Interaction Equivalence

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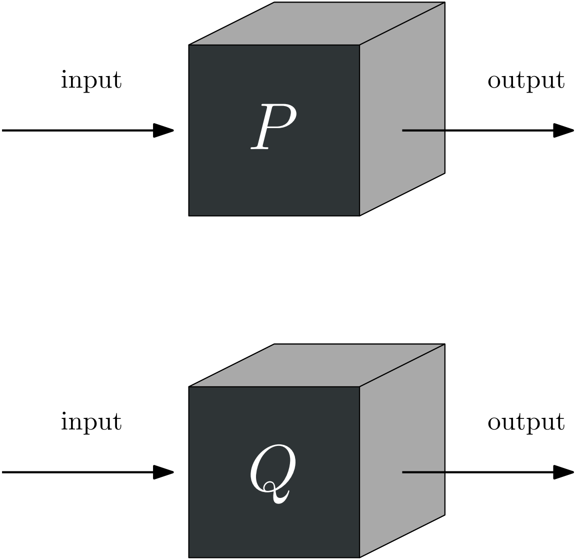
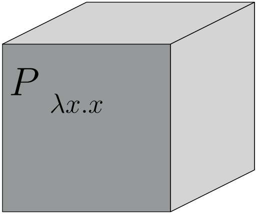
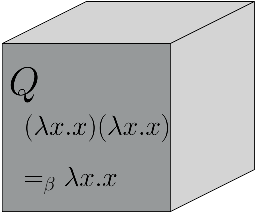
November 21th 2024 – Chocola

Contextual Equivalence

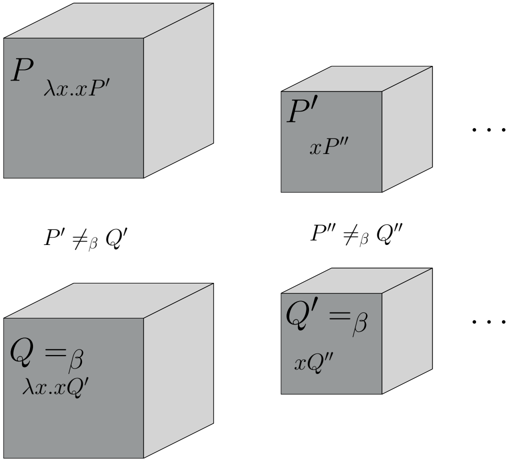


Intensional Equivalences





Coinductive Intensional Equivalences



Reconciling Intensional and Contextual?

When can contextual equivalence be rephrased as an intensional equivalence?

When are intensional equivalences fully abstract?

Can we add intensional information to contextual equivalence?

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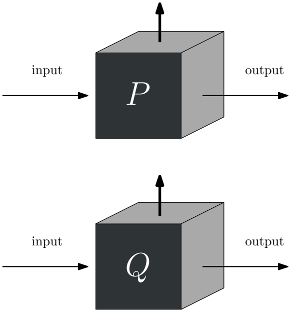

Contextual Equivalence

$$t \equiv^{\mathrm{ctx}} u \quad \text{if for all contexts } C \,.\ \ [\ C\langle t\rangle \Downarrow \ \ \Leftrightarrow \ \ C\langle u\rangle \Downarrow\ ]$$

Is an Equational Theory (for $\Downarrow := \Downarrow_{\mathrm{h}}$):

1. *Compatibility*: if $t \equiv^{\mathrm{ctx}} u$ then $C\langle t\rangle \equiv^{\mathrm{ctx}} C\langle u\rangle$ for all context C;
2. *Invariance*: if $t =_\beta u$ then $t \equiv^{\mathrm{ctx}} u$.

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Sands' improvement

$$t \equiv^{\text{cost}} u \quad \text{if for all contexts } C, \exists k \geq 0\,.\ \ [\ C\langle t\rangle \Downarrow^k \quad \Leftrightarrow \quad C\langle u\rangle \Downarrow^k\ ]$$

Not An Equational Theory!

