Higher coherence equations of semi-simplicial types as *n*-cubes of proofs

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April 22, 2023

HoTT/UF 2023 — Vienna, Austria

A brief reminder

Semi-simplicial (Δ_+) = Simplicial (Δ) − Degeneracies

- *Sets* $X_0, X_1, X_2, \ldots$
- Face maps $d_i : X_n \to X_{n-1}$ for any $0 \leq i \leq n$
- Satisfying the semi-simplicial identity…

$$d_i d_j = d_{j-1} d_i \quad \text{when} \quad i < j$$

- …And nothing more!

Straightforward to do in type theory by restricting ourselves to **h-Set**s

A brief reminder

Such a simple construction *should be* possible using **Type**s instead of **h-Set**s… right?

Problem

Define semi-simplicial *types* (SSTs) in HoTT

Raised during the Special Year on UF by Voevodsky and others

- **Type**s in HoTT are *weak* ∞-groupoids
- Constructions are no longer set-truncated
- Higher coherence issues!

The semi-simplicial identity $d_i d_j = d_{j-1} d_i$ induces "higher proof terms" that interfere with each other and need to be identified

Illustration

Suppose we want to show that $d_i d_j d_k =_{X_n \to X_{n-3}} d_{k-2} d_{j-1} d_i$ in type theory

- The semi-simplicial identity is the data of a term

$$\alpha_{i,j} : d_i d_j =_{X_n \to X_{n-2}} d_{j-1} d_i$$

- There are two ways to compose the $\alpha_{i,j}$'s together in order to inhabit the type

$$d_i d_j d_k =_{X_n \to X_{n-3}} d_{k-2} d_{j-1} d_i$$

- This is given by

$$\pi := \alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j} \quad \text{and} \quad \pi' := \alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}$$

- We now need the data of a term $\beta_{i,j,k} : \pi = \pi'$, and so on...

Possible approaches

- n-Truncate (e.g. **h-Set**s, **h-Grp**s, ...)

$$X : \Delta_+^{\mathrm{op}} \to \mathbf{h\text{-}Level}(n+2)$$

- For any *externally fixed n*, stop the construction at stage n

$$\begin{aligned}
X_0 &: \mathbf{Type} \\
X_1 &: X_0 \to X_0 \to \mathbf{Type} \\
X_2 &: \textstyle\prod_{a\,b\,c\,:\,X_0} X_1(a, b) \to X_1(b, c) \to X_1(a, c) \to \mathbf{Type} \\
&\vdots \\
X_n &: \textstyle\prod \cdots \to \mathbf{Type}
\end{aligned}$$

- Add an "outer" equality which is strict (2LTT, Altenkirch, Capriotti, Kraus)

A general solution is believed to be impossible in "plain" HoTT

Ten years of investigations

2012–13 **Special Year on UF**

2012–13 *Homotopy Type System* (Voevodsky)

2014–15 *Indexed* + **h-Set**s (Herbelin)

2015 *Logic-enriched HoTT* (Part, Luo)

2015–16 2-*Level Type Theory* (Altenkirch, Capriotti, Kraus)

2022 MLTT + *Type Streams* (Kolomatskaia)

2023 *Indexed* + **h-Set**s + *Parametricity* (Herbelin, Ramachandra)

Strict composition

TLCA 2015 — Warsaw, Poland

What if we write down all the coherence equations explicitly?

The explicit form taken by the higher-order coherence equations is well described in the case of (strict) ω-categories, i.e. when associativity and the exchange law hold on the nose for path composition

- Ross Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49(3):283–335, 1987.
- Ian R. Aitchison. The geometry of oriented cubes. Macquarie University Research Report No: 86–0082, 1986.

n-Cubes of proofs

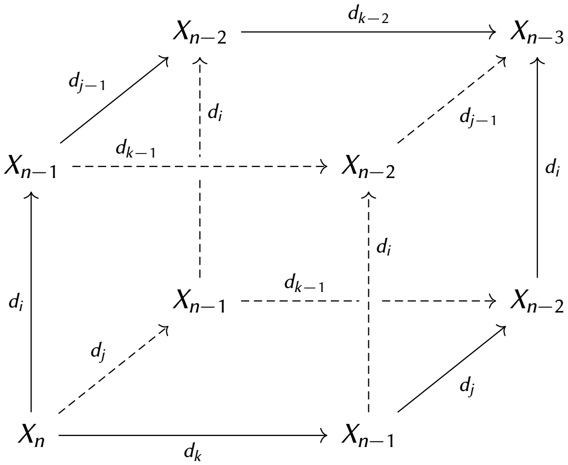
Our goal

Reformulate Aitchison's constructions in type theory, representing higher-order cubes as higher-order equality proofs

- Composition of n face maps corresponds to the "spine" of an n-cube
- (Higher) proof terms correspond to (composition of) *faces* of dimension ≥ 2
- The *k*-hemispheres are special cases of these compositions, as *sources* and *targets* of the equalities

How to give a recursive formulation of all the compositions involved?

