Towards a Quantitative Contextual Equivalence

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Interaction Quantities in Programming

How many $\leq$ does it take to check that a list is sorted?

```
let rec is_sorted  l compare = match l with
|a::b::t -> if compare(a,b) then false
    else is_sorted (b::t) compare
|_-> let n = fibonacci(42) in true
```

```
let rec my_compare a b = # returns false if b < a
    if a < 0 and b < 0 then my_compare(-a,-b)
    elif b < 0 then
        let k = fact(42) in
        false
    elif a < 0 then true
    else (b-a >= 0)
```

Argument from Authority

"The meaning of a program should express its history of access to resources which are not local to it."

– Milner 1975

Equivalence questions rather than interpretation

Interpretation: Program t $\rightsquigarrow$ $[\![t]\!]$
How many calls to compare does t make?

Comparison: Programs t and u $\rightsquigarrow$ $[\![t]\!] = [\![u]\!]$
Do t and u make a similar amount of calls to compare?

Tagging lambda terms

How can I count the interaction steps between a program and its environment?

We propose to tag abstractions and applications with colors, that determine who owns them (the program or the context)

$$\Lambda \ni t, u \quad := \quad x \mid \lambda x.t \mid tu$$

$$\Lambda_{br} \ni \mathtt{t}, \mathtt{u} \quad := \quad x \mid \lambda_{b} x.\mathtt{t} \mid \lambda_{r} x.\mathtt{t} \mid \mathtt{t} \bullet_{b} \mathtt{u} \mid \mathtt{t} \bullet_{r} \mathtt{u}$$

Intuition: we want to look at $C\langle t \rangle$

Colored Reduction

Operational Redexes: $(\lambda_{r} x.\mathtt{t}) \bullet_{r} \mathtt{u} \mapsto_{\beta\tau} \mathtt{t}\{x \leftarrow \mathtt{u}\}$

$$(\lambda_{b} x.\mathtt{t}) \bullet_{b} \mathtt{u} \mapsto_{\beta\tau} \mathtt{t}\{x \leftarrow \mathtt{u}\}$$

Interaction Redexes: $(\lambda_{r} x.\mathtt{t}) \bullet_{b} \mathtt{u} \mapsto_{\beta\bullet} \mathtt{t}\{x \leftarrow \mathtt{u}\}$

$$(\lambda_{b} x.\mathtt{t}) \bullet_{r} \mathtt{u} \mapsto_{\beta\bullet} \mathtt{t}\{x \leftarrow \mathtt{u}\}$$

$\mathtt{t} \Downarrow_{\mathrm{h}}^{\bullet = k}$ means $\mathtt{t}$ head-normalizes with k interaction steps

Colored Contextual Equivalence

Definition

Let $\mathtt{t}, \mathtt{t}' \in \Lambda_{\mathsf{br}}$

- **Head Quantitative Contextual Pre-Order:** $\mathtt{t} \precsim_{\mathcal{C}}^{\bullet} \mathtt{t}'$ if, for all contexts $\mathtt{C}$, if there exists k such that $\mathtt{C}\langle\mathtt{t}\rangle \Downarrow_{\mathrm{h}}^{\bullet=k}$ then $\mathtt{C}\langle\mathtt{t}'\rangle \Downarrow_{\mathrm{h}}^{\bullet=k}$;
- **Head Quantitative Contextual Equivalence:** $\mathtt{t} \simeq_{\mathcal{C}}^{\bullet} \mathtt{t}'$ is the equivalence relation induced by $\precsim_{\mathcal{C}}^{\bullet}$, that is, $\mathtt{t} \simeq_{\mathcal{C}}^{\bullet} \mathtt{t}' \iff \mathtt{t} \precsim_{\mathcal{C}}^{\bullet} \mathtt{t}'$ and $\mathtt{t}' \precsim_{\mathcal{C}}^{\bullet} \mathtt{t}$.