$\mathtt{I} =_\beta \mathtt{I}\,\mathtt{I}$ but $\mathtt{I} \not\equiv^{\text{cost}} \mathtt{I}\mathtt{I}$

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The best of both worlds?

Can we build a cost-sensitive equational theory?

How can we measure the interaction between a program and a context modulo the internal dynamics?

Our contribution: a framework to identify internal and interaction steps for the untyped λ-calculus
$\rightarrow$ checkers λ-calculus

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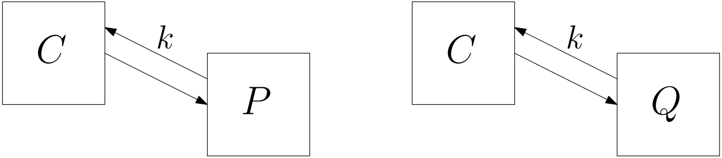

$$P \simeq Q$$

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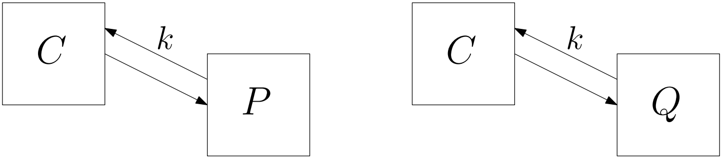

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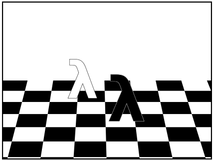

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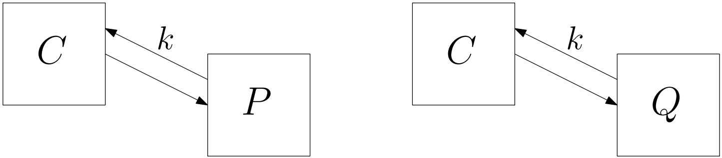
What should be the meaning of a program

Interaction Equivalence is an equational theory!



$$P \simeq Q$$

- Duality between Program/Context reminiscent of Game Semantics
- Modeling communication $P \mid C$ akin to π-calculus and LTS

"The meaning of a program should express its history of access to resources which are not local to it." – Milner 1975

Inspecting Black Boxes

Contextual equivalences are hard to check! $\forall$-quantifier ☹

Intensional equivalences are easy to check! ☺
$\rightarrow$ Böhm tree equivalence, normal form bisimilarity, set of approximants, etc.

Second contribution: interaction equivalence is exactly Böhm tree equivalence

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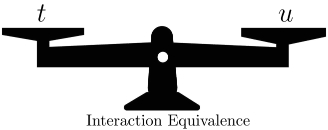
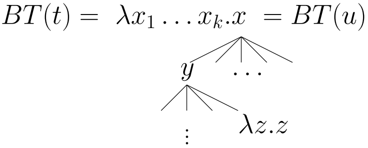
Second contribution: interaction equivalence is exactly Böhm tree equivalence

