Interaction Equivalence

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Program Equivalences

When are two programs equivalent?

In the λ-calculus, natural equivalences arise by identifying terms with the same (possibly infinite) normal form (aka Böhm tree equivalence).

This is not enough, some programs behave in the same way but do not have exactly the same (infinite) normal form.

Contextual Equivalence

t

u

When are two λ-terms equivalent? [Mor68]

$$t \equiv^{\mathrm{ctx}} u \quad \text{if for all contexts } C\,.\ \ [\ C\langle t\rangle \Downarrow \ \Leftrightarrow\ C\langle u\rangle \Downarrow\ ]$$

Head contextual equivalence is an **equational theory**, where $\mathcal{O}$:= having a head normal form.

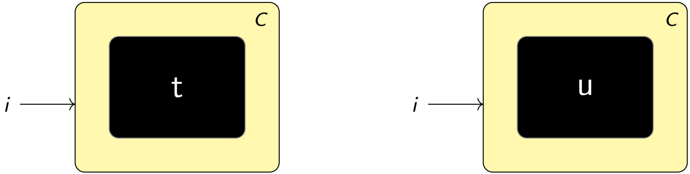
1. *Equivalence Relation*;
2. *Context-Closed*:

   $$\text{if } t \equiv^{\mathrm{ctx}} u \text{ then } \forall C,\ C\langle t\rangle \equiv^{\mathrm{ctx}} C\langle u\rangle;$$

3. *Invariance*:

   $$\text{if } t =_\beta u \text{ then } t \equiv^{\mathrm{ctx}} u.$$

Contextual Equivalence



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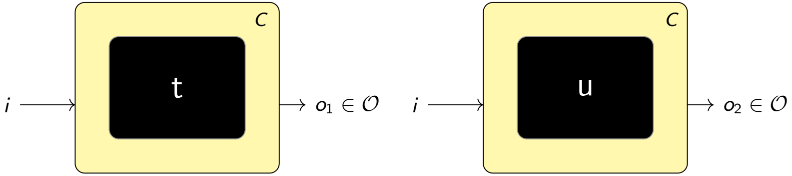

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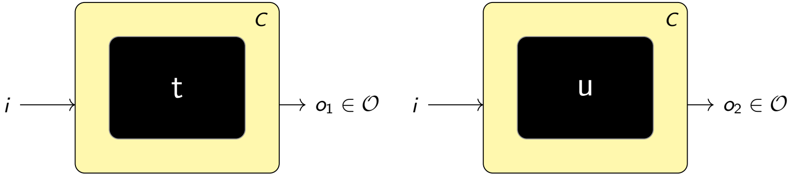

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Sands' improvement & cost equivalence

Sands' **cost equivalence** [San99] is defined as:

$$t \equiv^{\mathrm{cost}} u \quad \text{if } \forall C, \forall k \geq 0\,. \quad [\ C\langle t\rangle {\Downarrow}^k_{\mathcal{O}} \quad \Leftrightarrow \quad C\langle u\rangle {\Downarrow}^k_{\mathcal{O}}\ ]$$

Again, we specify to the λ-calculus, where $\mathcal{O} := \Downarrow_{\mathrm{h}}$.

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The best of both worlds?

Can we build a cost-sensitive equational theory?

How can we measure the interaction between a program and a context modulo the internal dynamics?

Our contribution: a framework to identify internal and interaction steps for the untyped λ-calculus
$\to$ checkers λ-calculus

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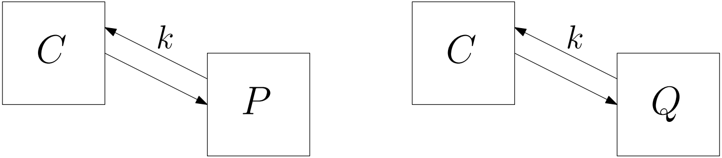

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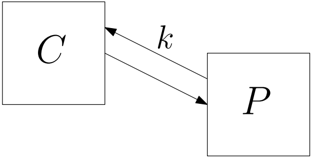

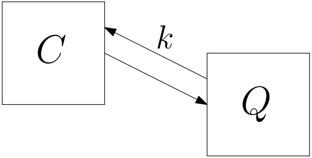

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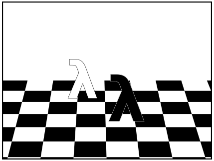

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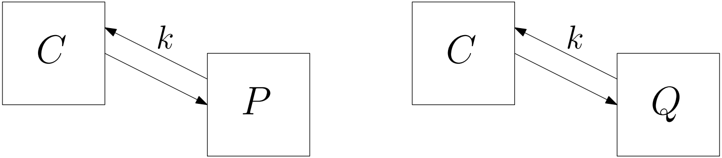
What should be the meaning of a program

Interaction Equivalence is an equational theory!



$$P \simeq Q$$

- Duality between Program/Context reminiscent of Game Semantics
- Modeling communication $P \mid C$ akin to π-calculus and LTS

"The meaning of a program should express its history of access to resources which are not local to it." – Milner 1975

Inspecting Black Boxes

Contextual equivalences are hard to check! $\forall$-quantifier

Intensional equivalences are easy to check!
$\rightarrow$ Böhm tree equivalence, normal form bisimilarity, set of approximants, etc.

