Interaction Equivalence

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Advances in Interactive and Quantitative Semantics

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Program Equivalences

When are two programs equivalent?

In the λ-calculus, natural equivalences arise by identifying terms with the same (possibly infinite) normal form (aka Böhm tree equivalence).

This is not enough, some programs behave in the same way but do not have exactly the same (infinite) normal form.

Contextual Equivalence

t

u

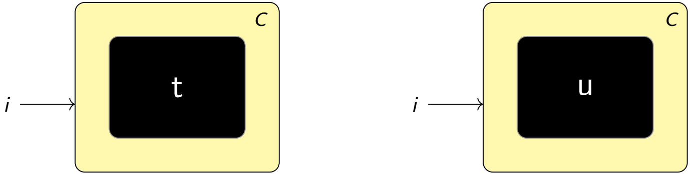
When are two λ-terms equivalent? [Mor68]

$$t \equiv^{\mathrm{ctx}} u \quad \text{if for all contexts } C\,.\ \ [\ C\langle t\rangle \Downarrow \ \Leftrightarrow \ C\langle u\rangle \Downarrow\ ]$$

Head contextual equivalence is an **equational theory**,
where $\mathcal{O}$ $:=$ having a head normal form.

1. *Equivalence Relation*;
2. *Context-Closed*:
$$\text{if } t \equiv^{\mathrm{ctx}} u \text{ then } \forall C,\ C\langle t\rangle \equiv^{\mathrm{ctx}} C\langle u\rangle;$$
3. *Invariance*:
$$\text{if } t =_\beta u \text{ then } t \equiv^{\mathrm{ctx}} u.$$

Contextual Equivalence



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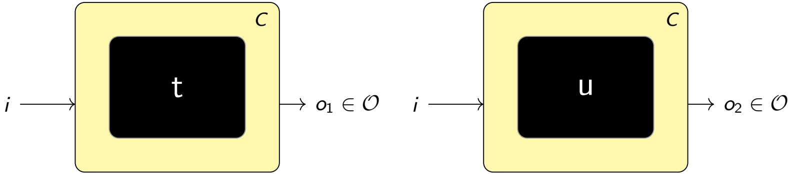

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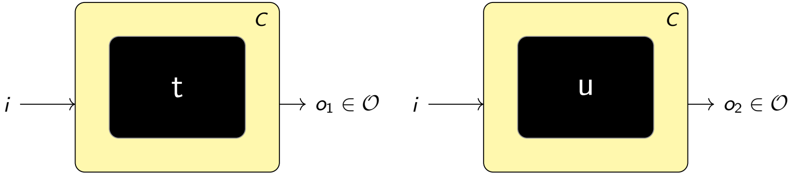

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Sands' improvement & cost equivalence

Sands' **cost equivalence** [San99] is defined as:

$$t \equiv^{\text{cost}} u \quad \text{if } \forall C, \forall k \geq 0 \,. \quad [\ C\langle t\rangle \Downarrow^k_{\mathcal{O}} \quad \Leftrightarrow \quad C\langle u\rangle \Downarrow^k_{\mathcal{O}}\ ]$$

Again, we specify to the λ-calculus, where $\mathcal{O} := \Downarrow_{\mathrm{h}}$.

Not an equational theory!
$\mathtt{I} =_\beta \mathtt{I}\,\mathtt{I}$ but $\mathtt{I} \not\equiv^{\text{cost}} \mathtt{I}\mathtt{I}$

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The best of both worlds?

Can we build a cost-sensitive equational theory?

How can we measure the interaction between a program and a context modulo the internal dynamics?