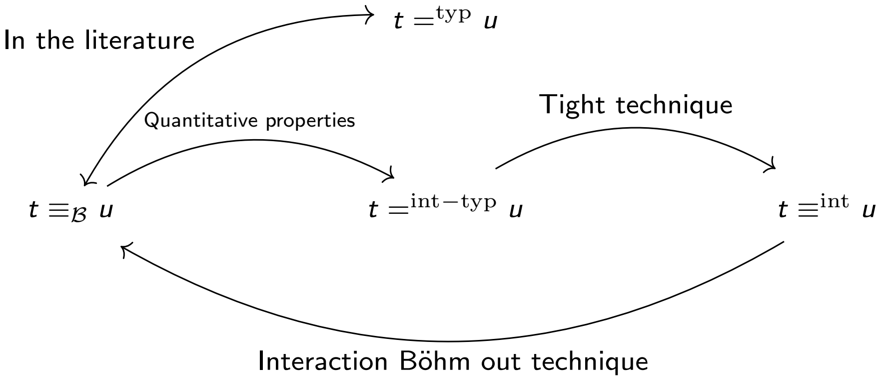


$\Leftrightarrow$

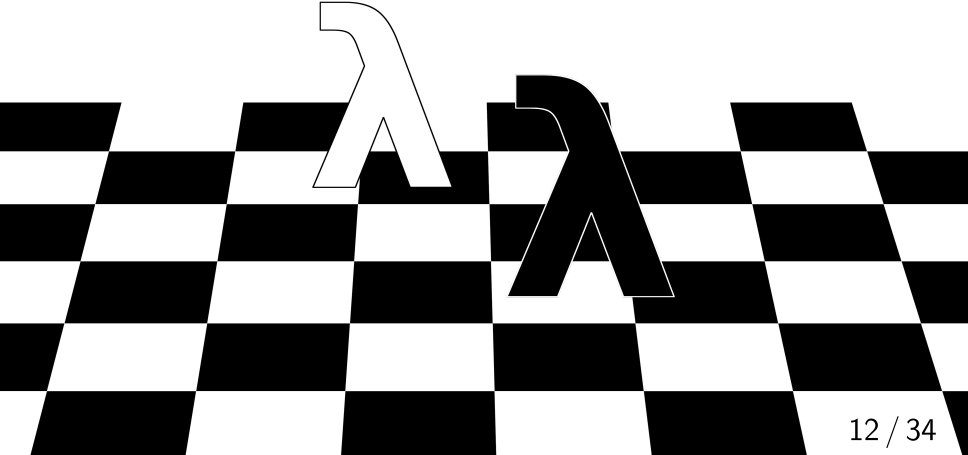


Technical Development

Intersection Types



The Checkers λ-Calculus



The Checkers λ-Calculus

$$
\begin{array}{rcllll}
\Lambda \ni t, u & := & x & \mid \lambda x.t & \mid tu \\
\Lambda_{\bullet\circ} \ni \mathfrak{t}, \mathfrak{u} & := & x & \mid \lambda_{\bullet} x.\mathfrak{t} & \mid \mathfrak{t} \bullet \mathfrak{u} \\
 & & & \mid \lambda_{\circ} x.\mathfrak{t} & \mid \mathfrak{t} \circ \mathfrak{u}
\end{array}
$$

Silent Steps:

$(\lambda_{\circ} x.\mathfrak{t}) \circ \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

$(\lambda_{\bullet} x.\mathfrak{t}) \bullet \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

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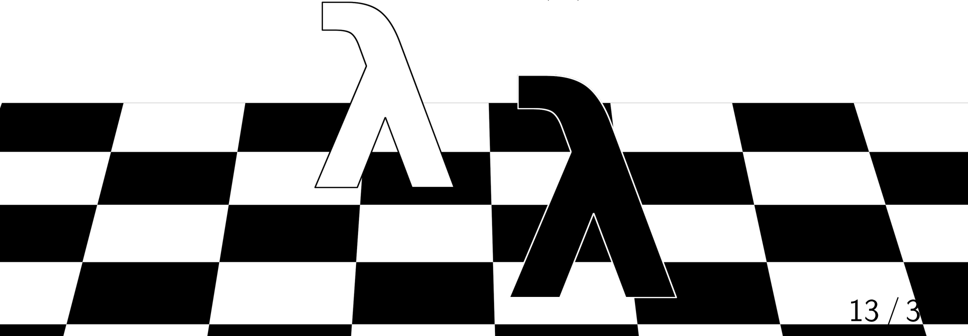
Intuition: $\overline{C}^{\circ}\langle t^{\bullet}\rangle$