Interaction Equivalence

Let $t, u \in \Lambda$. TFAE:

- *Böhm tree equivalence:* $t \simeq_{BT} u$
- *Relational Equivalence:* $t \simeq_{Rel} u$
- *Colored Relational Equivalence:* $t^{\mathsf{b}} \simeq_{RelC} u^{\mathsf{b}}$
- *Colored Contextual Equivalence:* $t^{\mathsf{b}} \simeq_{CtxC} u^{\mathsf{b}}$

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Relational (Colored) Semantics

$$\begin{array}{lrcll} \textsc{Types} & L, L' & ::= & X \mid M \xrightarrow{\mathsf{cd}} L & \mathsf{c}, \mathsf{d} \in \{\mathsf{r}, \mathsf{b}\} \\ \textsc{Multi types} & M, N & ::= & [L_1, \ldots, L_n] & n \geq 0 \end{array}$$

$$\frac{}{x : [L] \vdash^{0} x : L}\ \mathsf{ax} \qquad \frac{(\Gamma_i \vdash^{l_i} t : L_i)_{i \in I} \quad I \text{ finite}}{\uplus_{i \in I} \Gamma_i \vdash^{\sum_{i \in I} l_i} t : [L_i]_{i \in I}}\ \mathsf{many}$$

$$\frac{\Gamma, x : M \vdash^{k} t : L}{\Gamma \vdash^{k} \lambda_{\mathsf{c}} x.t : M \xrightarrow{\mathsf{cd}} L}\ \lambda$$

$$\frac{\Gamma \vdash^{l} t : M \xrightarrow{\mathsf{cc}^{\perp}} L' \quad \Delta \vdash^{h} u : M}{\Gamma \uplus \Delta \vdash^{l+h+1} t \bullet_{\mathsf{c}^{\perp}} u : L}\ @_{\bullet} \qquad \frac{\Gamma \vdash^{l} t : M \xrightarrow{\mathsf{cc}} L' \quad \Delta \vdash^{h} u : M}{\Gamma \uplus \Delta \vdash^{l+h} t \bullet_{\mathsf{c}} u : L}\ @_{\tau}$$

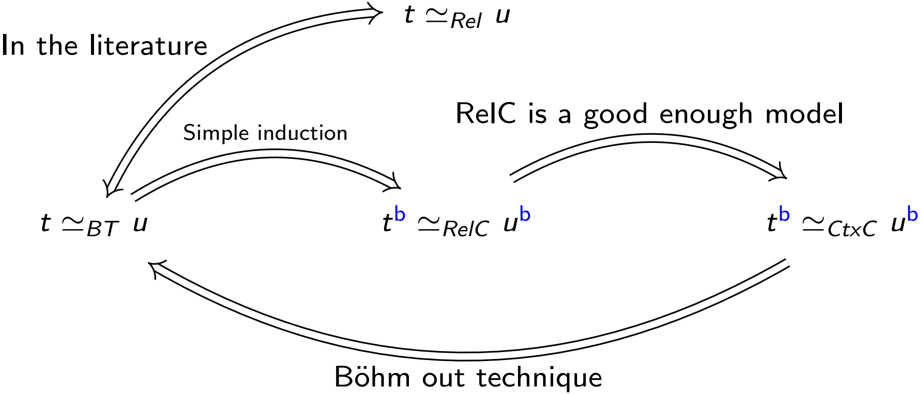
- $\mathtt{t} \precsim_{RelC} \mathtt{u}$ if $\forall (\Gamma, L)\ \Gamma \vdash^{k} \mathtt{t} : L \implies \Gamma \vdash^{k} \mathtt{u} : L$