Our contribution: a framework to identify internal and interaction steps for the untyped λ-calculus
$\rightarrow$ checkers λ-calculus

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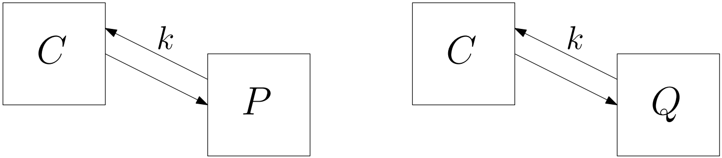

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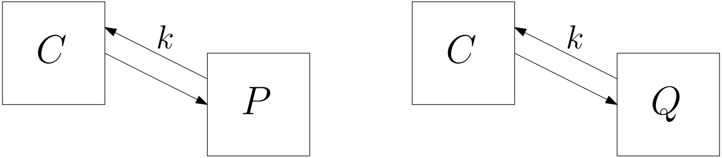

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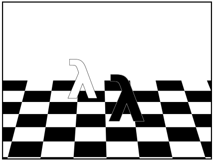

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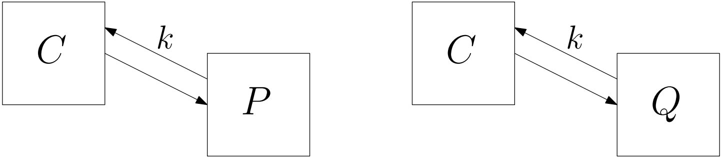
What should be the meaning of a program

Interaction Equivalence is an equational theory!



$$P \simeq Q$$

- Duality between Program/Context reminiscent of Game Semantics
- Modeling communication $P \,|\, C$ akin to π-calculus and LTS

"The meaning of a program should express its history of access to resources which are not local to it." – Milner 1975

Inspecting Black Boxes

Contextual equivalences are hard to check! $\forall$-quantifier

Intensional equivalences are easy to check!
$\rightarrow$ Böhm tree equivalence, normal form bisimilarity, set of approximants, etc.

Second contribution:
interaction equivalence is exactly Böhm tree equivalence

Also answering an open question in the λ-calculus: which contextual equivalence can match Böhm tree equivalence?

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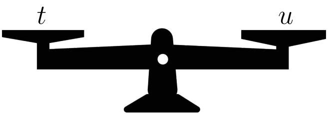

Interaction Equivalence

$\Leftrightarrow$

$$BT(t) = \lambda x_1 \dots x_k.x = BT(u)$$

$y \quad \dots$

$\vdots \quad \lambda z.z$

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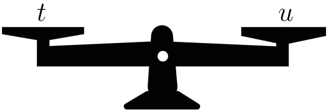

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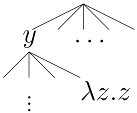

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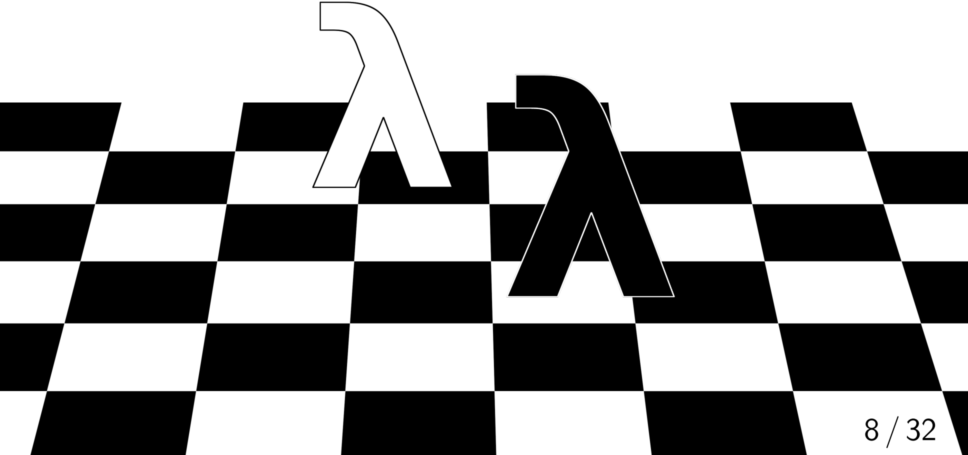
The Checkers λ-Calculus