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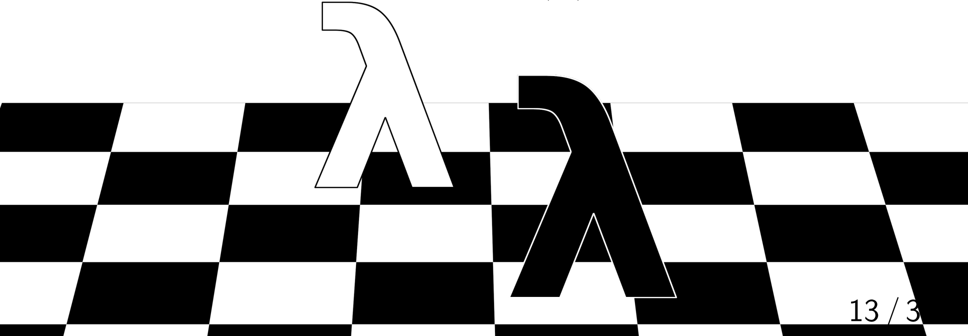
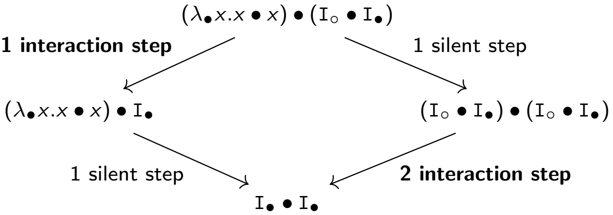


Interaction Cost?

Counting interactions depends on the reduction sequence.



We consider the **head interaction cost** :

$\mathfrak{t} \Downarrow^{\circ\bullet k}_{\mathtt{h}_{\circ\bullet}}$ means $\mathfrak{t}$ head-normalizes with k interaction steps

Interaction Equivalence

Definition

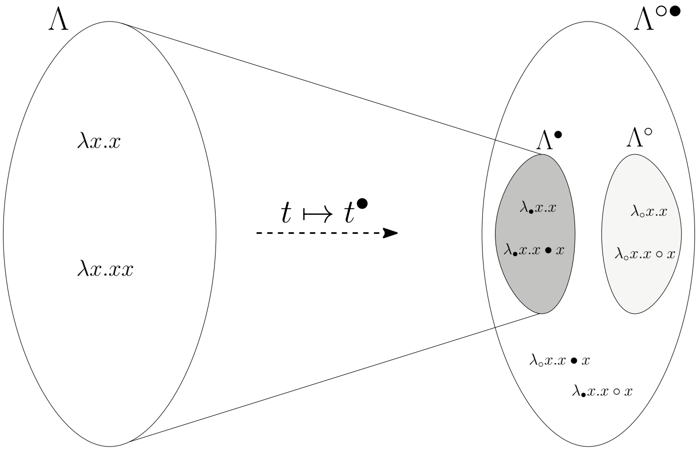
Let $\mathfrak{t}, \mathfrak{t}' \in \Lambda_{\bullet\circ}$

- **Checkers Quantitative Contextual Preorder:** $\mathfrak{t} \sqsubseteq^{\mathrm{ctx}}_{◐} \mathfrak{t}'$ if, for all contexts $\mathfrak{C}$, for all $k \geq 0$, if $\mathfrak{C}\langle\mathfrak{t}\rangle \Downarrow^{◐k}_{\mathtt{h}_{\circ\bullet}}$ then $\mathfrak{C}\langle\mathfrak{t}'\rangle \Downarrow^{◐k}_{\mathtt{h}_{\circ\bullet}}$;
- **Checkers Quantitative Contextual Equivalence:** $\mathfrak{t} \equiv^{\mathrm{ctx}}_{◐} \mathfrak{t}'$ is the equivalence relation induced by $\sqsubseteq^{\mathrm{ctx}}_{◐}$, that is, $\mathfrak{t} \equiv^{\mathrm{ctx}}_{◐} \mathfrak{t}' \iff \mathfrak{t} \sqsubseteq^{\mathrm{ctx}}_{◐} \mathfrak{t}'$ and $\mathfrak{t}' \sqsubseteq^{\mathrm{ctx}}_{◐} \mathfrak{t}$.

