated by the arrows in P_1 by formally adding all the finite paths as mentioned before, and s_0^*, t_0^* would be the extensions of s_0, t_0 to P_1^*. Of course, the generating arrows are in particular finite paths, so we have a canonical inclusion $i_1 : P_1 \longrightarrow P_1^*$ as well. How do we now introduce relations? A relation is merely the choice of two arrows of P_1^* that we want to be identified. We could further extend our diagram on the right:

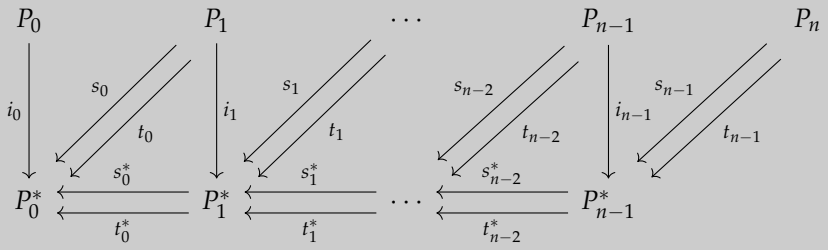


The set P_2 describes the relations we want on P_1^*. More concretely, given an element $r \in P_2$ we ask that $s_1(r) = t_1(r)$, and for that to make sense we need the domains and codomains of the selected arrows to coincide, i.e.:

$$s_0^* s_1 = s_0^* t_1 \qquad t_0^* s_1 = t_0^* t_1$$

Then again, P_2^* would be the "2-cells" of the "free 2-category" generated by quotienting P_1^* with the relations in P_2. If we keep iterating this process, we get the general definition of a *polygraph*:

DEFINITION 13. An *n-polygraph* is the data of a diagram in Set:



Elements of the P_k's are called *k-generators*. Additionally, the arrows on the diagram should satisfy the following relations:

- The maps s_k^*, t_k^* really are extensions of s_k, t_k through i_{k+1}:

$$s_k^* i_{k+1} = s_k \qquad t_k^* i_{k+1} = t_k$$

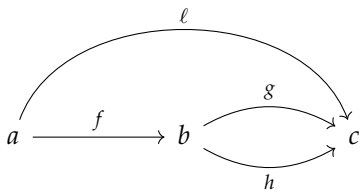
- Sources and targets should coincide at the previous level:

$$s_k^* s_{k+1} = s_k^* t_{k+1} \qquad t_k^* s_{k+1} = t_k^* t_{k+1}$$

Notice how I did not include P_n^* in the definition. To put it loosely, the idea is that an n-polygraph should be the description of a "free $(n-1)$-category" where we already *strictly* identified some of the "$(n-1)$-cells", instead of directly

looking at a "free n-category". A 2-polygraph really is, for example, an ordinary (1-)category which has a set P_1 of generators and a set P_2 of relations which are "strictly" considered (i.e. the diagrams commute "on the nose"). There are many other higher categorical aspects of polygraphs to consider which are completely out of the scope of this thesis, but which the interested reader can find in the substantial compilation of Ara et al. [2] to be published. In any case, I will really only look at 2-polygraphs for the time being.

Let us go back to our previous example, the category $\mathcal{M}$:



I can now give (at least) two descriptions of $\mathcal{M}$ as a 2-polygraph:

$$P_1 = \{ f, g, h, \ell \} \qquad\qquad P_1 = \{ f, g, h \}$$
$$P_2 = \{ \text{“}gf = \ell\text{”}, \text{“}\ell = gf\text{”} \} \qquad\qquad P_2 = \{ \text{“}gf = hf\text{”} \}$$

with $P_0 = \{ a, b, c \}$ in both cases. If we go even further back to our main goal of defining presheaves in type theory, then we should have a general procedure that turns 2-polygraphs (and more generally n-polygraphs in the "n-truncated" case) into either the indexed or the fibered presentation, together with the equivalence between the two. But there is actual room for intermediate steps.

3.4 Transitioning from fibered to indexed

Here is one of the possible fibered presentations for the category $\mathcal{M}$, that we already have seen earlier:

$$\begin{aligned} Xa, Xb, Xc &: \mathsf{Type} \\ Xf &: Xb \longrightarrow Xa \\ Xg &: Xc \longrightarrow Xb \\ Xh &: Xc \longrightarrow Xb \\ \alpha &: XfXg = XfXh \end{aligned}$$

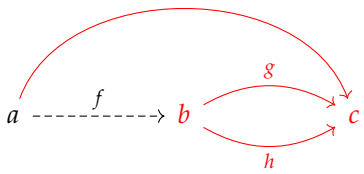
What we could do is successively "eliminate" the maps using the Grothendieck construction. Let me illustrate this by first eliminating Xf, for example:

$$
\begin{aligned}
Xa &: \mathsf{Type} \\
Xb &: Xa \longrightarrow \mathsf{Type} \\
Xc &: Xa \longrightarrow \mathsf{Type} \\
Xg &: \prod_{i\,:\,Xa} Xc\ i \longrightarrow Xb\ i \\
Xh &: \prod_{i\,:\,Xa} Xc\ i \longrightarrow Xb\ i \\
\alpha &: \pi_1 Xg = \pi_1 Xh
\end{aligned}
$$