The Checkers λ-Calculus

$$\Lambda \ni t, u \quad := \quad x \quad | \ \lambda x.t \quad | \ tu$$

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Silent Steps:

$(\lambda_{\bullet} x.\mathfrak{t}) \bullet \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

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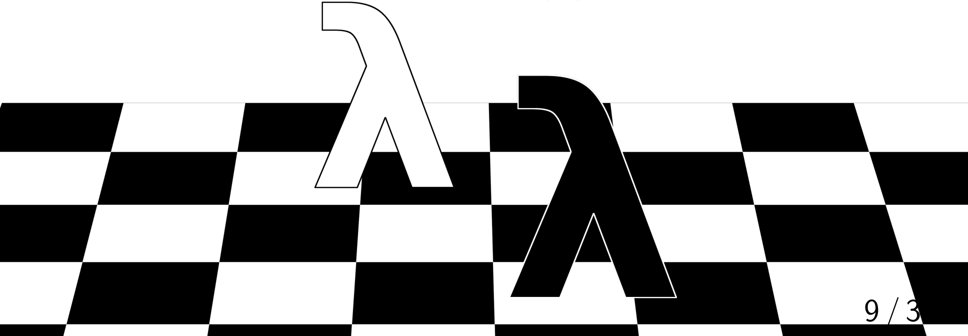
Intuition: $\overline{C}^{\circ}\langle t^{\bullet}\rangle$



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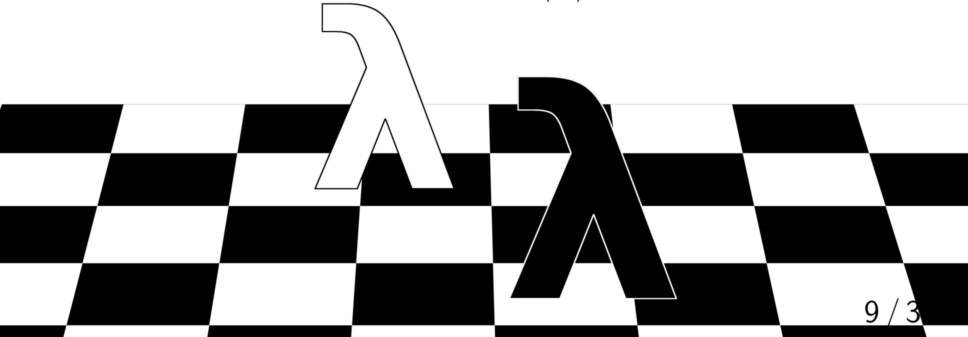

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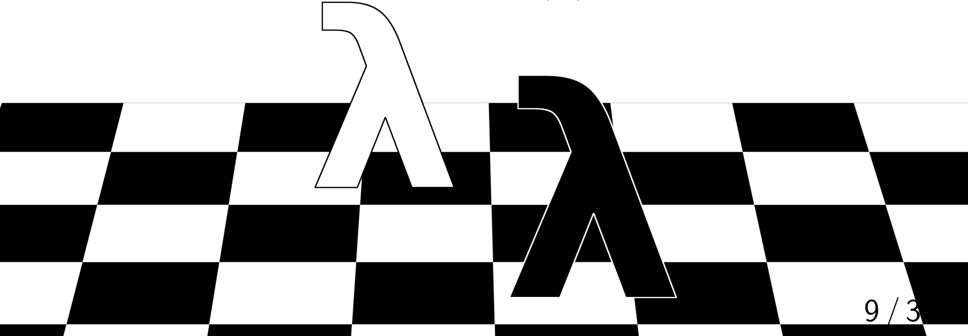