Transfering back to the plain λ-calculus, we can define interaction preorder (and equivalence):

$$t \sqsubseteq^{\mathrm{int}} u \text{ if } t^\bullet \sqsubseteq^{\mathrm{ctx}}_{◐} u^\bullet$$

Interaction Equivalence



$t \sqsubseteq^{\text{int}} u$ if $\forall \mathfrak{C}, \forall k,$
$\mathfrak{C}\langle t^\bullet \rangle \Downarrow^{\circ\!\!\bullet k}_{\mathtt{h}_{\circ\bullet}} \implies \mathfrak{C}\langle u^\bullet \rangle \Downarrow^{\circ\!\!\bullet k}_{\mathtt{h}_{\circ\bullet}}$

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Interaction Equivalence is an Equational Theory

1. *Compatibility*: if $t \sqsubseteq^{\mathrm{int}} u$ then $C\langle t\rangle \sqsubseteq^{\mathrm{int}} C\langle u\rangle$ for all context C;
   $\rightarrow$ Straightforward, as $\mathfrak{C}\langle C^{\bullet}\rangle$ is a colored context.

2. *Invariance*: if $t =_{\beta} u$ then $t \sqsubseteq^{\mathrm{int}} u$.
   $\rightarrow$ As in the plain case, it is not immediate: we need to work out theorems between head and beta.
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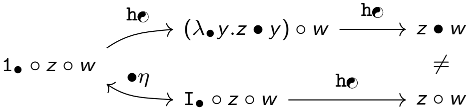
But... What terms are interaction (in)equivalent?

Interaction equivalence is not extensional!

$$\mathtt{I} := \lambda x.x \not\equiv^{\mathrm{int}} \lambda x.\lambda y.xy =: \mathtt{1}$$

- Let $\mathfrak{C} := \langle \cdot \rangle \circ z \circ w$



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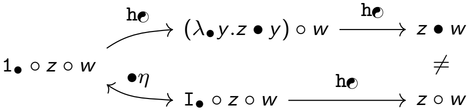

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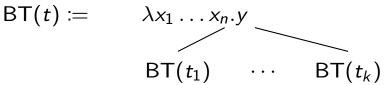
Böhm trees

Definition (Böhm tree of a term)

Let $t \in \Lambda$.

- If $t \to_{\mathrm{h}}^{*} \lambda x_1 \ldots x_n.y\ t_1 \cdots t_k$:



- Otherwise, t is head diverging:

$$\mathrm{BT}(t) \coloneqq \bot.$$

Preorder on Böhm trees: $\mathrm{BT}(t) \leq_{\bot} \mathrm{BT}(u)$ if

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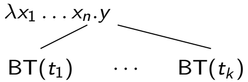
  $\mathrm{BT}(t) :=$ 

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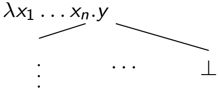
$\mathrm{BT}(t) =$ 

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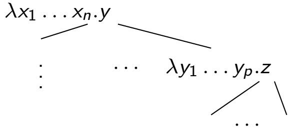
$\mathrm{BT}(u) =$ 

Böhm trees up to η equivalence

$\mathrm{BT}(\mathtt{I}) = \lambda x.x$

$\mathrm{BT}(\mathtt{1}) = \lambda x.\lambda y.x$
$\quad |$
$\quad y$

$\mathrm{BT}(\mathtt{J}) = \lambda x.\lambda y_0.x$
$\quad |$
$\quad \lambda y_1.y_0$
$\quad |$
$\quad \lambda y_2.y_1$
$\quad \vdots$

Theorem (Hyland75/Wadsworth76)

For all λ-terms t, u, we have $t \sqsubseteq^{\mathrm{ctx}} u$ if and only if $t \sqsubseteq_{\mathcal{B}\eta^\infty} u$.
Therefore $t \equiv^{\mathrm{ctx}} u$ if and only if $t =_{\mathcal{B}\eta^\infty} u$.

