Towards a Quantitative Contextual Equivalence

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Interaction Quantities in Programming

How many \( \leq \) does it take to check that a list is sorted?

```ocaml
let rec is_sorted l compare = match l with
|a::b::t -> if compare(a,b) then false
    else is_sorted (b::t) compare
|_-> let n = fibonacci(42) in true
```

```ocaml
let rec my_compare a b = # returns false if b < a
    if a < 0 and b < 0 then my_compare(-a,-b)
    elif b < 0 then
        let k = fact(42) in
        false
    elif a < 0 then true
    else (b-a >= 0)
```
Argument from Authority

"The meaning of a program should express its history of access to resources which are not local to it."
– Milner 1975
Equivalence questions rather than interpretation

**Interpretation:** Program $t \leadsto [t]

*How many calls to compare does $t$ make?*

**Comparison:** Programs $t$ and $u \leadsto [t] = [u]

*Do $t$ and $u$ make a similar amount of calls to compare?*
Tagging lambda terms

How can I count the interaction steps between a program and its environment?

We propose to tag abstractions and applications with colors, that determine who owns them (the program or the context)

$$\Lambda \ni t, u := x | \lambda x. t | tu$$

$$\Lambda_{br} \ni t, u := x | \lambda_b x. t | \lambda_r x. t | t_b u | t_r u$$

**Intuition:** we want to look at $C\langle t \rangle$
Colored Reduction

Operational Redexes: \[(\lambda_r x. t) \bullet_r u \mapsto_{\beta} t\{x \leftarrow u\}\]

\[(\lambda_b x. t) \bullet_b u \mapsto_{\beta} t\{x \leftarrow u\}\]

Interaction Redexes: \[(\lambda_r x. t) \bullet_b u \mapsto_{\beta} t\{x \leftarrow u\}\]

\[(\lambda_b x. t) \bullet_r u \mapsto_{\beta} t\{x \leftarrow u\}\]

\[t \Downarrow_h^{=k}\] means \(t\) head-normalizes with \(k\) interaction steps
Colored Contextual Equivalence

Definition
Let $t, t' \in \Lambda_{br}$

- **Head Quantitative Contextual Pre-Order:** $t \lesssim^C t'$ if, for all contexts $C$, if there exists $k$ such that $C\langle t \rangle \Downarrow^h = k$ then $C\langle t' \rangle \Downarrow^h = k$;

- **Head Quantitative Contextual Equivalence:** $t \simeq^C t'$ is the equivalence relation induced by $\lesssim^C$, that is, $t \simeq^C t'$ if and only if $t \lesssim^C t'$ and $t' \lesssim^C t$. 
Interaction Equivalence

Let $t, u \in \Lambda$. TFAE:

- **Böhm tree equivalence**: $t \simeq_{BT} u$
- **Relational Equivalence**: $t \simeq_{Rel} u$
- **Colored Relational Equivalence**: $t^b \simeq_{RelC} u^b$
- **Colored Contextual Equivalence**: $t^b \simeq_{CtxC} u^b$
Interaction Equivalence

Let $t, u \in \Lambda$. TFAE:

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Relational (Colored) Semantics

**Types**

\[ L, L' ::= X \mid M \xrightarrow{cd} L \quad c, d \in \{r, b\} \]

**Multi types**

\[ M, N ::= [L_1, \ldots, L_n] \quad n \geq 0 \]

\[ x : [L] \vdash^0 x : L \quad \text{ax} \]

\[ (\Gamma_i \vdash^l t : L_i)_{i \in I} \quad I \text{ finite} \]

\[ \bigoplus_{i \in I} \Gamma_i \vdash \sum_{i \in I} L_i : [L_i]_{i \in I} \quad \text{many} \]

\[ \Gamma, x : M \vdash^k t : L \]

\[ \frac{}{\Gamma \vdash^k \lambda_c x.t : M \xrightarrow{cd} L} \]

\[ \Gamma \vdash^l t : M \xrightarrow{cc} L' \]

\[ \Delta \vdash^h u : M \]

\[ \Gamma \uplus \Delta \vdash^{l+h+1} t \bullet_c u : L \quad \text{@} \]

\[ \Gamma \vdash^l t : M \xrightarrow{cc} L' \]

\[ \Delta \vdash^h u : M \]

\[ \Gamma \uplus \Delta \vdash^{l+h} t \bullet_c u : L \quad \text{@}_{\tau} \]

\[ t \preceq_{RelC} u \text{ if } \forall (\Gamma, L) \quad \Gamma \vdash^k t : L \implies \Gamma \vdash^k u : L \]
Good Properties of RelC

- **Subject Reduction and Expansion**: \( t \rightarrow_h u \) then for all \((\Gamma, L)\) \( \Gamma \vdash^k t : L \iff \Gamma \vdash^{k'} u : L \) and \( k' = k \) or \( k - 1 \) (if \( \rightarrow_h \) is interactional)

- **Built-in Tightness**: for all colored head-normal form \( h \), there exists \((\Gamma, L)\) such that \( \Gamma \vdash^0 t : L \)

- **Stability by Colored Contexts**: if \( t \lesssim_{RelC} u \) then for all \( C \) \( C\langle t \rangle \lesssim_{RelC} C\langle u \rangle \)

No formal relationship with Rel..
In the literature

$t \simeq_{BT} u$

Simple induction

$t \simeq_{Rel} u$

RelC is a good enough model

$t^b \simeq_{RelC} u^b$

Böhm out technique

$t^b \simeq_{CtxC} u^b$
Optimize the number of interactions

Why do we impose that contextually equivalent terms have the same number of interaction?

- **Head Quantitative Contextual Pre-Order:** \( t \sim_C u \) if, for all contexts \( C \), if there exists \( k \) such that \( C\langle t \rangle \Downarrow_h^\bullet = k \) then \( C\langle u \rangle \Downarrow_h^\bullet = k' \) with \( k' \leq k \);

- It does not change the associated equivalence relation.

This contextual preorder includes \( \eta \)-reduction:

\[
\lambda_b y. x \bullet_b y \sim_C x
\]

So does Rel!
RelC is $\eta$-flawed

Our current proof technique cannot go through: RelC does not support $\eta$-reduction.

\[
\begin{align*}
\frac{x: [\ ] \xrightarrow{rb} L \vdash^0 x: [ ] \xrightarrow{rb} L}{\vdash^0 y: [ ]} \\
\frac{x: [ ] \xrightarrow{rb} L \vdash^1 x \bullet_b y : L}{x: [ ] \xrightarrow{rb} L \vdash^1 \lambda_b y.x \bullet_b y: [ ] \xrightarrow{bc} L}
\end{align*}
\]

but \( x: [ ] \xrightarrow{rb} L \vdash^k x: [ ] \xrightarrow{bc} L \)
In the literature

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