Tags Get Mixed

Intuition: $\overline{C}^{\circ}\langle t^{\bullet}\rangle$

$$C \coloneqq (\lambda x.\langle\cdot\rangle)\mathtt{I}\,\mathtt{I} \qquad t \coloneqq \lambda y.yx$$

$$\begin{aligned}
\overline{C}^{\circ}\langle t^{\bullet}\rangle \quad &= (\lambda_{\circ}x.\lambda_{\bullet}y.y \bullet x) \circ \mathtt{I}_{\circ} \circ \mathtt{I}_{\circ} \\
&\to_{\beta\tau} (\lambda_{\bullet}y.y \bullet \mathtt{I}_{\circ}) \circ \mathtt{I}_{\circ} \\
&\to_{\beta\bullet\!\circ} \mathtt{I}_{\circ} \bullet \mathtt{I}_{\circ} \\
&\to_{\beta\bullet\!\circ} \mathtt{I}_{\circ}
\end{aligned}$$

Interaction Cost?

Counting interactions depends on the reduction sequence.

$$
\begin{tikzcd}[column sep=small]
& (\lambda_\bullet x.x \bullet x) \bullet (\mathtt{I}_\circ \bullet \mathtt{I}_\bullet) \arrow[dl, "\textbf{1 interaction step}"'] \arrow[dr, "\text{1 silent step}"] & \\
(\lambda_\bullet x.x \bullet x) \bullet \mathtt{I}_\bullet \arrow[dr, "\text{1 silent step}"'] & & (\mathtt{I}_\circ \bullet \mathtt{I}_\bullet) \bullet (\mathtt{I}_\circ \bullet \mathtt{I}_\bullet) \arrow[dl, "\textbf{2 interaction step}"] \\
& \mathtt{I}_\bullet \bullet \mathtt{I}_\bullet &
\end{tikzcd}
$$

The checkers calculus is **confluent** but does not preserve interaction steps.

We consider the **head interaction cost** :

$\mathfrak{t} \Downarrow_{\mathtt{h}}^{\bullet k}$ means $\mathfrak{t}$ $\mathtt{head}$-normalizes with k interaction steps

Counting Interactions in Contextual Equivalence

We define quantitative contextual relations for the checkers calculus.
Let $\mathfrak{t}, \mathfrak{t}' \in \Lambda_{\bullet\circ}$

- **Quantitative Contextual Equivalence:**
  $\mathfrak{t} \equiv^{\mathrm{ctx}}_{◐} \mathfrak{u}$ if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle \mathfrak{t} \rangle \Downarrow^{◐ k}_{\mathtt{h}} \iff \mathfrak{C}\langle \mathfrak{u} \rangle \Downarrow^{◐ k}_{\mathtt{h}}\ ];$$
- **Quantitative Contextual Preorder:**
  $\mathfrak{t} \sqsubseteq^{\mathrm{ctx}}_{◐} \mathfrak{u}$ [$\mathfrak{u}$ simulates $\mathfrak{t}$] if,
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Key Point: we ignore silent steps and count interaction steps.

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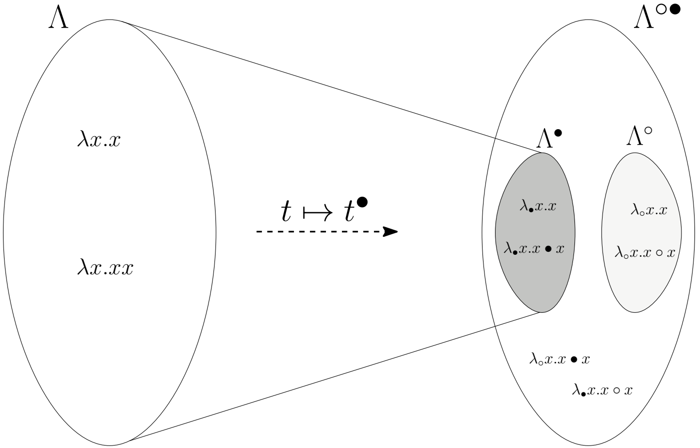

$t \sqsubseteq^{\mathrm{int}} u$ if $t^\bullet \sqsubseteq^{\mathrm{ctx}}_{◐} u^\bullet$

Note that contexts have both **black or white constructors**.