Interaction ⇒ Böhm

- *Interaction refines Contextual:*
  If $t \sqsubseteq^{\mathrm{int}} u$ then $t \sqsubseteq^{\mathrm{ctx}} u$
- *Interaction does not allow any kind of eta:*
  If $t \sqsubseteq^{\mathrm{int}} u$ and $t \sqsubseteq_{\mathcal{B}\eta^\infty} u$ then $t \sqsubseteq_{\mathcal{B}} u$ **[Technical Lemma]**
  → goes by contrapositive:
  $\not\sqsubseteq_{\mathcal{B}} \subseteq \not\sqsubseteq^{\mathrm{int}} \cup \not\sqsubseteq_{\mathcal{B}\eta^\infty}$
  $\not\sqsubseteq_{\mathcal{B}} \cap \sqsubseteq_{\mathcal{B}\eta^\infty} \subseteq (\not\sqsubseteq^{\mathrm{int}} \cup \not\sqsubseteq_{\mathcal{B}\eta^\infty}) \cap \sqsubseteq_{\mathcal{B}\eta^\infty} = \not\sqsubseteq^{\mathrm{int}} \cap \sqsubseteq_{\mathcal{B}\eta^\infty}$
- *Hyland & Wadsworth:*
  If $t \sqsubseteq^{\mathrm{ctx}} u$ then $t \sqsubseteq_{\mathcal{B}\eta^\infty} u$ **[Standard Böhm out]**

Therefore $\sqsubseteq^{\mathrm{int}} \subseteq \sqsubseteq_{\mathcal{B}}$.

Using coinduction, one would like to say:
$\sqsubseteq^{\mathrm{int}} \cap \sqsubseteq_{\mathcal{B}\eta^\infty}$ is a Böhm simulation,
i.e. $\sqsubseteq^{\mathrm{int}} \cap \sqsubseteq_{\mathcal{B}\eta^\infty} \subseteq \langle \sqsubseteq^{\mathrm{int}} \cap \sqsubseteq_{\mathcal{B}\eta^\infty} \rangle_{\mathcal{B}}$
By contrapositive:
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Interaction $\Rightarrow$ Böhm

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Interaction $\iff$ Böhm

We know that interaction equivalence may only equate at most Böhm tree equivalent terms.
Now we will show that in fact $\sqsubseteq^{\text{int}} = \sqsubseteq_{\mathcal{B}}$.

Through Intersection Types !

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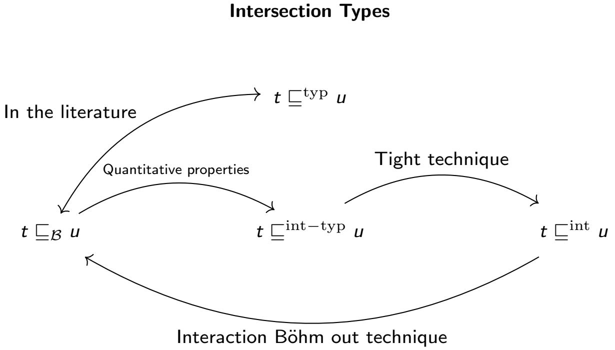

Intersection Types for Plain λ-Calculus

Terms can multiple types:
$x : \beta_1 \cap \beta_2, \ldots, y : \alpha \vdash t : \alpha \cap \beta$

All terminating terms are typable:
$x : \beta \to \alpha, x : \beta \vdash xx : \alpha$

The application rule of intersection types:

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Intersection Types for Plain λ-Calculus

Terms can multiple types:
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$$\begin{array}{rcll} \textsc{Types} & \mathsf{L}, \mathsf{L}' & ::= & \mathsf{A} \mid \mathsf{M} \xrightarrow{\mathsf{pq}} \mathsf{L} \qquad \mathsf{p}, \mathsf{q} \in \{\circ, \bullet\} \\ \textsc{Multi types} & \mathsf{M}, \mathsf{N} & ::= & [\mathsf{L}_1, \ldots, \mathsf{L}_n] \qquad n \geq 0 \end{array}$$

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Checkers Type Equivalence is an Equational Theory

1. *Compatibility*: if $\mathfrak{t} \sqsubseteq_{\bullet}^{\mathrm{typ}} \mathfrak{u}$ then $\mathfrak{C}\langle\mathfrak{t}\rangle \sqsubseteq_{\bullet}^{\mathrm{ctx}} \mathfrak{C}\langle\mathfrak{u}\rangle$ for all black and white context $\mathfrak{C}$;
   $\rightarrow$ Straightforward, as types are defined compositionally.

