# Light Genericity

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# Outline

Computable Functions & the Lambda Calculus

From Barendregt's Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion

# Partial Recursive Functions

Partial Recursive Functions model which mathematical functions are computable.

There is a natural *extensional preorder* on partial functions

$$f \leq_{\mathrm{PRF}} g \text{ if } \forall n \in \mathbb{N},\ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n)$$

$f_\bot : n \mapsto \bot$ is the minimum PRF function for $\leq_{\mathrm{PRF}}$

# Lambda Calculus

PRF do not look at how to compute, hence the preorder can only be extensional.

Instead, in the lambda calculus, how to compute is a critical concept.

There are a rich number of possible equivalences (or preorders) of lambda terms, both extensional or intensional.

# Computable Functions & Lambda Calculus

Partial recursive functions embed in the lambda calculus.

What is the lambda term that represents **undefined**?

A computation that never ends? $\Omega$!

Now, what is the equivalence class of **undefined**/$\Omega$ ?

# A first naive attempt

Undefined represents a computation that never ends.

- **undefined** terms $= \beta$-diverging terms?

Surprisingly, this would lead to an inconsistency.

If all $\beta$-diverging terms are equated in an equational theory, then this theory equates all terms.

# $\beta$-diverging terms may be very different

Indeed, let us look at two $\beta$-diverging terms

$$\begin{array}{ccc}
\mathit{fix} & \text{and} & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (\mathit{fix}\, f) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (f\ (\mathit{fix}\, f)) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (f\ (f\ (\mathit{fix}\, f))) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (f\ (f\ (f \ldots))) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\vdots & & \vdots
\end{array}$$

Recursion does not carry the same meaning as looping on itself.

# A second attempt

Instead, one might consider a more restrained reduction

- **undefined** terms $=$ **head**-diverging terms?

The equational theory that identifies **head**-diverging terms is consistent.

$>>$ This theory does not equate all terms.

# $\beta$-diverging terms may be very different

Fixpoint combinators are **head**-normalizing.

$$\begin{array}{ccc} \mathit{fix} & \text{and} & \Omega \\ \downarrow_h & & \downarrow_h \\ \lambda f.f\ (\mathit{fix}\, f) & & \Omega \\ \not\downarrow_h & & \downarrow_h \\ & & \vdots \end{array}$$

Recursion and looping are nicely separated by **head** reduction.

# Consistency

A relation $\mathcal{R} \subseteq \Lambda \times \Lambda$ is consistent if there exists $t, u \in \Lambda$ such that $(t, u) \notin \mathcal{R}$.

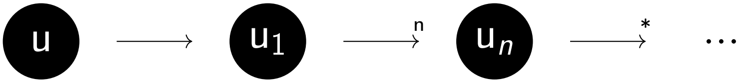
An equational theory is an equivalence relation $=_\mathcal{T}$ such that:

- *Stability by Computation*: if $t \to_\beta u$ then $t =_\mathcal{T} u$
- *Stability by Contexts*: if $t =_\mathcal{T} u$ then $\forall C$, $C\langle t\rangle =_\mathcal{T} C\langle u\rangle$.

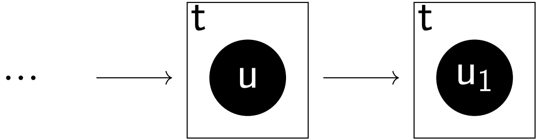
**To validate the choice of undefined terms:** Is there a consistent equational theory where undefined terms are collapsed?

# What is Genericity?

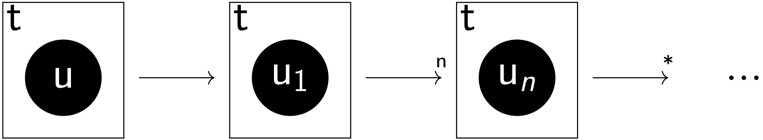
**Undefined** terms are black holes for the evaluation process.




