Mirroring Call-by-Need, or Values Acting Silly

Beniamino Accattoli[1], Adrienne Lancelot[12]

[1]Inria & LIX, École Polytechnique
[2]IRIF, Université Paris Cité & CNRS

June 6th 2024 – GT Syntax Meets Semantics

Outline

Call-by-Name and Call-by-Value

Evaluation strategies describe how to compute.

β-Reduction by Name: $(\lambda x.t)u \mapsto_{\beta} t\{x{\leftarrow}u\}$

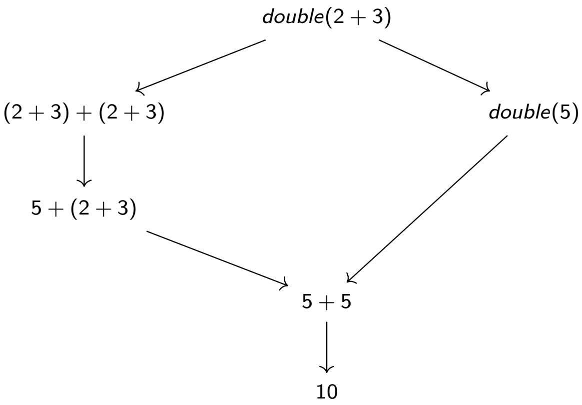
β-Reduction by Value: $(\lambda x.t)v \mapsto_{\beta_v} t\{x{\leftarrow}v\}$

For values v that are *answers*, i.e. computations that ended.

Call-by-Name vs. Call-by-Value

Let $double := x \mapsto x + x$. How do you calculate $double(2+3)$?

Call-by-Name | Call-by-Value



Feed any argument to any function | Only feed functions with *values*

Call-by-Name vs. Call-by-Value

Computations may never end: $\Omega := (\lambda x.xx)(\lambda x.xx)$

However, consider a constant function $always\_1 : x \mapsto 1$.
How to compute $always\_1(\Omega)$?

Call-by-Name | Call-by-Value

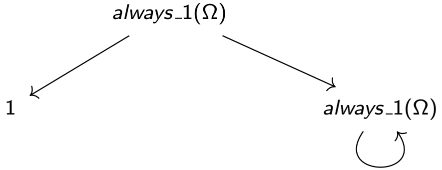
Feed any argument to any function | Only feed functions with *values*

Call-by-Name vs. Call-by-Value

Computations may never end: $\Omega := (\lambda x.xx)(\lambda x.xx)$

However, consider a constant function $always\_1 : x \mapsto 1$.
How to compute $always\_1(\Omega)$?

Call-by-Name | Call-by-Value



Feed any argument to any function | Only feed functions with *values*

Call-by-Name vs. Call-by-Value

Call-by-Name and Call-by-Value compute quite differently.

Efficiency: Call-by-Value computes *faster* than Call-by-Name.

Erasability: Call-by-Value gets stuck on *erasable* arguments.

Efficiency and Erasability can be combined: Call-by-Need!

Call-by-Name vs. Call-by-Value

Call-by-Name and Call-by-Value compute quite differently.

Efficiency: Call-by-Value computes *faster* than Call-by-Name.

Erasability: Call-by-Value gets stuck on *erasable* arguments.

Efficiency and Erasability can be combined: Call-by-Need!

Evaluation Strategies combining Name and Value

	Duplication by Name *Silly Duplication*	Duplication by Value *Wise Duplication*
Erasure by Name *Wise Erasure*	Call-by-Name	Call-by-Need
Erasure by Value *Silly Erasure*	**Call-by-Silly**	Call-by-Value

Evaluation Strategies combining Name and Value

	Duplication by Name *Silly Duplication*	Duplication by Value *Wise Duplication*
Erasure by Name *Wise Erasure*	Call-by-Name	Call-by-Need
Erasure by Value *Silly Erasure*	**Call-by-Silly**	Call-by-Value

Evaluation Strategies combining Name and Value

	Duplication by Name *Silly Duplication*	Duplication by Value *Wise Duplication*
Erasure by Name *Wise Erasure*	Call-by-Name	Call-by-Need
Erasure by Value *Silly Erasure*	**Call-by-Silly**	Call-by-Value

Evaluation Strategies combining Name and Value

	Duplication by Name *Silly Duplication*	Duplication by Value *Wise Duplication*
Erasure by Name *Wise Erasure*	Call-by-Name	Call-by-Need
Erasure by Value *Silly Erasure*	**Call-by-Silly**	Call-by-Value