3-Cube of proofs



$$\beta_{i,j,k} : \overbrace{\alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j}}^{\text{Backmost}} =_{\left(d_i d_j d_k =_{(X_n \to X_{n-3})} d_{k-2} d_{j-1} d_i\right)} \underbrace{\alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}}_{\text{Frontmost}}$$

Combinatorial structure

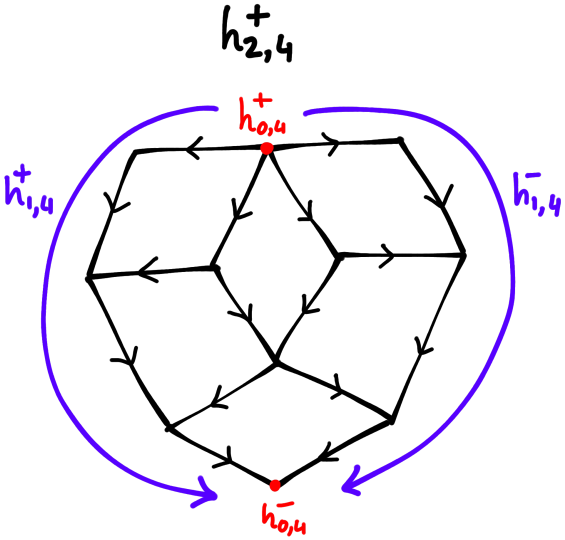
- When $n \geq 4$ associativity and the exchange law are explicitly required
- There are nontrivial proof terms that are not equivalent to k-hemispheres…
- …But are generated by hemispheres interfering at different levels!

Problem

How to describe the k-hemispheres?

- k-Hemispheres are made up of k-faces composed together in some order (not linear *a priori*!)
- We write $h_{k,n}^{\pm}$ for the k-hemispheres of the n-cube

Illustration



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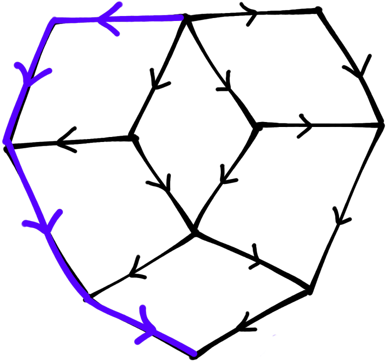

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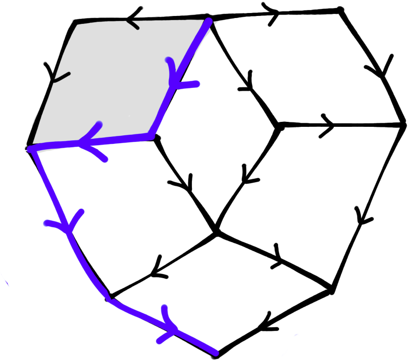

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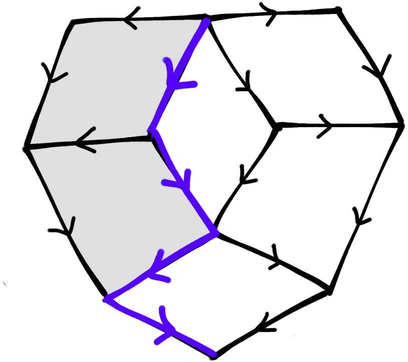

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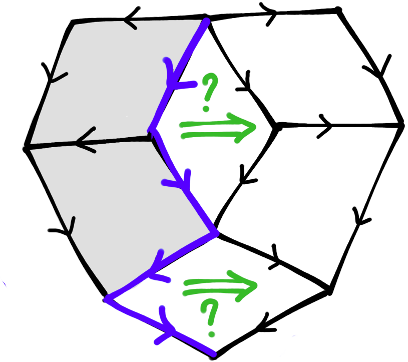


Combinatorial structure

Definition

D_k^n is the poset of *increasing* sequences $x_1 \ldots x_n$ of length n with values $0 \leq x_i \leq k$ equipped with the "pointwise" order

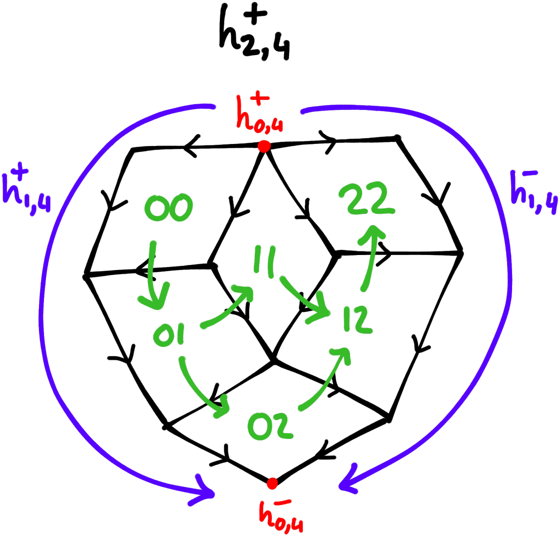
Hasse diagram of D_2^2

$$\begin{array}{ccccc} 00 & & & & \\ \downarrow & & & & \\ 01 & \longrightarrow & 11 & & \\ \downarrow & & \downarrow & & \\ 02 & \longrightarrow & 12 & \longrightarrow & 22 \end{array}$$