The maps π_1 are defined as the obvious "mere" projections, through implicit curryification. For example, in the previous case:

$$
\begin{aligned}
\pi_1 : \prod_{i\,:\,Xa} Xb\ i &\longrightarrow Xa \\
(i, j) &\longmapsto i
\end{aligned}
$$

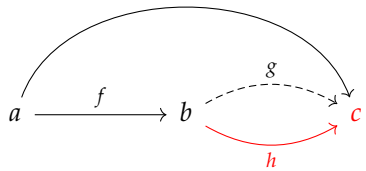
Everything that comes "after" f in the base category $\mathcal{M}$ has to be updated to depend on a choice of term in Xa:



i.e. both Xb and Xc now depend on Xa, and the canonical projections correspond to the now-eliminated respective maps Xf and $XfXg = XfXh$. Another possibility would have been to eliminate Xg first:

$$
\begin{aligned}
Xa, Xb &: \mathsf{Type} \\
Xf &: Xb \longrightarrow Xa \\
Xc &: Xb \longrightarrow \mathsf{Type} \\
Xh &: \prod_{j\,:\,Xb} Xc\ j \longrightarrow Xb \\
\alpha &: Xf\pi_1 = XfXh
\end{aligned}
$$

Again, we only have to update what comes "after" g in the base category:



Now if I eliminate Xf *then* Xg, I would get:

$$
\begin{aligned}
&Xa : \mathsf{Type} \\
&Xb : Xa \longrightarrow \mathsf{Type} \\
&Xc : \prod_{i\,:\,Xa} Xb\ i \longrightarrow \mathsf{Type} \\
&Xh : \prod_{i\,:\,Xa} \prod_{j\,:\,Xb\ i} Xc\ i\ j \longrightarrow Xb\ i \\
&\alpha : \pi_1\pi_1 = \pi_1 Xh
\end{aligned}
$$

This is starting to look a lot like the indexed presentation. If I eliminate Xh:

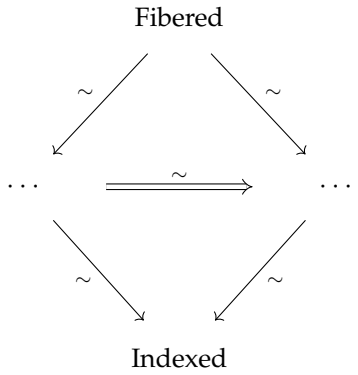
$$
\begin{aligned}
&Xa : \mathsf{Type} \\
&Xb : Xa \longrightarrow \mathsf{Type} \\
&Xc : \prod_{i\,:\,Xa} Xb\ i \longrightarrow Xb\ i \longrightarrow \mathsf{Type} \\
&\alpha : \pi_1\pi_1 = \pi_1\pi_1
\end{aligned}
$$

Then indeed, I do get the indexed presentation — up to an extra trivial term α.

Overall, it thus seems possible to decompose the equivalence of the fibered and indexed presentations as a succession of "basic" eliminations, corresponding to either direction of the Grothendieck construction. The other direction would be using Σ-types to turn the mere projections π_1 back into full-fledged maps. As eliminating maps almost inevitably introduces dependent products (just as above), maybe an easy slogan would be that "fibered and indexed presentations are equivalent, up to replacing Σ's with Π's and vice versa". I will now try to make a summary of everything we have seen so far, and outline possible areas of research.

3.5 Conclusion

As there is no reason *not* to take the order of the eliminations into account, we could argue that the possible paths of successive eliminations *should be themselves equivalent* at a higher level:



Furthermore we only have covered the case of 2-polygraphs here, ignoring the delicate subtleties of fully dealing with the identity type — which would be inevitable in plain HoTT for example. At last, we only really looked at the discrete version of the Grothendieck construction, effectively limiting us in the level of generality we could achieve.

The central intuition is the following: it would be reasonable to expect that "equivalences of equivalences" of possible paths of successive eliminations as in the diagram above, should be again equivalent *and higher up*. The way these higher equivalences emerge should be reflected in the "higher formulations" of the Grothendieck construction, translated into type theory. In the end, we want a general procedure that turns arbitrary n-polygraphs (or "∞-polygraphs" if we can make sense of the notion...?) encoded in type theory in some way and describing a Reedy category, into a "spectrum" — ranging from fibered to indexed — of type-theoretic presentations of presheaves, together with proofs using the Grothendieck construction that these presentations are all equivalent, and proofs that the proofs are all equivalent, and so on.