Evaluation Strategies: duplication and erasure rules

Duplication by Name: $(\lambda x.t)u \mapsto_\beta t\{x{\leftarrow}u\}$ where $x \in \mathtt{fv}(t)$

Duplication by Value: $(\lambda x.t)v \mapsto_\beta t\{x{\leftarrow}v\}$ where $x \in \mathtt{fv}(t)$

Erasure by Name: $(\lambda x.t)u \mapsto_\beta t\{x{\leftarrow}u\}$ where $x \notin \mathtt{fv}(t)$

Erasure by Value: $(\lambda x.t)v \mapsto_\beta t\{x{\leftarrow}v\}$ where $x \notin \mathtt{fv}(t)$

Main results about Call-by-Silly:

- Rewriting Properties and Multi Types
- CbSilly induces the same contextual equivalence than CbV
  - Mirroring the main theorem about CbNeed
  - A proof method for CbV contextual equivalence
  - CbV contextual equivalence is blind wrto efficiency
- Quantitative study of types: Call-by-Silly really is inefficient

Mirroring Call-by-Need, Or Values Acting Silly

Contributions

Main results about Call-by-Silly:

- Rewriting Properties and Multi Types
- CbSilly induces the same contextual equivalence than CbV
  - Mirroring the main theorem about CbNeed
  - A proof method for CbV contextual equivalence
  - CbV contextual equivalence is blind wrto efficiency
- Quantitative study of types: Call-by-Silly really is inefficient

Contextual Equivalences

CbN Contextual Equivalence

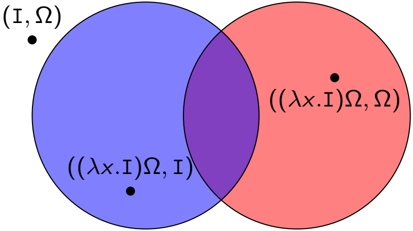
=

CbNeed Contextual Equivalence

CbV Contextual Equivalence

=

CbSilly Contextual Equivalence



$\mathtt{I} =_{CbN} \quad (\lambda x.\mathtt{I})\Omega \quad =_{CbV} \Omega$

$\mathtt{I} \neq_{CbN} \quad \Omega \quad \neq_{CbV} \mathtt{I}$

Contextual Equivalences

CbN Contextual Equivalence

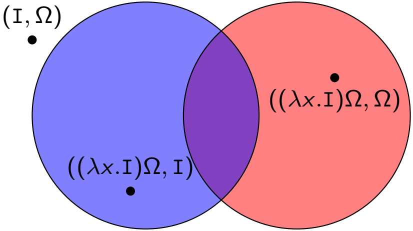
=

CbNeed Contextual Equivalence

CbV Contextual Equivalence

=

CbSilly Contextual Equivalence



$$\mathtt{I} =_{CbN} (\lambda x.\mathtt{I})\Omega =_{CbV} \Omega$$

$$\mathtt{I} \neq_{CbN} \Omega \neq_{CbV} \mathtt{I}$$

Outline

Micro-steps

$$
\begin{array}{rrcl}
\textsc{Terms} & t,u,s & ::= & x \mid \lambda x.t \mid tu \mid t[x{\leftarrow}u] \\
\textsc{Values} & v,v' & ::= & \lambda x.t \\
\textsc{Sub. Ctxs} & S,S' & ::= & \langle\cdot\rangle \mid S[x{\leftarrow}u] \\
\textsc{Weak contexts} & W & ::= & \langle\cdot\rangle \mid Wt \mid tW \mid t[x{\leftarrow}W] \mid W[x{\leftarrow}u]
\end{array}
$$

$$
\begin{array}{rcl}
(\lambda x.t)u & \mapsto_{\mathtt{m}} & t[x{\leftarrow}u] \\
\ldots x \ldots [x{\leftarrow}u] & \mapsto_{\mathtt{e}} & \ldots u \ldots [x{\leftarrow}u] \\
t[x{\leftarrow}v] & \mapsto_{\mathtt{gcv}} & t \quad \text{if } x \notin \mathtt{fv}(t)
\end{array}
$$

$\to_{\mathtt{w}}$ is the weak contextual closure of $\mapsto_{\mathtt{m}}$, $\mapsto_{\mathtt{e}_W}$ and $\mapsto_{\mathtt{gcv}}$.