Theorem

The k-hemispheres of the n-cube are described by D_{n-k}^k in the sense that there is a bijection sending any $h_{k,n}^{\pm}$ onto a linear extension of D_{n-k}^k

Illustration



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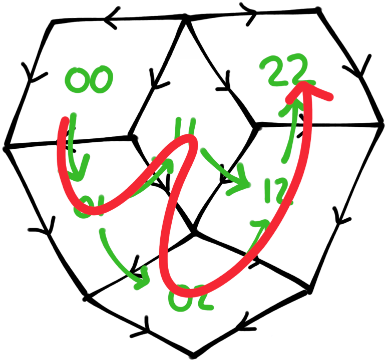

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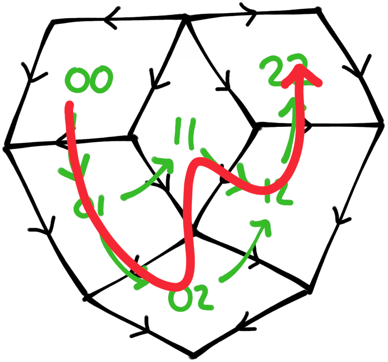


Combinatorial structure

D_k^n has a rich structure—so does $h_{k,n}^{\pm}$ as a result

- There is (essentially) one canonical choice of linear extension of D_k^n given by

$$x \preceq y \iff (x_n, x) \leq (y_n, y)$$

  In our previous examples with D_2^2, it chooses $11 \prec 02$
- D_k^n can be recursively constructed with maps

$$d_* : D_{k-1}^n \to D_k^n \quad \text{and} \quad R : D_k^{n-1} \to D_k^n$$

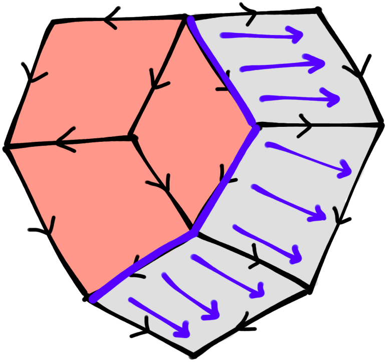
  Then $D_k^n = d_* D_{k-1}^n \amalg R D_k^{n-1}$
- This construction preserves $\preceq$ defined above!

Observation

$h_{k,n}^{\pm}$ has a similar recursive formula

Extrusion of the 3-cube

$$h^+_{2,4} = \text{``}h^+_{2,3} + h^-_{1,3}\text{''}$$



Present and future work

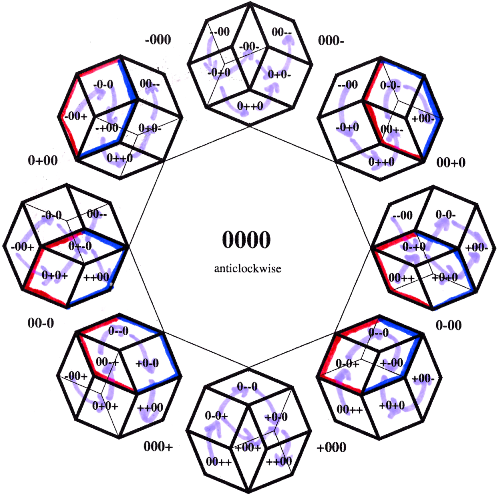
Aitchison's report "only" contains formulae for the k-hemispheres

$$\begin{aligned} \psi_k[n] &= \mu\psi_{k-1}[n-1] \cup \lambda\psi_k[n-1] \\ \omega_k[n] &= \nu\omega_k[n-1] \cup \mu\omega_{k-1}[n-1] \\ &\vdots \end{aligned}$$

- Give an explicit description of the "internal" compositions, which have a structure different from k-hemispheres (see Appendix)
- Make explicit the role of parenthesizing (associativity, exchange law, identity, ...)
- Combine these results to attempt a definition of semi-simplicial types with the coherence equations made explicit at every level

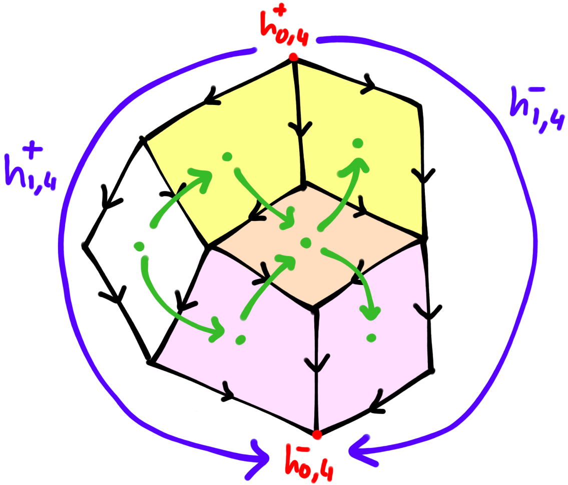
Questions?

Appendix



“Octagon of octagons” — Extracted from Aitchison’s report

Appendix



Appendix