What is Lévy-Longo Tree Equivalence ?
A Sequel to Interaction Equivalence

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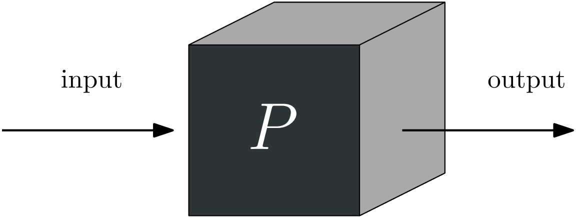
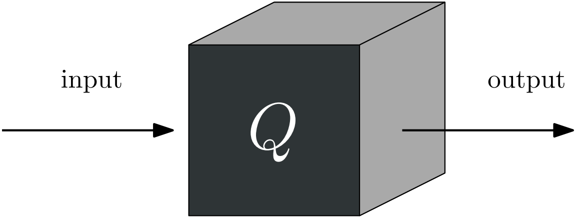
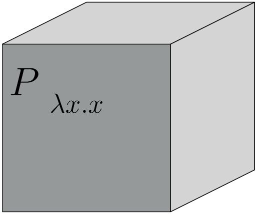
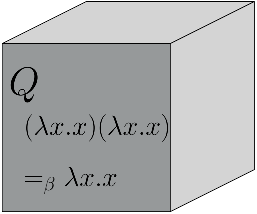
Contextual Equivalence



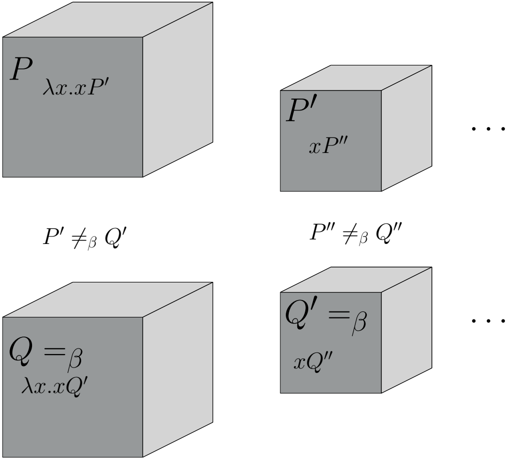


Intensional Equivalences





Coinductive Intensional Equivalences



Reconciling Intensional and Contextual?

When can contextual equivalence be rephrased as an intensional equivalence?

When are intensional equivalences fully abstract?

Can we add intensional information to contextual equivalence?
$\rightarrow$ **Interaction Equivalence** – Accattoli, Lancelot, Manzonetto and Vanoni

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Contextual Equivalence

$$t \equiv^{\mathrm{ctx}} u \quad \text{if for all contexts } C\,.\quad [\ C\langle t\rangle \Downarrow \quad \Leftrightarrow \quad C\langle u\rangle \Downarrow\ ]$$

Is an Equational Theory (for $\Downarrow := \Downarrow_{\mathrm{h}}$):

1. *Compatibility*: if $t \equiv^{\mathrm{ctx}} u$ then $C\langle t\rangle \equiv^{\mathrm{ctx}} C\langle u\rangle$ for all context C;
2. *Invariance*: if $t =_\beta u$ then $t \equiv^{\mathrm{ctx}} u$.

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Sands' improvement

$$t \equiv^{\text{cost}} u \quad \text{if for all contexts } C, \exists k \geq 0 \,.\ \ [\ C\langle t\rangle \Downarrow^k \quad \Leftrightarrow \quad C\langle u\rangle \Downarrow^k\ ]$$

Not An Equational Theory!

$\mathtt{I} =_\beta \mathtt{I}\,\mathtt{I}$ but $\mathtt{I} \not\equiv^{\text{cost}} \mathtt{I}\mathtt{I}$

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The best of both worlds?

Can we build a cost-sensitive equational theory?

How can we measure the interaction between a program and a context modulo the internal dynamics?