Interaction Equivalence is an Equational Theory

1. *Equivalence Relation*;
2. *Context-Closed*: if $t \sqsubseteq^{\mathrm{int}} u$ then $\forall C$, $C\langle t\rangle \sqsubseteq^{\mathrm{int}} C\langle u\rangle$;
   $\rightarrow$ Straightforward, as $\mathfrak{C}\langle C^\bullet\rangle$ is a colored context.
3. *Invariance*: if $t =_\beta u$ then $t \sqsubseteq^{\mathrm{int}} u$.
   $\rightarrow$ As in the plain case, it requires some work: rewriting theorems between head and beta and specialized to the checkers calculus.
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But... What terms are interaction (in)equivalent?

Interaction equivalence is not stable by η-equivalence!
η-equivalence: $\lambda x.tx =_\eta t$ if $x \notin \mathtt{fv}(t)$

$$\mathtt{I} \coloneqq \lambda x.x \not\equiv^{\mathrm{int}} \lambda x.\lambda y.xy \eqqcolon 1$$

- The **separating** context $\mathfrak{C} \coloneqq \langle\cdot\rangle \circ z \circ w$ distinguishes 1 and $\mathtt{I}$:

$$\begin{tikzcd}
 & (\lambda_\bullet y.z \bullet y) \circ w \arrow[r, "\text{h◐}"] & z \bullet w \arrow[d, phantom, "\neq"] \\
1_\bullet \circ z \circ w \arrow[ur, bend left, "\text{h◐}"] \arrow[dr, leftrightarrow, bend right, "\bullet\eta"] & & \\
 & \mathtt{I}_\bullet \circ z \circ w \arrow[r, "\text{h◐}"] & z \circ w
\end{tikzcd}$$

The interaction preorder strictly refines the contextual preorder.

$$t =_\beta u \implies t \sqsubseteq^{\mathrm{int}} u \underset{\not\Leftarrow}{\implies} t \sqsubseteq^{\mathrm{ctx}} u$$

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- The **separating** context $\mathfrak{C} := \langle\cdot\rangle \circ z \circ w$ distinguishes $\mathtt{1}$ and $\mathtt{I}$:

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Characterizing the Interaction Preorder

The interaction preorder is characterized by the preorder induced by Böhm trees $\sqsubseteq_{\mathcal{B}}$.

To prove that $\sqsubseteq_{\mathcal{B}} = \sqsubseteq^{\mathrm{int}}$, we develop interaction-based refinements of standard techniques for the λ-calculus:

$$\begin{tikzcd} t \sqsubseteq_{\mathcal{B}} u \arrow[rr, bend left, "\text{Black and White Intersection types}"] & & t \sqsubseteq^{\mathrm{int}} u \arrow[ll, bend left, "\text{Interaction Böhm-out}"] \end{tikzcd}$$

Böhm trees

Definition (Böhm tree of a term)

Let $t \in \Lambda$.

- If $t \to_{\mathrm{h}}^{*} \lambda x_1 \ldots x_n.y\ t_1 \cdots t_k$:

$$\mathrm{BT}(t) := \begin{tikzcd}[column sep=tiny] & \lambda x_1 \ldots x_n.y \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{BT}(t_1) & \cdots & \mathrm{BT}(t_k) \end{tikzcd}$$