To be more precise, there are two levels of equivalence:

i) Given a fixed polygraph, all its presentations in type theory (fibered, indexed, and otherwise) should be equivalent

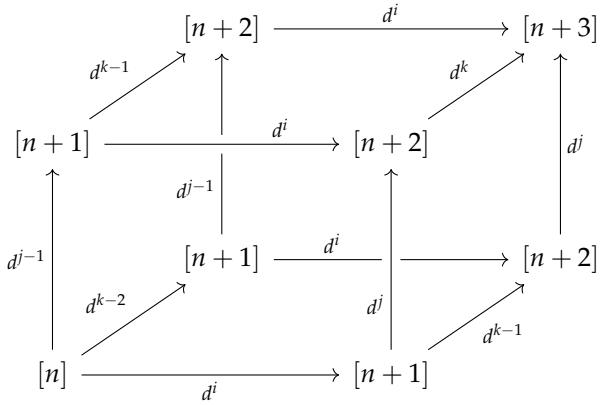
ii) All the polygraphs that describe the same category should be themselves equivalent in some sense

Then, presumably, i) the space of all the possible ways to build presheaves given a fixed polygraph should be contractible and contractibility is exactly the statement of the Grothendieck construction, and ii) the higher category of all the possible ways to build presheaves out of any polygraph describing the same higher category should be equivalent to its category of "higher presheaves".

4 A "Yoneda-esque" presentation

One issue that I have avoided so far is to keep track of the necessary "higher coherence witnesses" I mentioned in the previous introduction. Another aspect I have not touched yet is that, in general, the data of a 2-polygraph is "opaque" in the sense that it is not *externally fixed*, but a parameter of the procedure we wish to have. Let me illustrate the first point with the semi-simplex category.

Part of this is already explained in the introduction of [4]. Consider the following commutative diagram in Δ_+, assuming $i < j < k$:



The commutation is given by the fact that:

$$d^y d^x = d^x d^{y-1} \qquad \text{when } x < y$$

One can think of this relation as "continuously deforming" the path $d^y d^x$ into the path $d^x d^{y-1}$ and vice versa, on one of the faces of the cube above. Then, still looking at the cube, it seems pretty clear that there are at least two ways to "continuously deform" the path $d^k d^j d^i$ into the path $d^i d^{j-1} d^{k-2}$, either by crossing "at the top" or by crossing "at the bottom" of the cube.

If we tediously named all the faces of the cube, we could formally express this in type theory by showing that there are two distinct proofs of the equality:

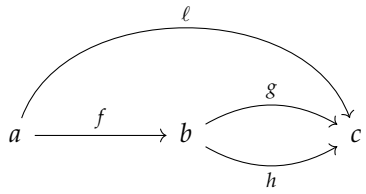
$$\pi, \pi' : d^k d^j d^i = d^i d^{j-1} d^{k-2}$$

Then we would need a "higher coherence" proof, witnessing the fact that these two proofs should be equal too (i.e. $\pi = \pi'$) in order for us to define presheaves properly. Of course, this would keep going at higher levels if we draw an hypercube, a "5-cube", ...

One solution seems to render these coherence issues entirely "opaque", at the cost of introducing new difficulties with dependent typing.

4.1 Staying at Hom

Consider our now-favorite example, the category $\mathcal{M}$:



We have seen that the indexed presentation for $\mathcal{M}$ is the following:

$$\begin{aligned} &Xa : \mathsf{Type} \\ &Xb : Xa \longrightarrow \mathsf{Type} \\ &Xc : \prod_{i \,:\, Xa} Xb\ i \longrightarrow Xb\ i \longrightarrow \mathsf{Type} \end{aligned}$$

Notice how the *number of dependencies* of each type do coincide with the number of ingoing morphisms in the base category, up to equality. For example, on the one hand the only (nontrivial) morphism ending with b is $f : a \longrightarrow b$, while on the other hand Xb "happens" to only depend on Xa. Similarly, the only morphisms ending with c are g, h, ℓ and at the same time Xc only depends on two copies of Xb over the same point i (witnessing the fact that $gf = \ell = hf$).

Let us try to factor the dependencies together using the hom "functor" of $\mathcal{M}$:

$$\begin{aligned} &Xa : \mathsf{Type} \\ &Xb : \forall\, s_0 : \left[\prod_{f_0 \,:\, \mathrm{Hom}(a,b)} Xa\right], \\ &\qquad \mathsf{Type} \\ &Xc : \forall\, s_0 : \left[\prod_{f_0 \,:\, \mathrm{Hom}(a,c)} Xa\right], \\ &\qquad \forall\, s_1 : \left[\prod_{f_1 \,:\, \mathrm{Hom}(b,c)} Xb\ (\lambda f_0 \,.\, s_0(f_1 \circ f_0))\right], \\ &\qquad \mathsf{Type} \end{aligned}$$