Second contribution:
interaction equivalence is exactly Böhm tree equivalence

Also answering an open question in the λ-calculus: which contextual equivalence can match Böhm tree equivalence?

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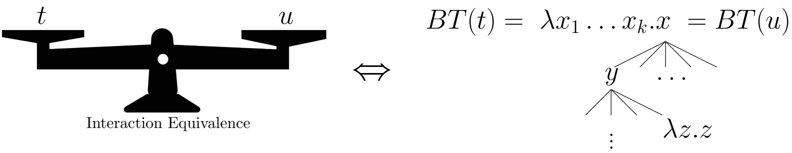

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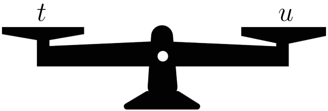
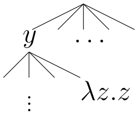


Interaction Equivalence

$\Leftrightarrow$

$$BT(t) = \lambda x_1 \dots x_k.x = BT(u)$$



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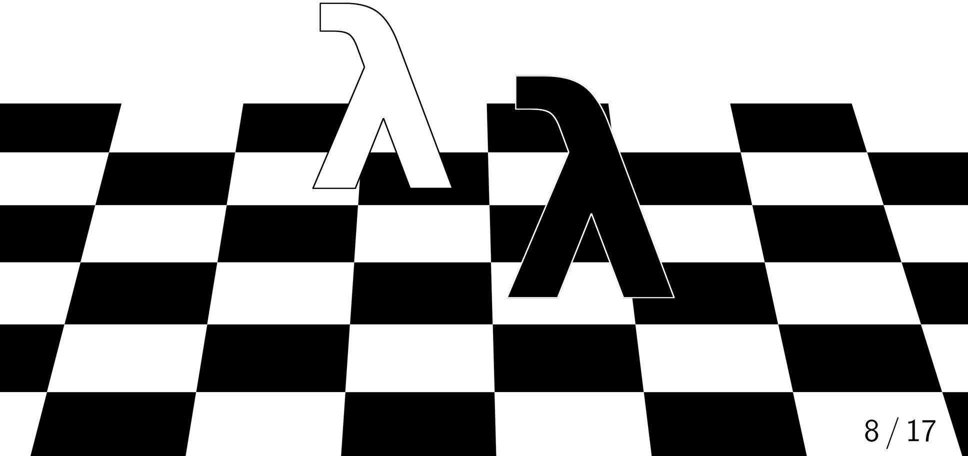
The Checkers λ-Calculus



The Checkers λ-Calculus

$$\Lambda \ni t, u \quad := \quad x \quad \mid \lambda x.t \quad \mid tu$$

$$\Lambda_{\bullet\circ} \ni \mathfrak{t}, \mathfrak{u} \quad := \quad x \quad \mid \lambda_\bullet x.\mathfrak{t} \quad \mid \mathfrak{t} \bullet \mathfrak{u}$$
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Silent Steps:

$(\lambda_\bullet x.\mathfrak{t}) \bullet \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

$(\lambda_\circ x.\mathfrak{t}) \circ \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

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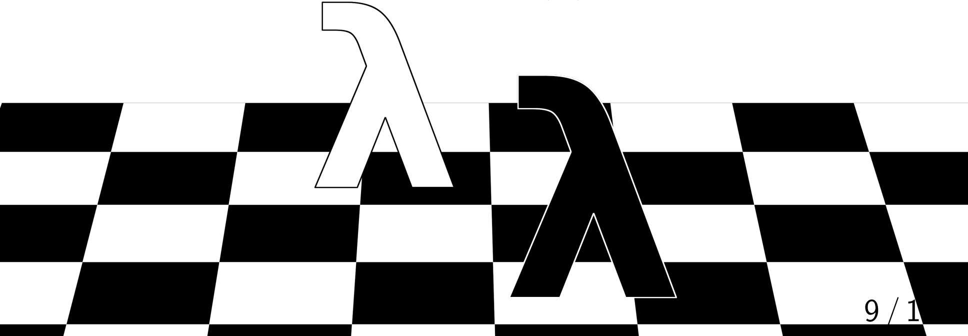
Intuition: $\overline{C}^\circ \langle t^\bullet \rangle$