Our contribution: a framework to identify internal and interaction steps for the untyped λ-calculus
$\rightarrow$ checkers λ-calculus

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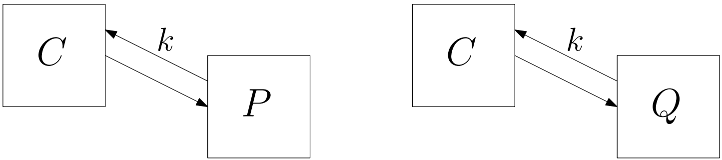

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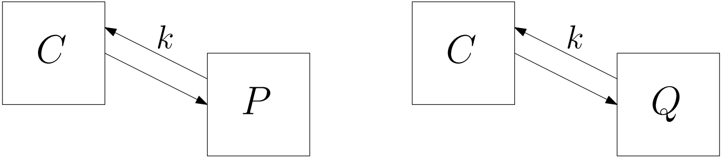

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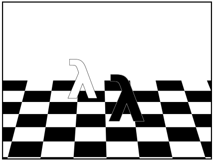

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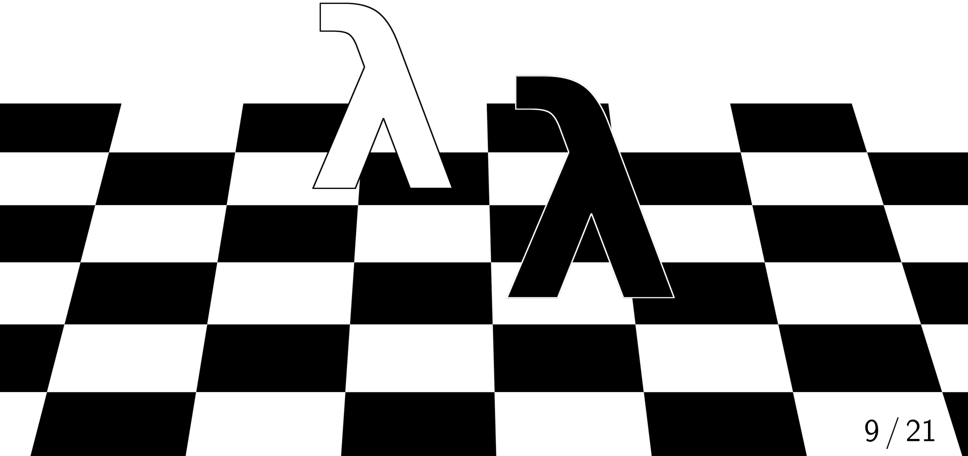
The Checkers λ-Calculus



The Checkers λ-Calculus

$$\begin{array}{rcl} \Lambda \ni t,u & := & x \mid \lambda x.t \mid tu \\ \Lambda_{\bullet\circ} \ni \mathfrak{t},\mathfrak{u} & := & x \mid \lambda_\bullet x.\mathfrak{t} \mid \mathfrak{t} \bullet \mathfrak{u} \\ & & \quad \mid \lambda_\circ x.\mathfrak{t} \mid \mathfrak{t} \circ \mathfrak{u} \end{array}$$

Silent Steps:

$(\lambda_\circ x.\mathfrak{t}) \circ \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

$(\lambda_\bullet x.\mathfrak{t}) \bullet \mathfrak{u} \mapsto_{\beta\tau} \mathfrak{t}\{x := \mathfrak{u}\}$

Interaction Steps:

$(\lambda_\circ x.\mathfrak{t}) \bullet \mathfrak{u} \mapsto_{\beta\text{◐}} \mathfrak{t}\{x := \mathfrak{u}\}$

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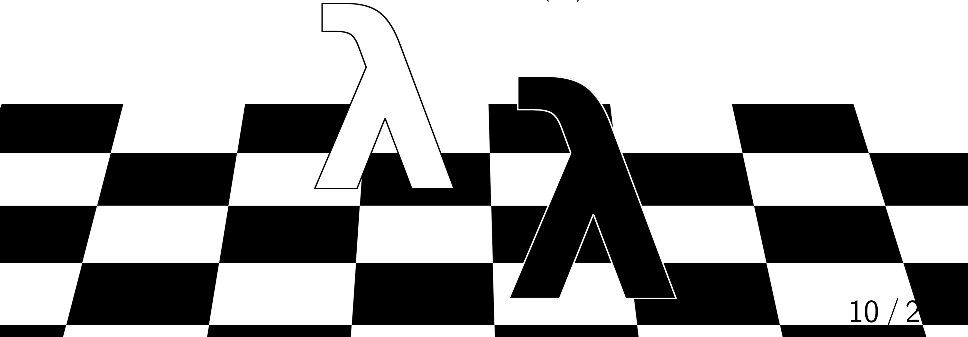
Intuition: $\overline{C}^\circ\langle t^\bullet\rangle$