For the sake of readability, I wrote some of the dependent products with $\forall$ instead of Π. Let me carefully unpack this:

- As nothing happens at the base level a, the type Xa is left unchanged.
- At level b, every arrow $f_0 : a \longrightarrow b$ in the base category should add a new dependency on Xa. To choose the dependencies "all at once", we simply require the data of a dependent function:

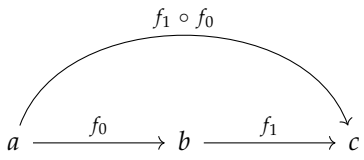
$$s_0 : \prod_{f_0 \,:\, \mathrm{Hom}(a,b)} Xa$$

In our case, $\mathrm{Hom}(a, b)$ really only contains f so this is equivalent to choosing a single element in Xa, i.e. just as if we wrote $Xb : Xa \longrightarrow \mathsf{Type}$.

- The last one is the most interesting. We first apply to Xc exactly the same reasoning to choose the dependencies on Xa:

$$s_0 : \prod_{f_0 \,:\, \mathrm{Hom}(a,c)} Xa$$

Again, as $\mathrm{Hom}(a, c)$ only contains ℓ this is equivalent to choosing a single element in Xa. But now, Xb also depends on Xa so we must ensure that *no matter which path we choose*, the choices we already made are consistent:



This is why we need to choose the dependencies on Xb "all at once" the following way:

$$s_1 : \prod_{f_1 \,:\, \mathrm{Hom}(b,c)} Xb\ (\lambda f_0\ .\ s_0(f_1 \circ f_0))$$

If we break down the definition of s_1 we might see how it makes sense, and more importantly how it type checks:

$$\begin{aligned}
f_0 &: \mathrm{Hom}(a, b) \\
f_1 &: \mathrm{Hom}(b, c) \\
s_0 &: \textstyle\prod_{f_0 \,:\, \mathrm{Hom}(a,c)} Xa \\
s_0(f_1 \circ f_0) &: Xa \\
\lambda f_0\ .\ s_0(f_1 \circ f_0) &: \textstyle\prod_{f_0 \,:\, \mathrm{Hom}(a,b)} Xa \\
Xb &: \forall\, s_0 : \left[\textstyle\prod_{f_0 \,:\, \mathrm{Hom}(a,b)} Xa\right], \\
&\qquad \mathsf{Type} \\
Xb\ (\lambda f_0\ .\ s_0(f_1 \circ f_0)) &: \mathsf{Type}
\end{aligned}$$

Since we always have that $f_0 = f$ and $f_1 = g$ or $f_1 = h$ in the definition above, then by definition of $\mathcal{M}$ the composition $f_1 \circ f_0$ will always be equal to ℓ, and thus $Xb\ (\lambda f_0\ .\ s_0(f_1 \circ f_0))$ will always be the same "fiber", i.e. it only depends on s_0. It therefore seems that we have moved the complexity of dealing with the relations of $\mathcal{M}$ "inside" the Hom!

I will now vastly generalize this to any direct category.

4.2 Definition

Suppose we have the data of a direct category in type theory, encoded as:

$$\begin{aligned} A &: \mathsf{nat} \longrightarrow \mathsf{Type} \\ \mathrm{Hom} &: \prod_{i\ j\ :\ \mathsf{nat}} A_i \longrightarrow A_j \longrightarrow \mathsf{Type} \\ &\vdots \end{aligned}$$

i.e. A is a family such that A_i contains the objects of degree i, and $\mathrm{Hom}(-,-)$ is the underlying hom "functor". The details of composition, associativity and identity are left out as they are not relevant for now. All the subtleties of generators, relations, and higher coherence witnesses are so far hidden "inside" the computation of A_i and $\mathrm{Hom}(x,y)$. To make this more convincing, note that the following easy lemma holds:

LEMMA 5. A direct category is exactly the data of a category whose objects can be partitioned into a family $(A_i)_{i\in\mathbb{N}}$ such that:

i) All the arrows within any A_i are trivial, i.e. there are only identities

ii) A nontrivial arrow can only go from A_i to A_j, with $i < j$

As a consequence, if we can formulate an indexed presentation of presheaves only using these ingredients, then presumably we would have successfully overcome the higher coherence issues and made them entirely "opaque". Here is how we could do it, by extending what we did with the category $\mathcal{M}$ before:

$$\begin{aligned} &X_0 : A_0 \longrightarrow \mathsf{Type} \\ &X_1 : \forall\, a_1 : A_1, \\ &\qquad \forall\, s_0 : \left[\textstyle\prod_{a_0\ :\ A_0} \prod_{f_0\ :\ \mathrm{Hom}(a_0,a_1)} X_0\ a_0\right], \\ &\qquad\qquad \mathsf{Type} \\ &X_2 : \forall\, a_2 : A_2, \\ &\qquad \forall\, s_0 : \left[\textstyle\prod_{a_0\ :\ A_0} \prod_{f_0\ :\ \mathrm{Hom}(a_0,a_2)} X_0\ a_0\right], \\ &\qquad \forall\, s_1 : \left[\textstyle\prod_{a_1\ :\ A_1} \prod_{f_1\ :\ \mathrm{Hom}(a_1,a_2)} X_1\ a_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\ &\qquad\qquad \mathsf{Type} \\ &\qquad\qquad \vdots \end{aligned}$$

$$
\begin{aligned}
X_n : \forall\, & a_n : A_n \\
& \forall\, s_0 : \left[\prod_{a_0\,:\,A_0} \prod_{f_0\,:\,\mathrm{Hom}(a_0,a_n)} X_0\ a_0\right], \\
& \forall\, s_1 : \left[\prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_n)} X_1\ a_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\
& \quad \vdots \\
& \forall\, s_i : \left[\prod_{a_i\,:\,A_i} \prod_{f_i\,:\,\mathrm{Hom}(a_i,a_n)} X_i\ a_i\ (\lambda f_0\,.\, s_0(f_i \circ f_0))\ \cdots\ (\lambda f_{i-1}\,.\, s_{i-1}(f_i \circ f_{i-1}))\right], \\
& \quad \vdots \\
& \forall\, s_{n-1} : \left[\prod_{a_{n-1}\,:\,A_{n-1}} \prod_{f_{n-1}\,:\,\mathrm{Hom}(a_{n-1},a_n)} X_{n-1}\ a_{n-1}\ (\lambda f_0\,.\, s_0(f_{n-1} \circ f_0))\ \cdots\right], \\
& \quad \mathsf{Type}
\end{aligned}
$$

We now depend on an additional parameter A_i at each level i, which corresponds to selecting the object of degree i we are looking at. Everything else is the same as with $\mathcal{M}$, even the intermediate "choices of dependencies":

$$\lambda f_k\,.\, s_k(f_i \circ f_k)$$

There were no reason *a priori* that its expression would be this simple at any level, but in practice if one were to implement the above definition in a proof assistant such as Coq [14] then one would observe that it indeed works, *provided the associativity of composition holds definitionally*! This last precision is absolutely crucial and I will come back to it in a moment. For now, I would like to explain the name.

4.3 Why "Yoneda-esque"?

Let us look at one of the s_i's of X_n:

$$s_i : \prod_{a_i\,:\,A_i} \prod_{f_i\,:\,\mathrm{Hom}(a_i,a_n)} X_i\ a_i\ (\lambda f_0\,.\, s_0(f_i \circ f_0))\ \cdots\ (\lambda f_{i-1}\,.\, s_{i-1}(f_i \circ f_{i-1}))$$

Given that a_n is the very first parameter of X_n, the dependence on $\mathrm{Hom}(a_i, a_n)$ should strongly remind us of the *representable functor* $\widehat{a_n}$. Then f_i could be thought as an "i-edge" of $\widehat{a_n}$, and we would be tempted to write instead:

$$s_i : \prod_{f_i\,:\,(\widehat{a_n})_i} \cdots$$

Let me better illustrate this by taking Δ_+ as an example of direct category. If we do the necessary simplifications in the definition, we are left with:

$$
\begin{aligned}
&X_0 : \mathsf{Type} \\
&X_1 : \left[(\Delta^1_+)_0 \longrightarrow X_0\right] \longrightarrow \mathsf{Type} \\
&X_2 : \forall\, s_0 : \left[(\Delta^2_+)_0 \longrightarrow X_0\right], \\
&\qquad \forall\, s_1 : \left[\textstyle\prod_{f_1\,:\,(\Delta^2_+)_1} X_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\
&\qquad\qquad \mathsf{Type} \\
&\qquad\qquad \vdots \\
&X_n : \forall\, s_0 : [(\Delta^n_+)_0 \longrightarrow X_0], \\
&\qquad \forall\, s_1 : \left[\textstyle\prod_{f_1\,:\,(\Delta^n_+)_1} X_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\
&\qquad\qquad \vdots \\
&\qquad \forall\, s_i : \left[\textstyle\prod_{f_i\,:\,(\Delta^n_+)_i} X_i\ (\lambda f_0\,.\, s_0(f_i \circ f_0))\ \cdots\ (\lambda f_{i-1}\,.\, s_{i-1}(f_i \circ f_{i-1}))\right], \\
&\qquad\qquad \vdots \\
&\qquad \forall\, s_{n-1} : \left[\textstyle\prod_{f_{n-1}\,:\,(\Delta^n_+)_{n-1}} X_{n-1}\ (\lambda f_0\,.\, s_0(f_{n-1} \circ f_0))\ \cdots\right], \\
&\qquad\qquad \mathsf{Type}
\end{aligned}
$$

The dependency on A_i has been made implicit, as in the case of Δ_+ there is only one object of each degree. The geometric intuition should help us understand what happens:

- The base type X_0 contains the *points*, or *vertices*.
- X_1 contains the *segments*. A segment has two endpoints that we can describe with the data of a map $(\Delta^1_+)_0 \longrightarrow X_0$. Indeed Δ^1_+ is the "standard segment", so $(\Delta^1_+)_0$ really is the abstract data of two endpoints.