Micro-steps

$$
\begin{array}{rrcl}
\textsc{Terms} & t, u, s & ::= & x \mid \lambda x.t \mid tu \mid t[x{\leftarrow}u] \\
\textsc{Values} & v, v' & ::= & \lambda x.t \\
\textsc{Sub. Ctxs} & S, S' & ::= & \langle\cdot\rangle \mid S[x{\leftarrow}u] \\
\textsc{Weak contexts} & W & ::= & \langle\cdot\rangle \mid Wt \mid tW \mid t[x{\leftarrow}W] \mid W[x{\leftarrow}u]
\end{array}
$$

$$
\begin{array}{rcl}
(\lambda x.t)u & \mapsto_{\mathtt{m}} & t[x{\leftarrow}u] \\
W\langle\!\langle x\rangle\!\rangle[x{\leftarrow}u] & \mapsto_{\mathtt{e}_W} & W\langle\!\langle u\rangle\!\rangle[x{\leftarrow}u] \\
t[x{\leftarrow}v] & \mapsto_{\mathtt{gcv}} & t \quad \text{if } x \notin \mathtt{fv}(t)
\end{array}
$$

$\to_{\mathtt{w}}$ is the weak contextual closure of $\mapsto_{\mathtt{m}}$, $\mapsto_{\mathtt{e}_W}$ and $\mapsto_{\mathtt{gcv}}$.

Micro-steps

$$
\begin{array}{rrcl}
\textsc{Terms} & t,u,s & ::= & x \mid \lambda x.t \mid tu \mid t[x{\leftarrow}u] \\
\textsc{Values} & v,v' & ::= & \lambda x.t \\
\textsc{Sub. Ctxs} & S,S' & ::= & \langle\cdot\rangle \mid S[x{\leftarrow}u] \\
\textsc{Weak contexts} & W & ::= & \langle\cdot\rangle \mid Wt \mid tW \mid t[x{\leftarrow}W] \mid W[x{\leftarrow}u]
\end{array}
$$

$$
\begin{array}{rcl}
S\langle\lambda x.t\rangle u & \mapsto_{\mathtt{m}} & S\langle t[x{\leftarrow}u]\rangle \\
W\langle\langle x\rangle\rangle[x{\leftarrow}u] & \mapsto_{\mathtt{e}_W} & W\langle\langle u\rangle\rangle[x{\leftarrow}u] \\
t[x{\leftarrow}S\langle v\rangle] & \mapsto_{\mathtt{gcv}} & S\langle t\rangle \quad \text{if } x \notin \mathtt{fv}(t)
\end{array}
$$

$\to_{\mathtt{w}}$ is the weak contextual closure of $\mapsto_{\mathtt{m}}$, $\mapsto_{\mathtt{e}_W}$ and $\mapsto_{\mathtt{gcv}}$.

The Silly Extra Copy

Let $\mathit{double} := x \mapsto x + x$. How to silly compute $\mathit{double}(2+3)$?

$$\mathit{double}(2+3)$$
$$\downarrow$$
$$x + x\ [x \leftarrow 2+3]$$
$$\swarrow \qquad \searrow$$
$$(2+3) + x\ [x \leftarrow 2+3] \qquad\qquad x + (2+3)\ [x \leftarrow 2+3]$$
$$\searrow \qquad \swarrow$$
$$(2+3) + (2+3)\ [x \leftarrow 2+3]$$
$$\downarrow$$
$$5 + (2+3)\ [x \leftarrow 2+3]$$
$$\downarrow$$
$$5 + 5\ [x \leftarrow 2+3]$$
$$\downarrow$$
$$10\ [x \leftarrow 2+3]$$

$\downarrow$ computing the silly extra copy

$$10\ [x \leftarrow 5]$$

$\downarrow$ garbage collector

$$10$$

Outline

Silly Multi Types

$$
\begin{array}{rrcl}
\textsc{Linear types} & L, L' & ::= & \mathtt{norm} \mid M \to L \\
\textsc{Multi types} & M, N & ::= & [L_i]_{i\in I} \quad \text{where } I \text{ is a finite set} \\
\textsc{Generic types} & T, T' & ::= & L \mid M
\end{array}
$$

$$
\frac{}{x : [L] \vdash x : L}\ \mathsf{ax}
\qquad
\frac{(\Gamma_i \vdash t : L_i)_{i\in I}}{\uplus_{i\in I}\Gamma_i \vdash t : [L_i]_{i\in I}}\ \mathsf{many}
$$