If a program awaits the evaluation of an undefined sub-term



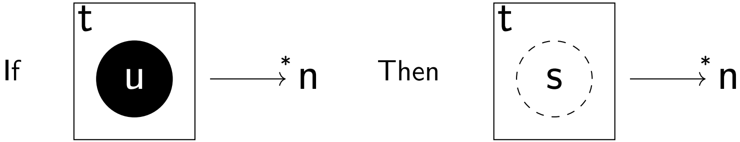

Then it will be unable to produce a result



# What is Genericity?

**Genericity** somehow specifies this fact dually:
If a program terminates while there were **undefined** sub-terms, then it never entered the black hole.

**Genericity** says: (n is a normal form and s is any term)




Anything can simulate the *generic* **undefined** sub-terms in a terminating term.

# Outline

# Genericity à la Barendregt & Consequences

**Heavy Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u\rangle \to_{\beta}^{*} n$ where $n$ is $\beta$-normal then $C\langle s\rangle \to_{\beta}^{*} n$ for all $s \in \Lambda$.

**Heavy Genericity** $\implies$ **Collapsibility**

**Collapsibility:** there exists an equational theory $\mathcal{T}$ such that undefined terms are equated in $\mathcal{T}$ and $\mathcal{T}$ is consistent

# Light Genericity

However, **Heavy Genericity** is very powerful and not all aspects are needed for the proof of **Collapsibility**.

$$\textbf{Light Genericity} \implies \textbf{Collapsibility}$$

We want to consider a lighter genericity statement:

- Use a simpler reduction than $\to_\beta$
- Do not compare normal forms

# This Paper

- A simplified form of **genericity**
- Explored in Call-by-Name and also in Call-by-Value
- Our development somehow abstracts the reduction strategy, $\to_{\mathtt{s}}$ instead of $\to_{CbN}$ or $\to_{CbV}$
- Highlighting the connection with program equivalences (or rather program *preorders*)

# Light Genericity

For a reduction $\to_{\mathtt{s}}$, we can state light genericity:

**Light Genericity:**
let $u$ be $\mathtt{s}$-diverging and $C$ such that $C\langle u\rangle$ is $\mathtt{s}$-normalizing
then $C\langle t\rangle$ is $\mathtt{s}$-normalizing for all $t \in \Lambda$.

Which **directly implies** that $\mathtt{s}$-diverging terms are minimum terms for the contextual preorder associated to $\mathtt{s}$.

If $u$ is $\mathtt{s}$-diverging, then $\forall s, \quad u \precsim_{\mathcal{C}}^{\mathtt{s}} s$

# Light vs. Heavy Genericity

**Heavy Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u\rangle \to_{\beta}^{*} n$ where $n$ is $\beta$-normal then $C\langle s\rangle \to_{\beta}^{*} n$ for all $s \in \Lambda$.

**Heavy Genericity** $\Rightarrow$ **Collapsibility:**

**Light Genericity** $\Rightarrow$ **Collapsibility:**

**Light Genericity:** let $u$ be s-diverging and $C$ such that $C\langle u\rangle$ is s-normalizing then $C\langle t\rangle$ is s-normalizing for all $t \in \Lambda$.

# Outline

# (Closed) Contextual Preorder

## and induced equivalence

The (closed) **contextual preorder** associated to a reduction $\rightarrow_{\mathtt{s}}$ is defined as:

- $t \precsim_{\mathcal{C}}^{\mathtt{s}} u$ if for all closing[1] contexts $C$, $C\langle t\rangle$ is $\mathtt{s}$-normalizing implies $C\langle u\rangle$ is $\mathtt{s}$-normalizing.

(Closed) contextual equivalence $\simeq_{\mathcal{C}}^{\mathtt{s}}$ is defined as the symmetric closure of the preorder.

**Light Genericity implies that** $\mathtt{s}$-diverging terms are minimum terms for the contextual preorder associated to $\mathtt{s}$.

[1] i.e. $C\langle t\rangle$ and $C\langle u\rangle$ are closed terms

# Open Contextual Preorder

## and induced equivalence

The **open contextual preorder** associated to a reduction $\to_{\mathtt{s}}$ is defined as:

- $t \precsim^{\mathtt{s}}_{\mathcal{CO}} u$ if **for all contexts** $C$, $C\langle t\rangle$ is $\mathtt{s}$-normalizing implies $C\langle u\rangle$ is $\mathtt{s}$-normalizing.

**Light Genericity exactly states that** $\mathtt{s}$-diverging terms are minimum terms for the open contextual preorder associated to $\mathtt{s}$.