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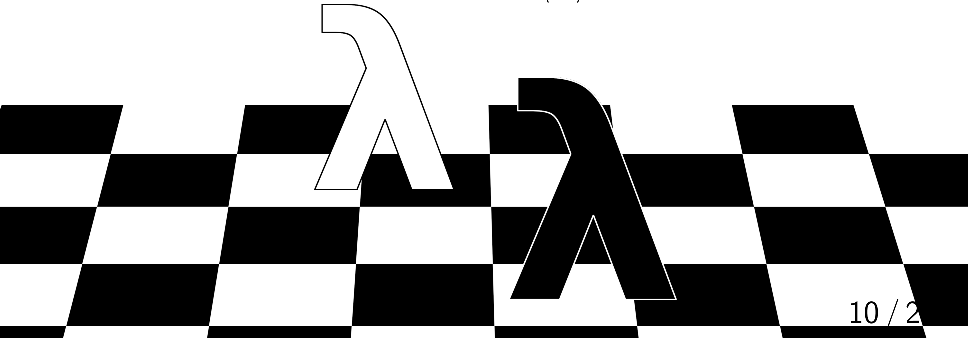
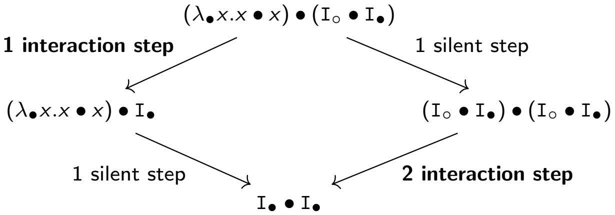


Interaction Cost?

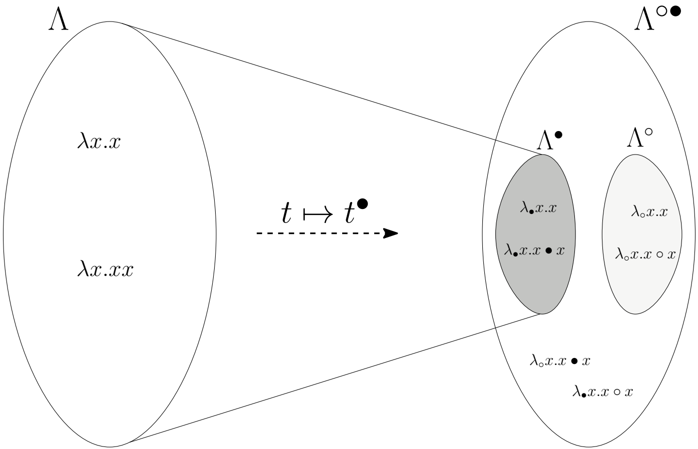
Counting interactions depends on the reduction sequence.



One needs to consider a specific *evaluation strategy*!

$\mathfrak{t} \Downarrow^{\bullet\!\circ k}_{\mathtt{h}_{\circ\bullet}}$ means $\mathfrak{t}$ head-normalizes with k interaction steps

Interaction Equivalence



$t \sqsubseteq^{\text{int}} u$ if $\forall \mathfrak{C}, \forall k,$
$\mathfrak{C}\langle t^\bullet \rangle \Downarrow^{\circ\bullet k}_{\mathtt{h}_{\circ\bullet}} \implies \mathfrak{C}\langle u^\bullet \rangle \Downarrow^{\circ\bullet k}_{\mathtt{h}_{\circ\bullet}}$

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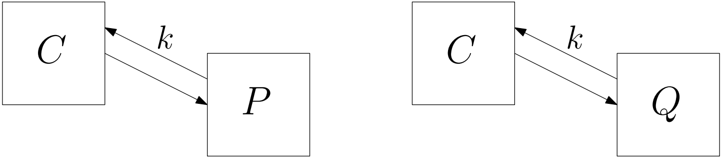
What should be the meaning of a program

Interaction Equivalence is an equational theory!