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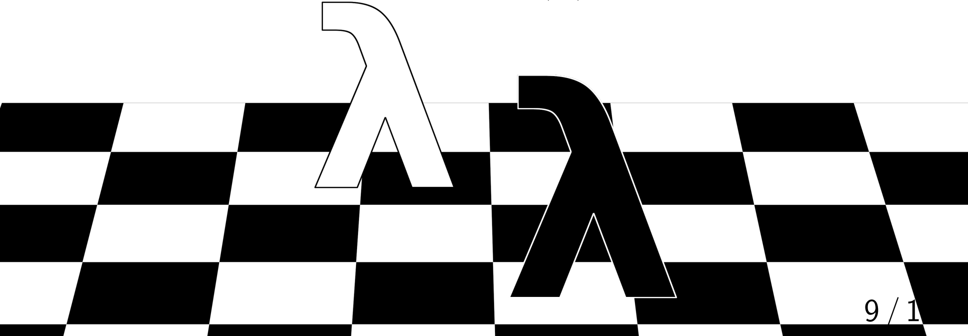

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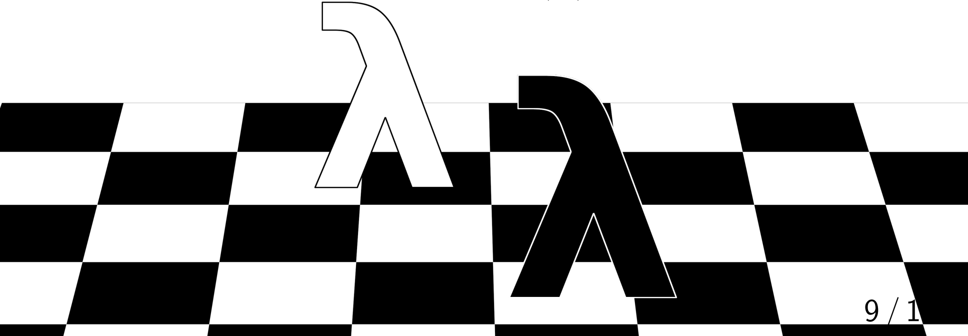
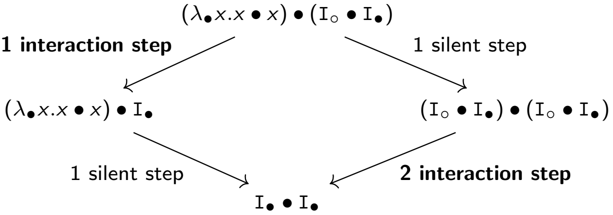


Interaction Cost?

Counting interactions depends on the reduction sequence.



The checkers calculus is **confluent** but does not preserve interaction steps.

We consider the **head interaction cost** :

$\mathfrak{t} \Downarrow_{\mathtt{h}}^{\bullet k}$ means $\mathfrak{t}$ $\mathtt{head}$-normalizes with k interaction steps

Counting Interactions in Contextual Equivalence

We define quantitative contextual relations for the checkers calculus.
Let $\mathfrak{t}, \mathfrak{t}' \in \Lambda_{\bullet\circ}$

- **Quantitative Contextual Equivalence:**
  $\mathfrak{t} \equiv_{\bullet}^{\mathrm{ctx}} \mathfrak{u}$ if,

  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle \mathfrak{t} \rangle \Downarrow_{\mathtt{h}}^{\bullet k} \iff \mathfrak{C}\langle \mathfrak{u} \rangle \Downarrow_{\mathtt{h}}^{\bullet k}\ ];$$

- **Quantitative Contextual Preorder:**
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Key Point: we ignore silent steps and count interaction steps.

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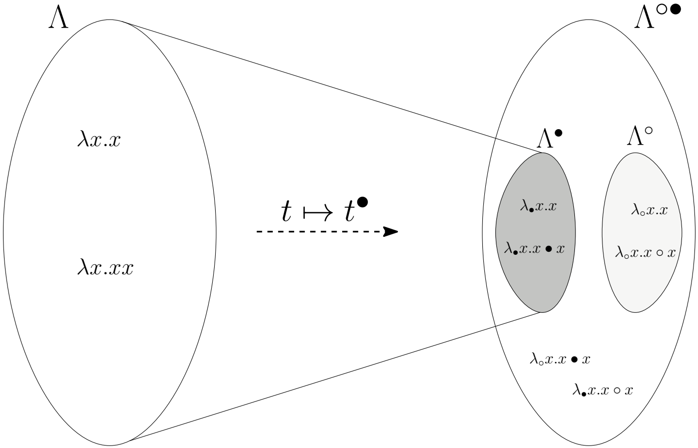

$$t \sqsubseteq^{\mathrm{int}} u \text{ if } t^\bullet \sqsubseteq^{\mathrm{ctx}}_{\circ\bullet} u^\bullet$$

Note that contexts have both **black or white constructors**.