  In the end, X_1 returns the type of all segments with given endpoints (i.e. "lying over $(\Delta^1_+)_0 \longrightarrow X_0$"), hence the signature:

  $$X_1 : \left[(\Delta^1_+)_0 \longrightarrow X_0\right] \longrightarrow \mathsf{Type}$$

- Now, X_2 contains the "filled" *triangles*. Just as with X_1, what we do is first select the vertices of a triangle through a map $(\Delta^2_+)_0 \longrightarrow X_0$. Then we need to select three segments *between the vertices we have already selected*! This what the expression $\lambda f_0\,.\, s_0(f_1 \circ f_0)$ is for.

If you stare at it long enough, the dependent product:

$$\prod_{f_1 \,:\, (\Delta^2_+)_1} X_1\ (\lambda f_0\ .\ s_0(f_1 \circ f_0))$$

guarantees that the selected f_1's have "compatible" endpoints, i.e. one starts where the other ends, and so on. This is what the computation of the composition $f_1 \circ f_0$ does. Overall, given three vertices and three compatible segments X_2 returns the type of all the "fillings" of the triangle it describes.

- In the general case, X_n contains the *n-simplices*. Each s_i encodes the data of the "i-edges" an n-simplex is made of. Because the "i-edges" themselves fit inside a "structure" of "$(i-1)$-edges" and so on, each s_i depends of course on all of the previous ones. This is witnessed by the expressions:

  $$\lambda f_k\ .\ s_k(f_i \circ f_k)$$

  f_i is one of the "i-edges" of the standard n-simplex. Pre-composing with f_k amounts to looking at one of the "k-edges" of f_i. By applying s_k, we "interpret" the resulting "k-edge" $f_i \circ f_k$ in the presheaf we already have constructed.

  To sum it all up, an easy slogan would be that "X_n takes the *boundary* of an n-simplex as an argument, and returns the type of all the n-simplices with such boundary".

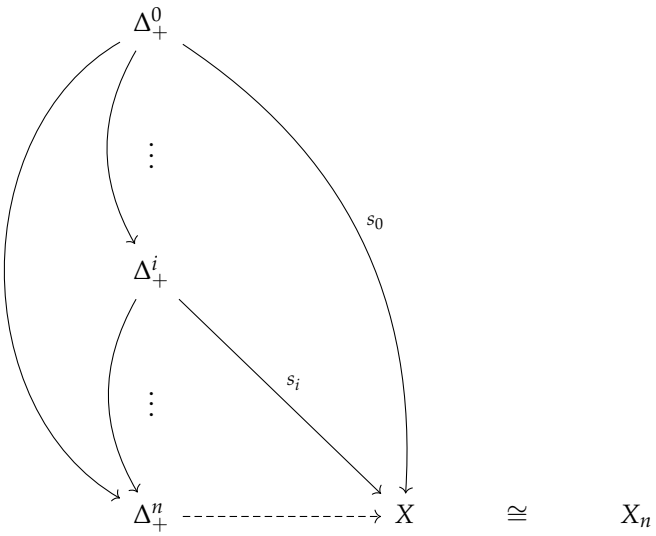
A different approach would have been to take the indexed presentation for "what it is", i.e. the way the n-simplices interact with each other is directly part of the definition of the presheaf:

$$\begin{aligned}
&X_0 : \mathsf{Type}\\
&X_1 : X_0 \longrightarrow X_0 \longrightarrow \mathsf{Type}\\
&X_2 : \prod_{a\ b\ c\ :\ X_0} X_1(a,b) \longrightarrow X_1(b,c) \longrightarrow X_1(a,c) \longrightarrow \mathsf{Type}\\
&\ \ \vdots
\end{aligned}$$

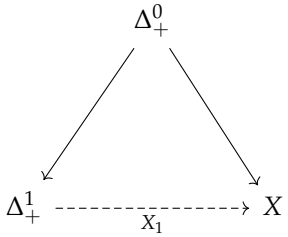
There is no trace of the standard n-simplex, or its structure. On the other hand, what we did above is look at the standard n-simplex, then "interpret" its structure with the s_i's. How could we not think of the Yoneda lemma:

$$\Delta^n_+ \longrightarrow X \qquad \cong \qquad X_n$$

More exactly, we first looked at the "i-edges" of Δ^n_+ so maybe the informal picture to keep in mind is the following:



For example in the most simple case $n = 1$, we can imagine something like:



i.e. X_1 would be exactly all the ways to fill the arrows $\Delta^0_+ \longrightarrow \Delta^1_+$ which are "parallel to" $\Delta^0_+ \longrightarrow X$. By applying the Yoneda lemma, we could reformulate this criterion of "parallelism" as the data of an arrow $(\Delta^1_+)_0 \longrightarrow X_0$. This is indeed the definition of X_1:

$$X_1 : \left[(\Delta^1_+)_0 \longrightarrow X_0\right] \longrightarrow \mathsf{Type}$$

At this stage, I have three open questions:

i) Is there a way to make this kind of diagram more precise and systematic, for any direct category? In particular, can we visually represent all of the complexity of the "Yoneda-esque" presentation with a diagram?

ii) As types are ∞-groupoids in HoTT, can we make "visually explicit" all the choices of paths and compositions that are made along the way?

iii) Suppose we look at "∞-presheaves" instead, from a base $(\infty, 1)$-category to the higher category of ∞-groupoids. How can the visual representation help us understand the associativity issues that I will explain next? And how would it relate to a hypothetical "weak ∞-Yoneda lemma"?

4.4 Coherence becomes associativity

The "Yoneda-esque" presentation does not type check starting at X_3. Let me write down the definition of the first few "slices":

$$
\begin{aligned}
&X_0 : A_0 \longrightarrow \mathsf{Type} \\
&X_1 : \forall\, a_1 : A_1, \\
&\qquad \forall\, s_0 : \left[\prod_{a_0\,:\,A_0} \prod_{f_0\,:\,\mathrm{Hom}(a_0,a_1)} X_0\ a_0\right], \\
&\qquad\qquad \mathsf{Type} \\
&X_2 : \forall\, a_2 : A_2, \\
&\qquad \forall\, s_0 : \left[\prod_{a_0\,:\,A_0} \prod_{f_0\,:\,\mathrm{Hom}(a_0,a_2)} X_0\ a_0\right], \\
&\qquad \forall\, s_1 : \left[\prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_2)} X_1\ a_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\
&\qquad\qquad \mathsf{Type} \\
&X_3 : \forall\, a_3 : A_3, \\
&\qquad \forall\, s_0 : \left[\prod_{a_0\,:\,A_0} \prod_{f_0\,:\,\mathrm{Hom}(a_0,a_3)} X_0\ a_0\right], \\
&\qquad \forall\, s_1 : \left[\prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_3)} X_1\ a_1\ (\lambda f_0\,.\, s_0(f_1 \circ f_0))\right], \\
&\qquad \forall\, s_2 : \left[\prod_{a_2\,:\,A_2} \prod_{f_2\,:\,\mathrm{Hom}(a_2,a_3)} X_2\ a_2\ (\lambda f_0\,.\, s_0(f_2 \circ f_0))\ (\lambda f_1\,.\, s_1(f_2 \circ f_1))\right], \\
&\qquad\qquad \mathsf{Type}
\end{aligned}
$$

The interesting bit happens in s_2:

$$s_2 : \prod_{a_2\,:\,A_2} \prod_{f_2\,:\,\mathrm{Hom}(a_2,a_3)} X_2\ a_2\ (\lambda f_0\,.\, s_0(f_2 \circ f_0))\ (\lambda f_1\,.\, s_1(f_2 \circ f_1))$$

What is the type of ϕ in the expression $\lambda \phi\,.\, X_2\ a_2\ (\lambda f_0\,.\, s_0(f_2 \circ f_0))\ \phi$? By doing the necessary substitutions, we get:

$$\phi : \prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_2)} X_1\ a_1\ (\lambda f_0\,.\, (\lambda f_0\,.\, s_0(f_2 \circ f_0))(f_1 \circ f_0))$$

The composition on the right β-reduces to $f_2 \circ (f_1 \circ f_0)$, that is:

$$\phi : \prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_2)} X_1\ a_1\ (\lambda f_0\,.\, s_0(f_2 \circ (f_1 \circ f_0)))$$

Now, if we look at the type of $\lambda f_1\,.\, s_1(f_2 \circ f_1)$:

$$\lambda f_1\,.\, s_1(f_2 \circ f_1) : \prod_{a_1\,:\,A_1} \prod_{f_1\,:\,\mathrm{Hom}(a_1,a_2)} X_1\ a_1\ (\lambda f_0\,.\, s_0((f_2 \circ f_1) \circ f_0))$$

As a result, we could only substitute ϕ with $\lambda f_1\,.\, s_1(f_2 \circ f_1)$ if the following equality held *definitionally*:

$$f_2 \circ (f_1 \circ f_0) = (f_2 \circ f_1) \circ f_0$$

This is, of course, not the case in general. A naive workaround would be to try and introduce an additional dependency on proofs of associativity. But then the construction would become quickly unmanageable, as we would also need to depend on proofs of equality of proofs, and so on. This is why it would be interesting to have some sort of diagram of the "Yoneda-esque" presentation that makes the associativity equations more explicit, as I suggested at the end of the previous subsection.