$>>$ **Do open and closed preorders coincide?** In the paper, we answer that question in both Call-by-Name and Call-by-Value

# Outline

# Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

**CbN Light Genericity:** head-diverging terms are minimum for the head open contextual preorder.

Which unfolds to:

**CbN Light Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u\rangle$ is head-normalizing then $C\langle t\rangle$ is head-normalizing for all $t \in \Lambda$.

Main difficulty: reasoning with contexts and reduction.

# Light Genericity in Call-by-Name

Takahashi proves heavy genericity with a very short proof [Tak94] and gives as a corollary light genericity.

**Key idea:** Reason with substitutions instead of contexts!

In the paper, we adapt **Takahashi's technique** to give a direct proof of light genericity.

**Light genericity as substitution:** let $u$ be $\mathbb{s}$-diverging and $t$ such that $t\{x{\leftarrow}u\}$ is $\mathbb{s}$-normalizing then $t\{x{\leftarrow}s\}$ is $\mathbb{s}$-normalizing for all $s \in \Lambda$.

# Light Genericity in CbN

**Light Genericity** $\implies$ **Collapsibility**

We use the head open contextual preorder $\precsim^{h}_{\mathcal{CO}}$ to prove it.

- It is consistent to collapse head-diverging terms:
  $\precsim^{h}_{\mathcal{CO}}$ equates head-diverging terms (by light genericity)
  and
  $\precsim^{h}_{\mathcal{CO}}$ is consistent ($\mathtt{I} \not\precsim^{h}_{\mathcal{CO}} \Omega$)

# Outline

# Computable Functions & Lambda Calculus

Computable functions embed in the **call-by-value** lambda calculus.

Composition of PRF is by definition call-by-value!
Given $f, g : \mathbb{N} \to \mathbb{N} \uplus \{\bot\}$, $f \circ g$ is defined as:

$$f \circ g(n) := \begin{cases} \bot & \text{if } g(n) = \bot \\ f(m) & \text{if } g(n) = m \end{cases}$$

In call-by-name, one cannot define $\overline{f \circ g} = \overline{f}\,\overline{g}$ because of this.
Let $f := m \mapsto 1$ and $g := n \mapsto \bot$

$$\overline{f}\,\overline{g} = (\lambda x.1)\overline{g} \to_\beta 1$$

and

but $f \circ g$ is extensionally equivalent to $n \mapsto \bot$

# Call-by-Value undefined terms

What is the lambda term that represents **undefined** in call-by-value?

$\Omega$ again!

What is the equivalence class of **undefined**/$\Omega$ ?

- $\beta_v$-diverging terms? No, $fix_v$ and $\Omega$ are different
- **head**-diverging terms? No, $\lambda x.\Omega$ and $\Omega$ are different
- **weak-head**-diverging terms? No, $x\Omega$ and $\Omega$ are equivalent
- **weak**-diverging terms? Yes, *on closed terms*

# Open terms in Call-by-Value

Call-by-Value has been formalized by Plotkin [Plo75], but Plotkin's Call-by-Value theory is only satisfactory on closed terms.

$\Omega_{nf} = (\lambda x.\Omega)(yz)$ is a meaningless normal form!

- **undefined** terms = **Pweak**-diverging terms? Does not work on open terms, $\Omega$ and $\Omega_{nf}$ are equivalent.

(Plotkin's CbV) open and closed contextual preorders do not agree:

$$\Omega_{nf} \simeq^{p_v}_{\mathcal{C}} \Omega \quad \text{but} \quad \Omega_{nf} \not\simeq^{p_v}_{\mathcal{CO}} \Omega$$

# Moving on to Open Call-by-Value

The good call-by-value contextual equivalence is Plotkin's closed.

$$\simeq_{\mathcal{C}}^{v} := \simeq_{\mathcal{C}}^{p_v} = \simeq_{\mathcal{C}}^{vsc}$$

We use a nicer calculus (the Value Substitution Calculus [AP12] that is closely related to Linear Logic and Proof Nets) that knows how to deal with open terms, but retains the same closed contextual equivalence.