$$P \simeq Q$$

- Duality between Program/Context reminiscent of Game Semantics
- Modeling communication $P \mid C$ akin to π-calculus and LTS

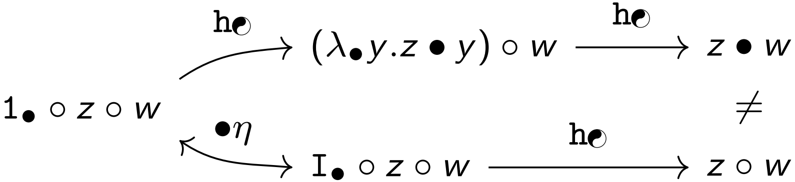
"The meaning of a program should express its history of access to resources which are not local to it." – Milner 1975

But... What terms are interaction (in)equivalent?

Interaction equivalence is not extensional!

$$\mathtt{I} := \lambda x.x \not\equiv^{\mathrm{int}} \lambda x.\lambda y.xy =: \mathtt{1}$$

- Let $\mathfrak{C} := \langle\cdot\rangle \circ z \circ w$



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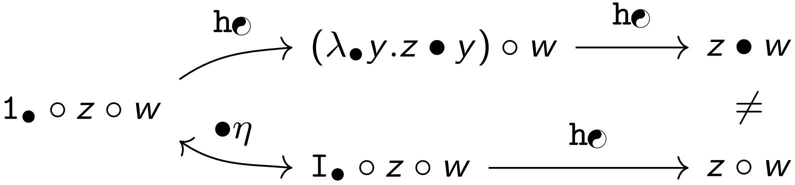

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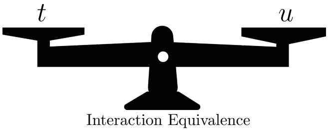
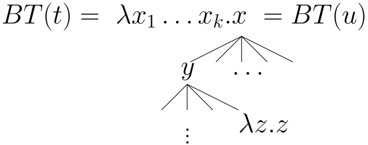
Inspecting Black Boxes

Second contribution: interaction equivalence is exactly Böhm tree equivalence



$\Leftrightarrow$



It even turns out that in the head case, it is the same to only look at white contexts.

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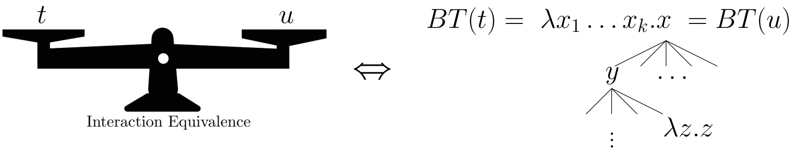

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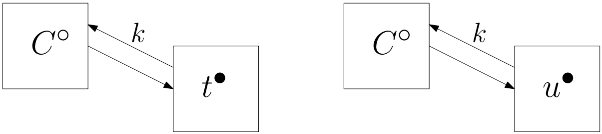


Interaction Equivalence in other paradigms

Now, what about other evaluation strategies?
WIP with Giulio Manzonetto: weak head evaluation

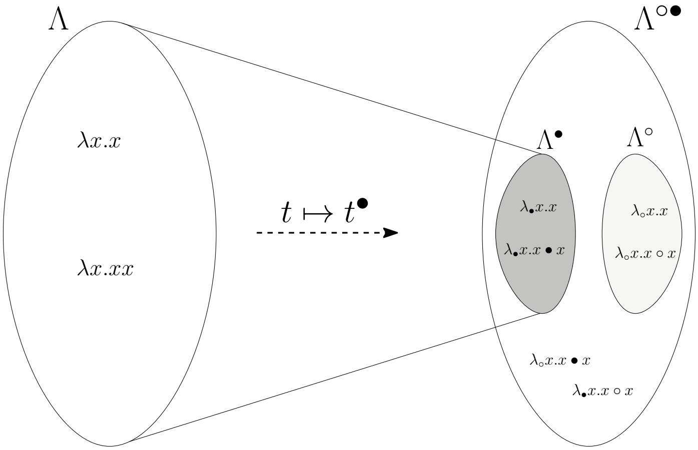
$$\begin{array}{rlcl} \textsc{Silent} & (\lambda_{\mathsf{p}} x.t) \cdot^{\mathsf{p}} u & \mapsto_{\beta_\tau} & t\{x := u\} \\ \textsc{Interaction} & (\lambda_{\mathsf{p}} x.t) \cdot^{\mathsf{p}^\perp} u & \mapsto_{\beta_\bullet} & t\{x := u\} \end{array}$$