- Otherwise, t is head diverging:

$$\mathrm{BT}(t) := \bot.$$

Preorder on Böhm trees: $\mathrm{BT}(t) \leq_{\bot} \mathrm{BT}(u)$ if

$$\mathrm{BT}(t) = \begin{tikzcd}[column sep=tiny] & \lambda x_1 \ldots x_n.y \arrow[dl, dash] \arrow[dr, dash] & \\ \vdots & \cdots & \bot \end{tikzcd} \quad \text{and} \quad \mathrm{BT}(u) = \begin{tikzcd}[column sep=tiny] & \lambda x_1 \ldots x_n.y \arrow[dl, dash] \arrow[dr, dash] & & \\ \vdots & \cdots & \lambda y_1 \ldots y_p.z \arrow[dl, dash] \arrow[dr, dash] & \\ & & \cdots & \end{tikzcd}$$

Böhm trees

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Normal Form Bisimulations (instead of Böhm trees)

Definition (Head Normal Form Simulation)

The relation $\mathcal{R} \in \Lambda \times \Lambda$ is a head normal form simulation if: whenever $t\mathcal{R}u$ we have that either:

- *Normal forms:* $t \to_{\mathsf{h}}^{*} \lambda x_1 \dots x_n . y\, t_1 \cdots t_k$ and $u \to_{\mathsf{h}}^{*} \lambda x_1 \dots x_n . y\, u_1 \cdots u_k$ such that $t_i \;\mathcal{R}\; u_i$ for all i.
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Böhm trees up to η equivalence

$$
\mathsf{BT}(\mathtt{I}) = \lambda x.x \qquad \mathsf{BT}(\mathtt{1}) = \begin{array}{c} \lambda x.\lambda y.x \\ | \\ y \end{array} \qquad \mathsf{BT}(\mathtt{J}) = \begin{array}{c} \lambda x.\lambda y_0.x \\ | \\ \lambda y_1.y_0 \\ | \\ \lambda y_2.y_1 \\ \vdots \end{array}
$$

Theorem (Hyland75/Wadsworth76)

For all λ-terms t, u, we have $t \sqsubseteq^{\mathrm{ctx}} u$ if and only if $t \sqsubseteq_{\mathcal{B}\eta^{\infty}} u$.
Therefore $t \equiv^{\mathrm{ctx}} u$ if and only if $t =_{\mathcal{B}\eta^{\infty}} u$.

Interaction $\Rightarrow$ Böhm

- *Interaction refines Contextual:*
  If $t \sqsubseteq^{\mathrm{int}} u$ then $t \sqsubseteq^{\mathrm{ctx}} u$
- *Interaction does not allow any kind of eta:*
  If $t \sqsubseteq^{\mathrm{int}} u$ and $t \sqsubseteq_{\mathcal{B}\eta^\infty} u$ then $t \sqsubseteq_{\mathcal{B}} u$ **[Checkers Böhm out]**
- *Hyland & Wadsworth:*
  If $t \sqsubseteq^{\mathrm{ctx}} u$ then $t \sqsubseteq_{\mathcal{B}\eta^\infty} u$ **[Standard Böhm out]**

Therefore $\sqsubseteq^{\mathrm{int}} \subseteq \sqsubseteq_{\mathcal{B}}$.

White Contexts are Enough

One can define a restrained interaction preorder, where contexts are all white:

$t \sqsubseteq^{\circ-\mathrm{int}} u$ if, for all contexts C, for all k such that $C^{\circ}\langle t^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k}$, then $C^{\circ}\langle u^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k}$;

White contexts are, in particular, checkers contexts: $\sqsubseteq^{\mathrm{int}} \subseteq \sqsubseteq^{\circ-\mathrm{int}}$
We show, by other means, that $\sqsubseteq_{\mathcal{B}} \subseteq \sqsubseteq^{\mathrm{int}}$.
Böhm-out contexts can be white contexts: $\sqsubseteq^{\circ-\mathrm{int}} \subseteq \sqsubseteq_{\mathcal{B}}$
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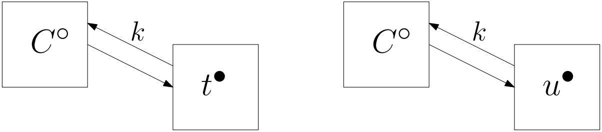