Good Properties of RelC

- **Subject Reduction and Expansion:** $\mathtt{t} \to_{\mathtt{h}} \mathtt{u}$ then for all (Γ, L) $\Gamma \vdash^{k} \mathtt{t} : L \iff \Gamma \vdash^{k'} \mathtt{u} : L$ and $k' = k$ or $k - 1$ (if $\to_{\mathtt{h}}$ is interactional)
- **Built-in Tightness:** for all colored head-normal form $\mathtt{h}$, there exists (Γ, L) such that $\Gamma \vdash^{0} \mathtt{t} : L$
- **Stability by Colored Contexts:** if $\mathtt{t} \precsim_{RelC} \mathtt{u}$ then for all $\mathtt{C}$ $\mathtt{C}\langle\mathtt{t}\rangle \precsim_{RelC} \mathtt{C}\langle\mathtt{u}\rangle$

No formal relationship with Rel..

The Complete Picture



Optimize the number of interactions

Why do we impose that contextually equivalent terms have the same number of interaction?

- **Head Quantitative Contextual Pre-Order:** $\mathtt{t} \precsim_{\mathcal{C}}^{\bullet} \mathtt{u}$ if, for all contexts C, if there exists k such that $C\langle\mathtt{t}\rangle \Downarrow_{\mathrm{h}}^{\bullet=k}$ then $C\langle\mathtt{u}\rangle \Downarrow_{\mathrm{h}}^{\bullet=k'}$ **with** $k' \leq k$;
- It does not change the associated equivalence relation.

This contextual preorder includes η-reduction:

$$\lambda_{\mathsf{b}} y.x \bullet_{\mathsf{b}} y \precsim_{\mathcal{C}}^{\bullet} x$$

So does Rel!

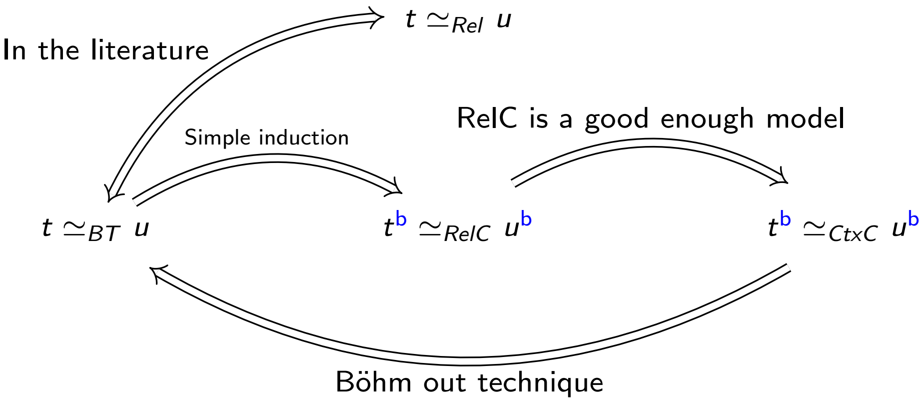
RelC is η-flawed

Our current proof technique cannot go through: RelC does not support η-reduction.

$$\dfrac{\dfrac{x:[[\,]\xrightarrow{\mathsf{rb}} L] \vdash^{0} x:[\,]\xrightarrow{\mathsf{rb}} L \qquad\qquad \vdash^{0} y:[\,]}{x:[[\,]\xrightarrow{\mathsf{rb}} L] \vdash^{1} x\bullet_{\mathsf{b}} y : L}}{x:[[\,]\xrightarrow{\mathsf{rb}} L] \vdash^{1} \lambda_{\mathsf{b}} y.x\bullet_{\mathsf{b}} y : [\,]\xrightarrow{\mathsf{bc}} L}$$

but $x:[[\,]\xrightarrow{\mathsf{rb}} L] \not\vdash^{k} x:[\,]\xrightarrow{\mathsf{bc}} L$

Thank you!



$t \simeq_{Rel} u$
In the literature
Simple induction
RelC is a good enough model
$t \simeq_{BT} u$
$t^{b} \simeq_{RelC} u^{b}$
$t^{b} \simeq_{CtxC} u^{b}$
Böhm out technique