$$
\frac{}{\vdash \lambda x.t : \mathtt{norm}}\ \mathsf{ax}_\lambda
\qquad
\frac{\Gamma \vdash t : M \to L \qquad \Delta \vdash u : M \uplus [\mathtt{norm}]}{\Gamma \uplus \Delta \vdash tu : L}\ @
$$

$$
\frac{\Gamma \vdash t : L}{\Gamma \setminus\!\!\setminus x \vdash \lambda x.t : \Gamma(x) \to L}\ \lambda
\qquad
\frac{\Gamma \vdash t : L \qquad \Delta \vdash u : \Gamma(x) \uplus [\mathtt{norm}]}{(\Gamma \setminus\!\!\setminus x) \uplus \Delta \vdash t[x \leftarrow u] : L}\ \mathsf{ES}
$$

Silly Multi Types

$$\begin{array}{rrcl} \text{LINEAR TYPES} & L, L' & ::= & \texttt{norm} \mid M \to L \\ \text{MULTI TYPES} & M, N & ::= & [L_i]_{i \in I} \quad \text{where } I \text{ is a finite set} \\ \text{GENERIC TYPES} & T, T' & ::= & L \mid M \end{array}$$

$$\frac{}{x : [L] \vdash^{(0,1)} x : L}\ \text{ax} \qquad \frac{(\Gamma_i \vdash^{(m_i, e_i)} t : L_i)_{i \in I}}{\uplus_{i \in I} \Gamma_i \vdash^{(\sum_{i \in I} m_i, \sum_{i \in I} e_i)} t : [L_i]_{i \in I}}\ \text{many}$$

$$\frac{}{\vdash^{(0,0)} \lambda x.t : \texttt{norm}}\ \text{ax}_\lambda \qquad \frac{\Gamma \vdash^{(m,e)} t : M \to L \qquad \Delta \vdash^{(m',e')} u : M \uplus [\texttt{norm}]}{\Gamma \uplus \Delta \vdash^{(m+m'+1, e+e')} tu : L}\ @$$

$$\frac{\Gamma \vdash^{(m,e)} t : L}{\Gamma \setminus\!\!\setminus x \vdash^{(m,e)} \lambda x.t : \Gamma(x) \to L}\ \lambda \qquad \frac{\Gamma \vdash^{(m,e)} t : L \qquad \Delta \vdash^{(m',e')} u : \Gamma(x) \uplus [\texttt{norm}]}{(\Gamma \setminus\!\!\setminus x) \uplus \Delta \vdash^{(m+m', e+e')} t[x \leftarrow u] : L}\ \text{ES}$$

Example of type derivation

$\mathtt{n} := \mathtt{norm}$

$$\dfrac{\vdash \lambda y.yy : [[L]\to L, L, \mathtt{n}]\to L \qquad \dfrac{\vdash \mathtt{II} : [L]\to L \quad \vdash \mathtt{II} : L \quad \vdash \mathtt{II} : \mathtt{n} \quad \vdash \mathtt{II} : \mathtt{n}}{\vdash \mathtt{II} : [[L]\to L, L, \mathtt{n}] \uplus [\mathtt{n}]}}{\vdash (\lambda y.yy)(\mathtt{II}) : L}$$

Silly Multi Types and the SSC

Proposition (Quantitative subject reduction for Weak SSC)

Let $\pi \triangleright \Gamma \vdash^{(m,e)} t : L$ *be a derivation.*

1. *Multiplicative: if* $t \to_{\mathtt{wm}} u$ *then* $m \geq 1$ *and there exists* $\rho \triangleright \Gamma \vdash^{(m',e)} u : L$ *with* $m > m'$.
2. *Exponential: if* $t \to_{\mathtt{we}} u$ *then* $e \geq 1$ *and there exists* $\rho \triangleright \Gamma \vdash^{(m,e')} u : L$ *with* $e > e'$.
3. *GC by value: if* $t \to_{\mathtt{wgcv}} u$ *then there exists* $\rho \triangleright \Gamma \vdash^{(m,e)} u : L$ *with* $|\pi| > |\rho|$.

Silly Multi Types and the SSC

Theorem

Let t be a term.