2. *Invariance*: if $t =_{\beta} u$ then $t \sqsubseteq_{\bullet}^{\mathrm{typ}} u$.
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Good Properties of Checkers Intersection Types

- **Subject Reduction and Expansion:**
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Tightness of Intersection Types

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Proof.
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Böhm ⇒ Type

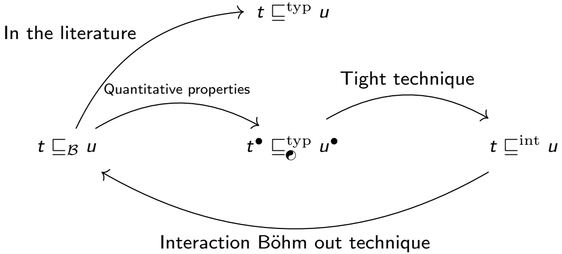
$$t \sqsubseteq_{\mathcal{B}} u \Rightarrow t^{\bullet} \sqsubseteq_{\bullet}^{\mathrm{typ}} u^{\bullet}$$

$t^{\bullet} \sqsubseteq_{\bullet}^{\mathrm{typ}} u^{\bullet}$:
for all $(\pi, \Gamma, \mathsf{L}, k)$ such that $\pi : \Gamma \vdash_{\bullet}^{k} t^{\bullet} : \mathsf{L} \Rightarrow$ there exists π' such that $\pi' : \Gamma \vdash_{\bullet}^{k} u^{\bullet} : \mathsf{L}$

Proof.
By induction on (the size of) the type derivation π.
It is crucial that the type derivation of the normal form of $t^{\bullet}$ is smaller.

Non Idempotent Intersection Types! □

Summing Up the results



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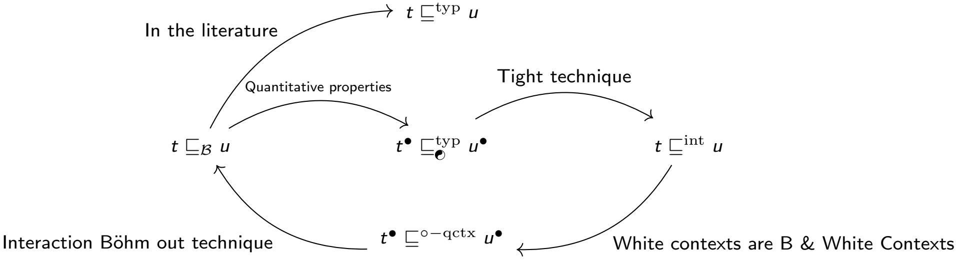

$t^{\bullet} \sqsubseteq^{\circ-\mathrm{qctx}} u^{\bullet}$ if, for all contexts C, there exists k such that $C^{\circ}\langle t^{\bullet}\rangle \Downarrow^{◐ k}_{\mathrm{h}_{\circ\bullet}}$ iff $C^{\circ}\langle u^{\bullet}\rangle \Downarrow^{◐ k}_{\mathrm{h}_{\circ\bullet}}$;