- **undefined** terms = **VSCweak**-diverging terms? Yes

Additionally, open and closed contextual preorders coincide for the VSC.

$$\simeq_{\mathcal{C}}^{v} := \simeq_{\mathcal{C}}^{p_v} = \simeq_{\mathcal{C}}^{vsc} = \simeq_{\mathcal{CO}}^{vsc}$$

# Call-by-Value Light Genericity & Collapsibility

Using this open calculus, we can show:

- Light Genericity: **VSCweak**-diverging terms are minimum for $\precsim^{vsc}_{\mathcal{CO}}$
- Collapsibility: $\precsim^{vsc}_{\mathcal{CO}}$ equates diverging terms and is consistent

Hence, we also have collapsibility in Plotkin's closed contextual preorder.

# Proofs of Call-by-Value Light Genericity

How to prove light genericity?

- Direct proof: *Takahashi's technique* adapts, but not very smoothly.
- *Using a good model of CbV:* relational semantics [Ehr12]
- *Applicative similarity* or any program preorder that is included in $\precsim^{\mathtt{s}}_{\mathcal{CO}}$ and has diverging terms as minimums.

# Outline

# Characterization of minimum terms

Light genericity says:

$$t \text{ is } \mathbb{s}\text{-diverging} \implies t \text{ is a minimum for } \precsim^{\mathrm{s}}_{\mathcal{CO}}$$

Adequacy ($t \mathrel{\mathcal{R}} u$ and $t$ is $\mathbb{s}$-normalizing then $u$ is $\mathbb{s}$-normalizing) implies the converse implication.

$$t \text{ is } \mathbb{s}\text{-diverging} \iff t \text{ is a minimum for } \precsim^{\mathrm{s}}_{\mathcal{CO}}$$

# Well, what about maximums?

$$t \text{ is } ?? \iff t \text{ is a maximum for } \precsim^{s}_{\mathcal{CO}}$$

- **Call-by-Name**: no maximum elements

The hammer proof is that call-by-name contextual preorder is characterized by program preorders that do not have maximums!

- Nakajima trees, applicative bisimilarity...

# Well, what about maximums?

$$t \text{ is } ?? \iff t \text{ is a maximum for } \precsim^{\mathrm{s}}_{\mathcal{CO}}$$

- **Call-by-Value**: *super terms!*

**Co-genericity:** super terms are maximum for $\precsim^{\mathrm{s}}_{\mathcal{CO}}$

# Super terms

A term $t$ is *$\mathtt{s}$-super* if, coinductively, $t \to_{\mathtt{s}}^{*} \lambda y.u$ and $u$ is $\mathtt{s}$-super.

Intuitively, $t$ infinitely normalizes to $\lambda y_1. \lambda y_2. \ldots \lambda y_k. \ldots$

An example:

$$\Omega_\lambda := \quad (\lambda x.\lambda y.xx)(\lambda x.\lambda y.xx) \downarrow \lambda y.\Omega_\lambda$$

# Co-genericity

In call-by-value:

**Co-genericity** $\Longrightarrow$ **Co-Collapsibility**

**Co-Collapsibility:** there exists an equational theory $\mathcal{T}$ such that super terms are equated in $\mathcal{T}$ and $\mathcal{T}$ is consistent

The open call-by-value contextual preorder $\precsim^{v}_{\mathcal{CO}}$ suffices.

It is consistent to equate diverging terms and to equate super terms, as $\precsim^{v}_{\mathcal{CO}}$ does it and is consistent.

# Proofs of co-genericity

How to prove co-genericity ?

- Direct proof: *Takahashi's technique* adapts, and the proof is easier than for light genericity.
- *Using a good model of CbV?* relational semantics [Ehr12] do not work, as $\mathfrak{s}$-super terms are not maximum elements in the model!
- *Applicative similarity* or any program preorder that is included in $\precsim^{\mathfrak{s}}_{\mathcal{CO}}$ and has super terms as maximums[2].

[2] I don't know of any other one

# Outline

# Conclusion

- **Light genericity** is a modular concept that is strong enough to imply **Collapsibility**, the main consequence of Barendregt's genericity.
- It is naturally dualizable as **co-genericity**. Both concepts are inspired and tied with contextual preorders.