1. *Correctness: if $\pi \triangleright \Gamma \vdash^{(m,e)} t : L$ then $t \in SN_{\mathtt{w}}$.*
2. *Quantitative info: if $\pi \triangleright \Gamma \vdash^{(m,e)} t : L$ and $d : t \rightarrow^*_{\mathtt{w}} n$ is a normalizing sequence then $|d|_{\mathtt{wm}} \leq m$ and $|d|_{\mathtt{we}} \leq e$.*
3. *Completeness: if $t \rightarrow^*_{\mathtt{w}} n$ and n is a weak normal form then there exists $\pi \triangleright \Gamma \vdash t : \mathtt{norm}$.*

Outline

Closed Call-by-Value

$$\begin{array}{lrcl} \textsc{Terms} & t, u, s & ::= & x \mid \lambda x.t \mid tu \\ \textsc{Values} & v, v' & ::= & \lambda x.t \end{array}$$

$$\frac{}{(\lambda x.t)v \to_{\beta_v} t\{x{\leftarrow}v\}} \qquad \frac{t \to_{\beta_v} t'}{tu \to_{\beta_v} t'u} \qquad \frac{t \to_{\beta_v} t'}{ut \to_{\beta_v} ut'}$$

Silly Multi Types and Closed CbV

Theorem

Let t be a closed λ-term.

1. *Correctness: if $\pi \triangleright\ \vdash^m t : L$ then there are an abstraction v and a reduction sequence $d\colon t \to^*_{\beta_v} v$ with $|d| \leq m$.*
2. *Completeness: if there exists a value v and a reduction sequence $d\colon t \to^*_{\beta_v} v$ then $\pi \triangleright\ \vdash t : \mathtt{norm}$.*

TFAE, for a closed term t:

- *Typability in Silly Multi Types:* $\pi \triangleright\ \vdash t : L$
- *CbV normalization:* $d\colon t \to^*_{\beta_v} v$
- *CbSilly normalization:* $d : t \to^*_{\mathtt{w}} n$

Contextual Equivalences

Definition (Contextual Equivalence)

We define contextual equivalence $\simeq_C^s$ for a rewriting relation $\rightarrow_s$:

$t \simeq_C t'$ if for all C contexts such that $C\langle t\rangle$ and $C\langle t'\rangle$ are closed terms, $C\langle t\rangle$ is $\rightarrow_s$-normalizing iff $C\langle t'\rangle$ is $\rightarrow_s$-normalizing.

Let us consider $\simeq_C^{\beta_v}$, Plotkin's CbV contextual equivalence induced by $\rightarrow_{\beta_v}$ and $\simeq_C^{silly}$, the contextual equivalence induced by $\rightarrow_{\mathtt{w}}$.

CbV and Silly induce the same contextual equivalence

Theorem

$\simeq_C^{\beta_v} = \simeq_C^{silly}$

Proof.

$t \simeq_C^{\beta_v} t'$ $\iff$ For all C contexts such that $C\langle t\rangle$ and $C\langle t'\rangle$ are closed terms, $C\langle t\rangle$ is $\to_{\beta_v}$-normalizing iff $C\langle t'\rangle$ is $\to_{\beta_v}$-normalizing.

[On closed terms, $\to_{\beta_v}$-normalization is equivalent to Silly typability]

$\iff$ For all C contexts such that $C\langle t\rangle$ and $C\langle t'\rangle$ are closed terms, $\pi \triangleright\ \vdash C\langle t\rangle : L$ iff $\pi' \triangleright\ \vdash C\langle t'\rangle : L'$.

[(On all terms,) Silly typability is equivalent to silly $\to_{\mathtt{w}}$-normalization]

$\iff$ For all C contexts such that $C\langle t\rangle$ and $C\langle t'\rangle$ are closed terms, $C\langle t\rangle$ is $\to_{\mathtt{w}}$-normalizing iff $C\langle t'\rangle$ is $\to_{\mathtt{w}}$-normalizing.

$\iff$ $t \simeq_C^{silly} t'$

□

CbSilly helps to prove CbV contextual equivalence

Let i any normal norm that does not look like an abstraction, for example $i = y\mathtt{I}$.
Consider the four following terms:

$$(\lambda x.xx)\, i \qquad (\lambda x.xi)\, i \qquad (\lambda x.ii)\, i \qquad ii$$

How to prove these terms are CbV contextually equivalent?