$$\frac{\mathfrak{t} \mapsto_{\beta_\tau} \mathfrak{u}}{\mathfrak{t} \to_{\mathrm{wh}_\tau} \mathfrak{u}} \qquad \frac{\mathfrak{t} \mapsto_{\beta_\bullet} \mathfrak{u}}{\mathfrak{t} \to_{\mathrm{wh}_\bullet} \mathfrak{u}}$$

$$\frac{\mathfrak{t} \to_{\mathrm{wh}_\tau} \mathfrak{t}'}{\mathfrak{t}\mathfrak{u} \to_{\mathrm{wh}_\tau} \mathfrak{t}'\mathfrak{u}} \qquad \frac{\mathfrak{t} \to_{\mathrm{wh}_\bullet} \mathfrak{t}'}{\mathfrak{t}\mathfrak{u} \to_{\mathrm{wh}_\bullet} \mathfrak{t}'\mathfrak{u}}$$

Functions $\lambda x.t$ are normal forms.

Weak Head Interaction Equivalence



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Lévy-Longo & Interaction

Lévy-Longo trees are the weak variant of Böhm trees.

[WIP] Interaction Equivalence $\Leftrightarrow$ Lévy-Longo Equivalence.

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Lévy-Longo vs. Contextual vs. Interaction

Some key examples:

η-equivalence: $\lambda x.\lambda y.xy$ and $\lambda x.x$

$\rightarrow$ these terms are already discriminated by weak head contextual equivalence: $C := \langle\cdot\rangle\Omega$

sound η-equivalence: $\lambda x.x\lambda y.xy$ and $\lambda x.xx$

This is the classical example of contextually equivalent terms that are not Lévy-Longo equivalent.
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Lévy-Longo matches Interaction and Relational

folklore, Lancelot 2024

scrutiny of known proofs

$t \sqsubseteq^{\mathrm{rel}} u$ $\qquad$ $t \sqsubseteq_{\mathcal{LLT}} u$ $\qquad$ $t \sqsubseteq^{\mathrm{int}} u$

Weak Head Interaction Böhm out technique, **WIP**

We cannot restrict to white contexts:

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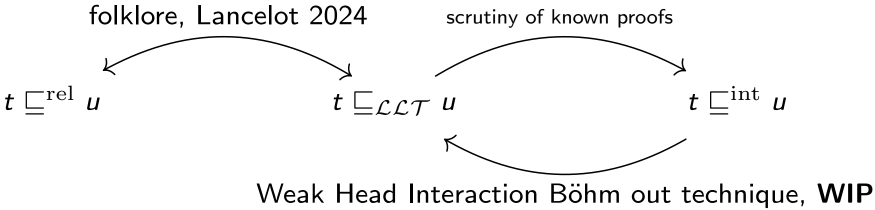

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Conclusion

Interaction Equivalence:

- Checkers Calculus: a new framework to represent interaction between programs
- Interaction Equivalence: a cost-sensitive equational theory
- The first contextual characterization of Böhm tree equivalence without effects (and simple!)

WIP and future work:

- Weak Head Interaction Equivalence, exactly matches Lévy-Longo equivalence
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Optimize the number of interactions

Why do we impose that interaction equivalent terms have the same number of interaction?

- **Interaction Improvement:** $\mathfrak{t} \sqsubseteq^{ctx}_{\bullet} \mathfrak{u}$ if, for all contexts C, if there exists k such that $C\langle\mathfrak{t}\rangle \Downarrow^{\bullet k}_{\mathrm{wh}_{\circ\bullet}}$ then $C\langle\mathfrak{u}\rangle \Downarrow^{\bullet k'}_{\mathrm{wh}_{\circ\bullet}}$ **with** $k' \leq k$;
- It does not change the associated equivalence relation.

Interaction improvement includes η-reduction:

$$\lambda_{\bullet} y.x \bullet y \sqsubseteq^{ctx}_{\bullet} x$$

So does the Plain Intersection Types Preorder!