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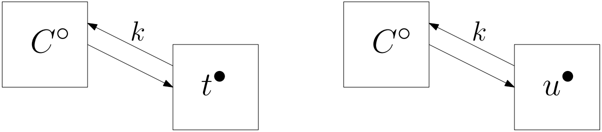

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Interaction $\iff$ Böhm

We know that interaction equivalence may only equate at most Böhm tree equivalent terms.
Now we will show that in fact $\sqsubseteq^{\mathrm{int}} = \sqsubseteq_{\mathcal{B}}$.

Through Intersection Types !

Intersection Types

$$t \equiv_{\mathcal{B}} u \xleftrightarrow{\text{In the literature}} t \equiv^{\mathrm{typ}} u$$

$$\begin{tikzcd}
t \sqsubseteq_{\mathcal{B}} u \arrow[r, bend left, "\text{Quantitative properties}"] & t \sqsubseteq^{\mathrm{int-typ}} u \arrow[r, bend left, "\text{Tight technique}"] & t \sqsubseteq^{\mathrm{int}} u \arrow[ll, bend left, "\text{Interaction Böhm out technique}"]
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Intersection Types for Plain λ-Calculus

Terms can multiple types:
$x : \beta_1 \cap \beta_2, \ldots, y : \alpha \vdash t : \alpha \cap \beta$

All terminating terms are typable:
$x : \beta \to \alpha, x : \beta \vdash xx : \alpha$

The application rule of intersection types:

$$\frac{\Gamma \vdash t : [\beta_1, \ldots, \beta_i] \multimap \alpha \quad \Delta_1 \vdash u : \beta_1 \quad \cdots \quad \Delta_i \vdash u : \beta_i}{\Gamma \uplus \Delta_1 \uplus \cdots \uplus \Delta_i \vdash tu : \alpha} @$$

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Checkers Type Equivalence is an Equational Theory

1. *Compatibility*: if $\mathfrak{t} \sqsubseteq_{\bullet}^{\mathrm{typ}} \mathfrak{u}$ then $\mathfrak{C}\langle\mathfrak{t}\rangle \sqsubseteq_{\bullet}^{\mathrm{ctx}} \mathfrak{C}\langle\mathfrak{u}\rangle$ for all black and white context $\mathfrak{C}$;
   $\rightarrow$ Straightforward, as types are defined compositionally.
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Good Properties of Checkers Intersection Types

- **Subject Reduction and Expansion:**
  If $\mathfrak{t} \rightarrow_{\mathtt{h}_\mathcal{T}} \mathfrak{u}$ then for all (Γ, L), $\quad \Gamma \vdash^{k}_{\bullet} \mathfrak{t} : \mathsf{L} \iff \Gamma \vdash^{k}_{\bullet} \mathfrak{u} : \mathsf{L}$
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- Intuition: $t \Downarrow^{\bullet k}_{\mathtt{h}} \iff \exists \Gamma, \mathsf{L}, \Gamma \vdash^{k}_{\bullet} t : \mathsf{L}$

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Böhm $\Rightarrow$ Type

$$t \sqsubseteq_{\mathcal{B}} u \Rightarrow t^{\bullet} \sqsubseteq_{\circledbullet}^{\mathrm{typ}} u^{\bullet}$$

$t^{\bullet} \sqsubseteq_{\circledbullet}^{\mathrm{typ}} u^{\bullet}$:
for all $(\pi, \Gamma, \mathsf{L}, k)$ such that $\pi : \Gamma \vdash_{\circledbullet}^{k} t^{\bullet} : \mathsf{L} \Rightarrow$ there exists π' such that $\pi' : \Gamma \vdash_{\circledbullet}^{k} u^{\bullet} : \mathsf{L}$

Proof.
By induction on (the size of) the type derivation π.
It is crucial that the type derivation of the normal form of $t^{\bullet}$ is smaller.