- $(\lambda x.xx)\, i$, $(\lambda x.xi)\, i$ and $(\lambda x.ii)\, i$ all reduce to the same silly normal form $ii[x{\leftarrow}i]$
- $(\lambda x.xx)\, i =_{\mathtt{w}} (\lambda x.xi)\, i =_{\mathtt{w}} (\lambda x.ii)\, i =_{\mathtt{w}} ii[x{\leftarrow}i]$
- **Proposition:** if $t =_{\mathtt{w}} u$ then $t \simeq_C^{silly} u$
- $(\lambda x.xx)\, i \simeq_C^{silly} (\lambda x.xi)\, i \simeq_C^{silly} (\lambda x.ii)\, i \simeq_C^{silly} ii[x{\leftarrow}i]$

CbSilly helps to prove CbV contextual equivalence

Let i any normal norm that does not look like an abstraction, for example $i = y\mathtt{I}$.
Consider the four following terms:

$$(\lambda x.xx)\, i \qquad (\lambda x.xi)\, i \qquad (\lambda x.ii)\, i \qquad ii$$

How to prove these terms are CbV contextually equivalent?

- $(\lambda x.xx)\, i$, $(\lambda x.xi)\, i$ and $(\lambda x.ii)\, i$ all reduce to the same silly normal form $ii[x{\leftarrow}i]$
- $(\lambda x.xx)\, i =_{\mathtt{w}} (\lambda x.xi)\, i =_{\mathtt{w}} (\lambda x.ii)\, i =_{\mathtt{w}} ii[x{\leftarrow}i]$
- **Proposition:** if $t =_{\mathtt{w}} u$ then $t \simeq_C^{silly} u$
- $(\lambda x.xx)\, i \simeq_C^{silly} (\lambda x.xi)\, i \simeq_C^{silly} (\lambda x.ii)\, i \simeq_C^{silly} ii[x{\leftarrow}i]$

CbSilly helps to prove CbV contextual equivalence

Let i any normal norm that does not look like an abstraction, for example $i = y\mathtt{I}$.
Consider the four following terms:

$$(\lambda x.xx)\, i \qquad (\lambda x.xi)\, i \qquad (\lambda x.ii)\, i \qquad ii$$

How to prove these terms are CbV contextually equivalent?

- $(\lambda x.xx)\, i$, $(\lambda x.xi)\, i$ and $(\lambda x.ii)\, i$ all reduce to the same silly normal form $ii[x{\leftarrow}i]$
- $(\lambda x.xx)\, i =_{\mathtt{w}} (\lambda x.xi)\, i =_{\mathtt{w}} (\lambda x.ii)\, i =_{\mathtt{w}} ii[x{\leftarrow}i]$
- **Proposition:** if $t =_{\mathtt{w}} u$ then $t \simeq_C^{silly} u$
- $(\lambda x.xx)\, i \simeq_C^{silly} (\lambda x.xi)\, i \simeq_C^{silly} (\lambda x.ii)\, i \simeq_C^{silly} ii[x{\leftarrow}i]$

Outline

Conclusion

In the paper:

- Introduce a degenerated reduction, call-by-silly
- Mirroring the main theorem about CbN and CbNeed: call-by-silly and call-by-value generate the same contextual equivalence
- **New Proof Method** fo CbV contextual equivalence: convertibility in call-by-silly
- Tight types and maximality: the type system exactly measures the reduction length of a call-by-silly strategy. We can also show thanks to the type system that this strategy is maximal in some sense.

Future work:

- Refining CbV contextual equivalence to forbid silly duplications and be aware of efficiency
- Categorical or Game Semantics for CbSilly and CbNeed?

Thank you for your attention!

	Duplication by Name *Silly Duplication*	Duplication by Value *Wise Duplication*
Erasure by Name *Wise Erasure*	Call-by-Name	Call-by-Need
Erasure by Value *Silly Erasure*	**Call-by-Silly**	Call-by-Value

The problem with variable as values

$$x[z{\leftarrow}ww] \;{}_{\mathtt{wgcv}}\!\!\leftarrow x[y{\leftarrow}z][z{\leftarrow}ww] \to_{\mathtt{e}_W} x[y{\leftarrow}ww][z{\leftarrow}ww]$$