Interaction Equivalence is an Equational Theory

1. *Equivalence Relation*;
2. *Context-Closed*: if $t \sqsubseteq^{\mathrm{int}} u$ then $\forall C$, $C\langle t\rangle \sqsubseteq^{\mathrm{int}} C\langle u\rangle$;
   $\rightarrow$ Straightforward, as $\mathfrak{C}\langle C^{\bullet}\rangle$ is a colored context.
3. *Invariance*: if $t =_{\beta} u$ then $t \sqsubseteq^{\mathrm{int}} u$.
   $\rightarrow$ As in the plain case, it requires some work: rewriting theorems between head and beta and specialized to the checkers calculus.
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But... What terms are interaction (in)equivalent?

Interaction equivalence is not stable by η-equivalence!
η-equivalence: $\lambda x.tx =_\eta t$ if $x \notin \mathtt{fv}(t)$

$$\mathtt{I} := \lambda x.x \not\equiv^{\mathrm{int}} \lambda x.\lambda y.xy =: 1$$

- The **separating** context $\mathfrak{C} := \langle\cdot\rangle \circ z \circ w$ distinguishes 1 and $\mathtt{I}$:

The interaction preorder strictly refines the contextual preorder.

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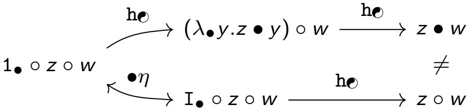

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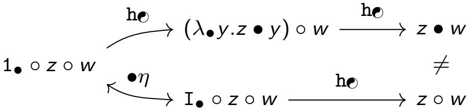

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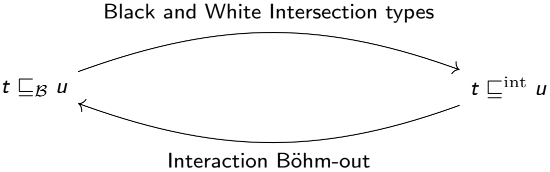
Characterizing the Interaction Preorder

The interaction preorder is characterized by the preorder induced by Böhm trees $\sqsubseteq_{\mathcal{B}}$.

To prove that $\sqsubseteq_{\mathcal{B}} = \sqsubseteq^{\mathrm{int}}$, we develop interaction-based refinements of standard techniques for the λ-calculus:



Optimize the number of interactions

- **Interaction Preorder:**
  $t \sqsubseteq^{\mathrm{int}} u$ [u simulates t] if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle t^\bullet\rangle \Downarrow_{\mathtt{h}}^{\bullet k} \implies \mathfrak{C}\langle u^\bullet\rangle \Downarrow_{\mathtt{h}}^{\bullet k}\ ];$$
- **Interaction Improvement:**
  $t \sqsubseteq^{\mathrm{int\cdot imp}} u$ [u improves t] if,
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η-reduction may improve terms, but η-expanding them can never lead to improvement:

$$\lambda y.xy \sqsubseteq^{\mathrm{int\cdot imp}} x \qquad \text{but} \qquad x \not\sqsubseteq^{\mathrm{int\cdot imp}} \lambda y.xy$$

Future work: characterize exactly the interaction improvement.

$$\sqsubseteq^{\mathrm{int}} \quad \underset{?}{\subseteq} \quad \sqsubseteq^{\mathrm{int\cdot imp}} \quad \subsetneq \quad \sqsubseteq^{\mathrm{ctx}}$$

$$\sqsubseteq_{\mathcal{B}} \quad \subseteq \quad\quad \subsetneq \quad \sqsubseteq_{\mathcal{B}\eta^\infty}$$

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$$\sqsubseteq_{\mathcal{B}} \quad \subseteq \quad \sqsubseteq_{\mathcal{B}} + \eta red \quad \subsetneq \quad \sqsubseteq_{\mathcal{B}\eta^\infty}$$

Conclusion

- Checkers Calculus: a new framework to represent interaction between programs
- Interaction Equivalence: a cost-sensitive equational theory
- The first contextual characterization of Böhm tree equivalence without non determinism in evaluation contexts! (and simple)

Future work:

- Work it out in Call-by-Value/Need, and in effectful extensions
  $\lambda f.f\,() \equiv_{\mathrm{ctx}} \lambda f.f\,()\,;f\,()$
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