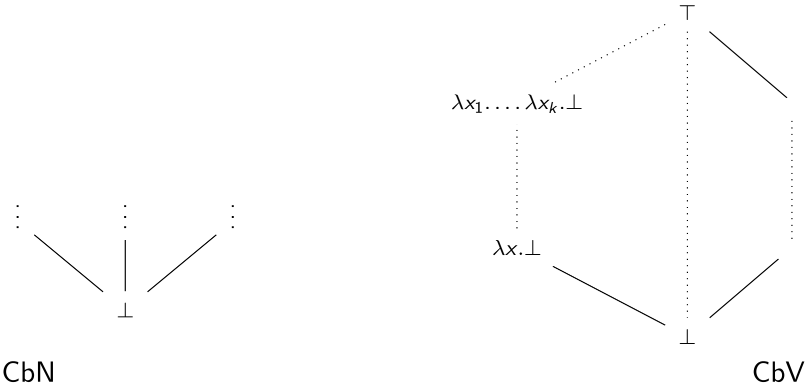
Also in the paper:

- another consequence of genericity is the **Maximality** of the open contextual preorder (any larger theory is inconsistent)
- Which in turns provides an elegant proof of the fact that closed and open contextual equivalences coincide.

**A question remains:** we named the two genericity statements Heavy and Light, but we don't know whether one implies the other or not.

# Bottom (and Top?) line

## Thank you!




**Contextual preorder for lambda terms**

$\bot :=$ equivalence class of $\Omega$

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# Equational theories

## Or rather inequational theories

### Definition (Inequational $\mathbb{s}$-theory)

Let $\mathbb{s}$ be a reduction. An inequational $\mathbb{s}$-theory $\leq^{\mathbb{s}}_{\mathcal{T}}$ is a compatible[3] preorder on terms containing $\mathbb{s}$-conversion.

Closed/Open $\mathbb{s}$-contextual preorders are $\mathbb{s}$-inequational theories.

The non-trivial point is that they contain $\mathbb{s}$-conversion.

[3] Stable by contextual closure: $t \leq^{\mathbb{s}}_{\mathcal{T}} u \implies \forall C,\ C\langle t\rangle \leq^{\mathbb{s}}_{\mathcal{T}} C\langle u\rangle$

# Inequational theories

## Generalization of sensible and semi-sensible

An inequational $\mathtt{s}$-theory $\leq^{\mathtt{s}}_{\mathcal{T}}$ is called:

- *Consistent*: whenever it does not relate all terms;
- $\mathtt{s}$-*ground*: if $\mathtt{s}$-diverging terms are minimum terms for $\leq^{\mathtt{s}}_{\mathcal{T}}$;
- $\mathtt{s}$-*adequate*: if $t \leq^{\mathtt{s}}_{\mathcal{T}} u$ and $t$ is $\mathtt{s}$-normalizing entails $u$ is $\mathtt{s}$-normalizing.

Groundness and Adequacy correspond (in CbN) with the order-variants of sensible and semi-sensible theories.

Adequacy implies: minimum terms for $\leq^{\mathtt{s}}_{\mathcal{T}}$ are $\mathtt{s}$-diverging.

# Maximality

For $\mathtt{s} \in \{\text{head CbN, weak CbV}\}$, we can state maximality uniformly.

The proof is not uniform as it relies on critical solvability concepts.

**Theorem**

*Maximality of $\precsim^{\mathtt{s}}_{\mathcal{CO}}$: $\precsim^{\mathtt{s}}_{\mathcal{CO}}$ is a maximal consistent inequational $\mathtt{s}$-theory, i.e.*

$$\text{if } \precsim^{\mathtt{s}}_{\mathcal{CO}} \subsetneq \mathcal{R} \text{ then } \mathcal{R} \text{ is inconsistent.}$$

An elegant proof that closed and open contextual equivalence coincides follows: $\precsim^{\mathtt{s}}_{\mathcal{CO}} \subseteq \precsim^{\mathtt{s}}_{\mathcal{C}}$ and $\precsim^{\mathtt{s}}_{\mathcal{C}}$ is consistent, hence $\precsim^{\mathtt{s}}_{\mathcal{CO}} = \precsim^{\mathtt{s}}_{\mathcal{C}}$