Similar issues arise in CbNeed:

$$\begin{array}{ccc}
x(\lambda z.wx)[x{\leftarrow}y][y{\leftarrow}\mathtt{I}] & \longrightarrow & x(\lambda z.wx)[x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{I}] \\
\downarrow & & \vdots \\
y(\lambda z.wx)[x{\leftarrow}y][y{\leftarrow}\mathtt{I}] \dashrightarrow \mathtt{I}(\lambda z.wx)[x{\leftarrow}y][y{\leftarrow}\mathtt{I}] & & \mathtt{I}(\lambda z.wx)[x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{I}]
\end{array}$$

The problem with variable as values

$$x[z\leftarrow ww] \;{}_{\mathtt{wgcv}}\!\leftarrow\; x[y\leftarrow z][z\leftarrow ww] \rightarrow_{\mathtt{e}_W} x[y\leftarrow ww][z\leftarrow ww]$$

Similar issues arise in CbNeed:

$$\begin{array}{ccc}
x(\lambda z.wx)[x\leftarrow y][y\leftarrow \mathtt{I}] & \longrightarrow & x(\lambda z.wx)[x\leftarrow \mathtt{I}][y\leftarrow \mathtt{I}] \\
\downarrow & & \vdots \\
y(\lambda z.wx)[x\leftarrow y][y\leftarrow \mathtt{I}] \dashrightarrow \mathtt{I}(\lambda z.wx)[x\leftarrow y][y\leftarrow \mathtt{I}] & & \mathtt{I}(\lambda z.wx)[x\leftarrow \mathtt{I}][y\leftarrow \mathtt{I}]
\end{array}$$

Why Silly Types do not work for Open CbV

For $n \geq 1$, we have the following derivation for the source term $z((\lambda x.y)u)$ in the silly type system:

$$\dfrac{\pi_z \triangleright\ \ldots \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{y:[A]\vdash y:A}\,\mathsf{ax}}{y:[A]\vdash \lambda x.y:0\to A}\,\lambda \quad \dfrac{\pi_u \triangleright\ \Gamma\vdash u:\mathtt{norm}}{\Gamma\vdash u:[\mathtt{norm}]}\,\mathsf{many}}{(y:[A],\Gamma\vdash(\lambda x.y)u:A)_{i=1,\ldots,n}}\,@ \qquad \dfrac{\dfrac{\dfrac{}{y:[\mathtt{norm}]\vdash y:\mathtt{norm}}\,\mathsf{ax}}{y:[\mathtt{norm}]\vdash\lambda x.y:0\to\mathtt{norm}}\,\lambda \quad \ldots}{y:[\mathtt{norm}],\Gamma\vdash(\lambda x.y)u:\mathtt{no}\ldots}}{y:[A^n,\mathtt{norm}],\Gamma^{n+1}\vdash(\lambda x.y)u:[A^n,\mathtt{norm}]}}{z:[[A^n,\mathtt{norm}]\to B],y:[A^n,\mathtt{norm}],\Gamma^{n+1}\vdash z((\lambda x.y)u):B}\,@$$

where $\pi_z \triangleright\ \ldots$ stands for:

$$\dfrac{}{\pi_z \triangleright\ z:[[A^n]\to B]\vdash z:[A^n]\to B}\,\mathsf{ax}$$

The term $(\lambda x.zy)u$, instead, can only be typed as follows, the key point being that Γ^{n+1} is replaced by Γ:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z:[[A^n]\to B]\vdash z:[A^n]\to B}\,\mathsf{ax} \quad \dfrac{\dfrac{}{(y:[A]\vdash y:A)_{i=1,\ldots,n}}\,\mathsf{ax} \quad \dfrac{}{y:[\mathtt{norm}]\vdash y:\mathtt{norm}}\,\mathsf{ax}}{y:[A^n,\mathtt{norm}]\vdash y:[A^n,\mathtt{norm}]}\,\mathsf{many}}{z:[[A^n]\to B],y:[A^n,\mathtt{norm}]\vdash zy:B}\,@}{z:[[A^n]\to B],y:[A^n,\mathtt{norm}]\vdash\lambda x.zy:0\to B}\,\lambda \qquad \dfrac{\pi_u\triangleright\ \Gamma\vdash u:\ldots}{\Gamma\vdash u:[\mathtt{no}\ldots}}{z:[[A^n]\to B],y:[A^n,\mathtt{norm}],\Gamma\vdash(\lambda x.zy)u:B}$$