Furthermore, the higher coherence issues we have seen at the beginning (with the "cube of d_i's") seem to have been replaced with higher associativity issues. The last question is therefore this: is there an actual, formal correspondence between the higher structure of coherence equations with the higher structure of associativity equations of an $(\infty, 1)$-category? Has "coherence become associativity"?

REMARK. Another workaround we tried with Hugo Herbelin is to apply the "Yoneda trick" (see [6] for example, called the "Yoneda translation"), which roughly amounts to using the equivalence:

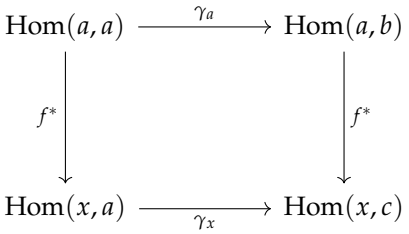
$$\prod_x \mathrm{Hom}(x, a) \longrightarrow \mathrm{Hom}(x, b) \qquad \cong \qquad \mathrm{Hom}(a, b)$$

modulo some additional details. In one direction, it sends dependent products γ to $\gamma_a(1_a)$, and in the other direction it sends morphisms g to the dependent product $\lambda x \,.\, \lambda f : x \longrightarrow a \,.\, g \circ f$.

What we supposedly gain with the "Yoneda trick" is that composition is "lifted" to the level of type theory, and thus holds definitionally. Indeed:

$$\begin{aligned} h : \textstyle\prod_x \mathrm{Hom}(x, b) &\longrightarrow \mathrm{Hom}(x, c) \\ g : \textstyle\prod_x \mathrm{Hom}(x, a) &\longrightarrow \mathrm{Hom}(x, b) \\ \lambda x \,.\, \lambda f : x \longrightarrow a \,.\, h_x(g_x(f)) : \textstyle\prod_x \mathrm{Hom}(x, a) &\longrightarrow \mathrm{Hom}(x, c) \end{aligned}$$

By substituting $\mathrm{Hom}(a_i, a_n)$ with $\prod_x \mathrm{Hom}(x, a_i) \longrightarrow \mathrm{Hom}(x, a_n)$ in the "Yoneda-esque" presentation, we would presumably overcome the associativity issues. But the equivalence would only hold if every dependent product $\gamma : \prod_x \mathrm{Hom}(x, a) \longrightarrow \mathrm{Hom}(x, b)$ was completely determined by its value at $\gamma_a(1_a)$! This is made obvious for any $f : x \longrightarrow a$, by looking at the following diagram:



where f^* is defined as the pre-composition by f. This diagram must commute for the equivalence to make sense, and then we conclude that:

$$\gamma_x(f) = f^*(\gamma_a(1_a))$$

It is essentially the same idea as in the proof of the Yoneda lemma. But now, our associativity issues seem to have come back as indeed we would probably need this equality at some point:

$$(f^* \circ (g^* \circ h^*))(\gamma_a(1_a)) = ((f^* \circ g^*) \circ h^*)(\gamma_a(1_a))$$

References

[1] Thorsten Altenkirch, Paolo Capriotti, and Nicolai Kraus. "Extending Homotopy Type Theory with Strict Equality". In: *25th EACSL Annual Conference on Computer Science Logic*. 2016.

[2] Dimitri Ara, Albert Burroni, Yves Guiraud, Philippe Malbos, François Métayer, and Samuel Mimram. *Polygraphs: From Rewriting to Higher Categories*. In preparation.

[3] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Birkhäuser, 2009.

[4] Hugo Herbelin. "A Dependently-Typed Construction of Semi-Simplicial Types". In: *Mathematical Structures in Computer Science* (2015).

[5] Hugo Herbelin. *A Stratified Dependently-Typed Definition of Simplicial Sets and of Similar Presheaves over a Reedy Category*. Unpublished.

[6] Guilhem Jaber, Gabriel Lewertowski, Pierre-Marie Pédrot, Matthieu Sozeau, and Nicolas Tabareau. "The Definitional Side of the Forcing". In: *Logics in Computer Science*. 2016.

[7] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford University Press, 2002.

[8] Peter LeFanu Lumsdaine. *Semi-Simplicial Types*. 2012. URL: `https://ncatlab.org/ufias2012/published/Semi-simplicial+types`.

[9] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[10] Jon Peter May. *Simplicial Objects in Algebraic Topology*. University of Chicago Press, 1967.

[11] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.

[12] Emily Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014.

[13] Emily Riehl. *Category Theory in Context*. Courier Dover Publications, 2017.

[14] *The Coq Proof Assistant*. 2022. URL: `https://coq.inria.fr/`.