Non Idempotent Intersection Types! $\square$

Summing Up

$$\begin{tikzcd}
t \equiv_{\mathcal{B}} u \arrow[r, leftrightarrow, "\text{In the literature}"] & t \equiv^{\mathrm{typ}} u
\end{tikzcd}$$

$$\begin{tikzcd}
t \sqsubseteq_{\mathcal{B}} u \arrow[r, bend left, "\text{Quantitative properties}"] & t^{\bullet} \sqsubseteq^{\mathrm{typ}}_{\LEFTcircle} u^{\bullet} \arrow[r, bend left, "\text{Tight technique}"] & t \sqsubseteq^{\mathrm{int}} u \arrow[dl, bend left, "\text{White contexts are B \& W Contexts}"] \\
& t^{\bullet} \sqsubseteq^{\circ-\mathrm{int}} u^{\bullet} \arrow[ul, bend left, "\text{Interaction Böhm out technique}"] &
\end{tikzcd}$$

Optimize the number of interactions

- **Interaction Preorder:**
  $t \sqsubseteq^{\mathrm{int}} u$ [u simulates t] if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle t^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k} \implies \mathfrak{C}\langle u^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k}\ ];$$
- **Interaction Improvement:**
  $t \sqsubseteq^{\mathrm{int\cdot imp}} u$ [u improves t] if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle t^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k} \implies \mathfrak{C}\langle u^{\bullet}\rangle \Downarrow_{\mathtt{h}}^{\bullet k'} \text{ with } k' \leq k\ ];$$
- It does not change the induced equivalence relation.

η-reduction may improve terms, but η-expanding them can never lead to improvement:

$$\lambda y.xy \sqsubseteq^{\mathrm{int\cdot imp}} x \qquad \text{but} \qquad x \not\sqsubseteq^{\mathrm{int\cdot imp}} \lambda y.xy$$

Future work: characterize exactly the interaction improvement.

$$\sqsubseteq^{\mathrm{int}} \ \underset{?}{\subseteq} \ \sqsubseteq^{\mathrm{int\cdot imp}} \ \subsetneq \ \sqsubseteq^{\mathrm{ctx}}$$

$$\sqsubseteq_{\mathcal{B}} \ \subsetneq \qquad\qquad \subsetneq \ \sqsubseteq_{\mathcal{B}\eta^{\infty}}$$

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Checkers Types are η-flawed

Our current proof technique cannot go through: Checkers Intersection Types do not support η-reduction.

$$\dfrac{\dfrac{x : [\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^{0}_{\bullet} x : \mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L} \qquad \vdash^{0}_{\bullet} y : \mathbf{0}}{x : [\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^{1}_{\bullet} x \bullet y : \mathsf{L}}}{x : [\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \vdash^{1}_{\bullet} \lambda_{\bullet} y . x \bullet y : \mathbf{0} \xrightarrow{\bullet\mathsf{p}} \mathsf{L}}$$

but $x : [\mathbf{0} \xrightarrow{\circ\bullet} \mathsf{L}] \nvdash^{k}_{\bullet} x : \mathbf{0} \xrightarrow{\bullet\mathsf{p}} \mathsf{L}$

Conclusion

- Checkers Calculus: a new framework to represent interaction between programs
- Interaction Equivalence: a cost-sensitive equational theory
- The first contextual characterization of Böhm tree equivalence without non determinism in evaluation contexts! (and simple)

Future work:

- Work it out in Call-by-Value/Need, and in effectful extensions
  $\lambda f.f\,() \equiv_{\mathrm{ctx}} \lambda f.f\,()\,;f\,()$
- How does it relate to Game Semantics? to process calculi?
- What does our interaction cost represent?
  Communication complexity?

Thank you!

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Thank you!

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