The Call-by-Silly Strategy

$$\textsc{Call-by-name strategy} \quad \to_{\mathtt{n}}$$

$$\begin{array}{rrcl|rcl}
\textsc{Name ctxs} & N, N' & ::= & \langle\cdot\rangle \mid Nt \mid N[x{\leftarrow}t] & \to_{\mathtt{nm}} & := & N\langle\mapsto_{\mathtt{m}}\rangle \\
 & & & & \to_{\mathtt{ne}} & := & N\langle\mapsto_{\mathtt{e}_N}\rangle \\
\textsc{Root GC} & t[x{\leftarrow}s] & \mapsto_{\mathtt{gc}} & t \quad \text{if } x \notin \mathtt{fv}(t) & \to_{\mathtt{ngc}} & := & N\langle\mapsto_{\mathtt{gc}}\rangle \\
 & & & & \to_{\mathtt{n}} & := & \to_{\mathtt{nm}} \cup \to_{\mathtt{ne}} \cup \to_{\mathtt{ngcv}}
\end{array}$$

$$\textsc{Call-by-silly strategy} \quad \to_{\mathtt{y}}$$

$$\begin{array}{rrcl|rcl}
 & & & & \to_{\mathtt{ym}} & := & Y\langle\mapsto_{\mathtt{m}}\rangle \\
\textsc{Answers} & a, a' & ::= & v \mid a[x{\leftarrow}a'] & \to_{\mathtt{ygcv}} & := & Y\langle\mapsto_{\mathtt{gcv}}\rangle \\
\textsc{Aux. ctxs} & A, A' & ::= & \langle\cdot\rangle \mid a[x{\leftarrow}A] \mid A[x{\leftarrow}t] & \to_{\mathtt{ye_{AY}}} & := & A\langle\mapsto_{\mathtt{e}_Y}\rangle \\
\textsc{Silly ctxs} & Y, Y' & ::= & A\langle N\rangle & \to_{\mathtt{ye_{YN}}} & := & Y\langle\mapsto_{\mathtt{e}_N}\rangle \\
 & & & & \to_{\mathtt{y}} & := & \to_{\mathtt{ym}} \cup \to_{\mathtt{ye_{AY}}} \cup \to_{\mathtt{ye_{YN}}} \cup \to_{\mathtt{ygcv}}
\end{array}$$

Figure 4 The call-by-name and call-by-silly strategies.

The Call-by-Silly Strategy: example

CbN evaluation:

$(\lambda y.yy)(\mathtt{II})$

$\to_{\mathtt{ym}}$ $yy[y{\leftarrow}\mathtt{II}]$		$\to_{\mathtt{ye_{YN}}}$ $(\mathtt{II})y[y{\leftarrow}\mathtt{II}]$
$\to_{\mathtt{ym}}$ $(x[x{\leftarrow}\mathtt{I}])y[y{\leftarrow}\mathtt{II}]$		$\to_{\mathtt{ye_{YN}}}$ $(\mathtt{I}[x{\leftarrow}\mathtt{I}])y[y{\leftarrow}\mathtt{II}]$
$\to_{\mathtt{ym}}$ $z[z{\leftarrow}y][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$		$\to_{\mathtt{ye_{YN}}}$ $y[z{\leftarrow}y][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$
		$\to_{\mathtt{ye_{YN}}}$ $\mathtt{II}[z{\leftarrow}y][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$
$\to_{\mathtt{ym}}$ $z'[z'{\leftarrow}\mathtt{I}][z{\leftarrow}y][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$		$\to_{\mathtt{ye_{YN}}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}y][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$

CbS extension:

	$\to_{\mathtt{ye_{AY}}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}\mathtt{II}][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$
$\to_{\mathtt{ym}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}z'[z'{\leftarrow}\mathtt{I}]][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$	$\to_{\mathtt{ye_{YN}}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}\mathtt{I}[z'{\leftarrow}\mathtt{I}]][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{II}]$
$\to_{\mathtt{ym}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}\mathtt{I}[z'{\leftarrow}\mathtt{I}]][x{\leftarrow}\mathtt{I}][y{\leftarrow}z'[z'{\leftarrow}\mathtt{I}]]$	$\to_{\mathtt{ye_{YN}}}$ $\mathtt{I}[z'{\leftarrow}\mathtt{I}][z{\leftarrow}\mathtt{I}[z'{\leftarrow}\mathtt{I}]][x{\leftarrow}\mathtt{I}][y{\leftarrow}\mathtt{I}[z'{\leftarrow}\mathtt{I}]]$