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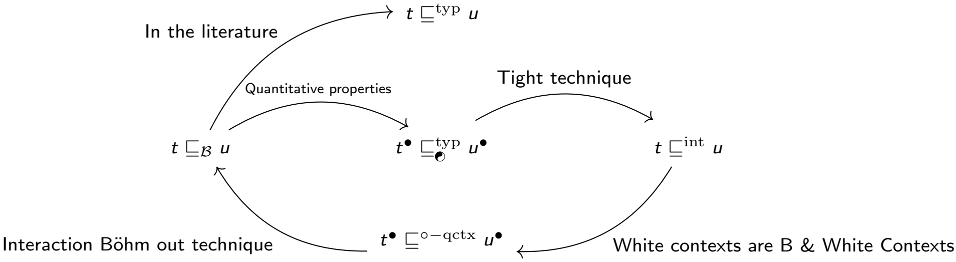

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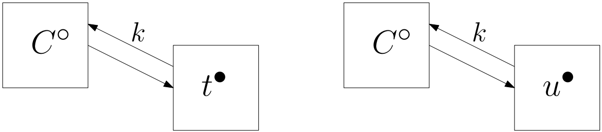

Optimize the number of interactions

Why do we impose that interaction equivalent terms have the same number of interaction?

- **Interaction Improvement:** $\mathfrak{t} \sqsubseteq_{\bullet}^{\text{ctx}} \mathfrak{u}$ if, for all contexts C, if there exists k such that $C\langle\mathfrak{t}\rangle \Downarrow_{\mathtt{h}_{\circ\bullet}}^{\bullet k}$ then $C\langle\mathfrak{u}\rangle \Downarrow_{\mathtt{h}_{\circ\bullet}}^{\bullet k'}$ **with** $k' \leq k$;
- It does not change the associated equivalence relation.

Interaction improvement includes η-reduction:

$$\lambda_{\bullet} y.x \bullet y \sqsubseteq_{\bullet}^{\text{ctx}} x$$

So does the Plain Intersection Types Preorder!

Checkers Types are η-flawed

Our current proof technique cannot go through: Checkers Intersection Types do not support η-reduction.

$$\dfrac{\dfrac{x:[\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^0_{\bullet} x:\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L} \qquad \vdash^0_{\bullet} y:\mathbf{0}}{x:[\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^1_{\bullet} x \bullet y:\mathsf{L}}}{x:[\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^1_{\bullet} \lambda_{\bullet} y.x \bullet y:\mathbf{0} \xrightarrow{\bullet \mathsf{p}} \mathsf{L}}$$

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Maybe the right subtyping relation could work..?

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Conclusion

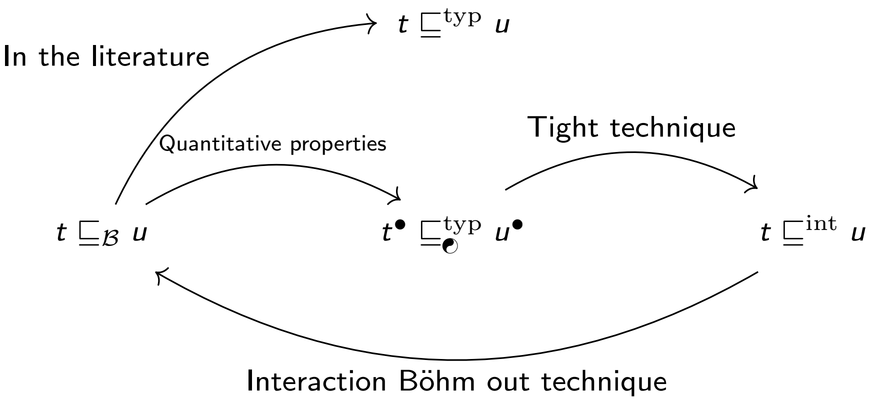
- Checkers Calculus: a new framework to represent interaction between programs
- Interaction Equivalence: a cost-sensitive equational theory
- The first contextual characterization of Böhm tree equivalence without effects! (and simple)

Future work:

- **A conjecture:** interaction improvement is exactly the Böhm tree preorder up to η-reduction
- Work it out in Call-by-Value, and in effectful extensions

  WIP: Weak Head Call-by-Name
- How does it relate to Game Semantics? to process calculi?
- What does our interaction cost represent?

Thank you!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In the literature
Quantitative properties
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