Light Genericity

Beniamino Accattoli[1], Adrienne Lancelot[12]

[1]Inria & LIX, École Polytechnique
[2]IRIF, Université Paris Cité & CNRS

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Outline

Lambda Calculus

In the beginning there was the Lambda Calculus

Then people asked "What are equivalent lambda terms?"

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Partial Recursive Functions

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And people knew what was the right *preorder* on partial functions

$$f \leq_{\mathrm{PRF}} g \text{ if } \forall n \in \mathbb{N},\ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n)$$

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Q1: What is the lambda term that represents **undefined** ($\perp$) ?

Answer: possibly $\Omega := \delta\delta$

Q2: If one is concerned with program equivalence, what is the class of **undefined** lambda terms ?

Answer: any term equivalent with Ω

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The Lambda-Calculus

Undefinedness, via Solvability

What should represent **undefined** in the lambda-calculus?
The story of solvability is partially inspired by that question.

Undefined is..	β-diverging	unsolvable	inscrutable[1]
Eq. Theory	Inconsistent ☹	Consistent [2]	Consistent

Induced Eq. Theory: the smallest equational theory equating undefined terms

Consistency: not equating everything

[1] synonym of non potentially valuable
[2] Why? Genericity Lemma

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Undefined is..	β-diverging	head-diverging	whead-diverging
Eq. Theory	Inconsistent ☹	Consistent [3]	Consistent

In **Call-by-Name**, there is an operational characterization of solvability.

$$t \text{ is solvable} \iff t \text{ is head-normalizing}$$
$$t \text{ is scrutable} \iff t \text{ is weakhead-normalizing}$$

[3]Why? Genericity Lemma

The Call-by-Value Lambda-Calculus

Undefinedness, a Mess

What should represent **undefined** in the Call-by-Value lambda-calculus?

Undefined is..	β_v-diverging	unsolvable	inscrutable
Eq. Theory	Inconsistent ☹	Inconsistent ☹[4]	Consistent [5]

No operational characterization. Open terms cause problems!

$\Omega_{nf} = (\lambda x.\delta)(yy)\delta$ is an inscrutable β_v-normal form.

We can recover operational characterizations in refinements of Plotkin's CbV lambda-calculus.

[4] Why? Short answer: all values should be *defined*. Long answer: [AG22]

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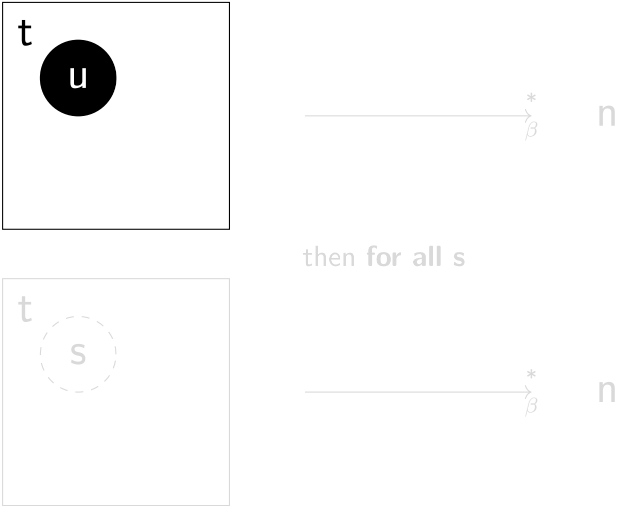
Barendregt's Genericity

Intuitively

Given t a term, if for some u in $\mathcal{U}$ (the set of undefined terms)



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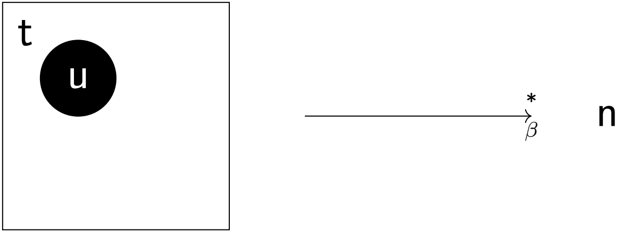

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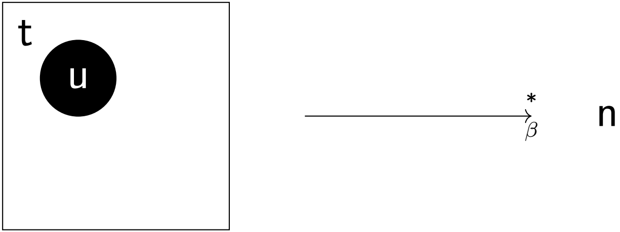

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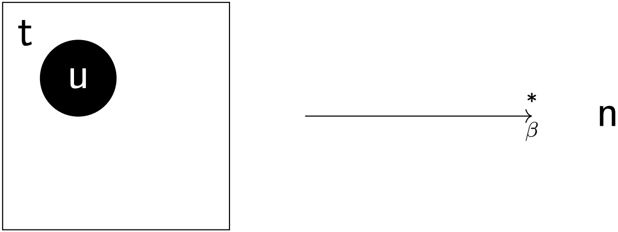

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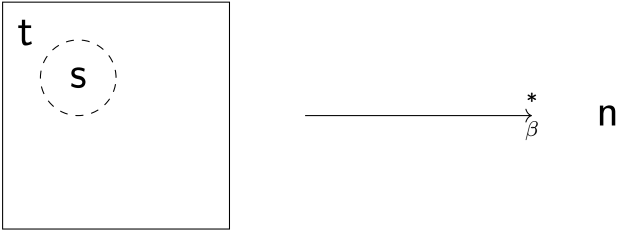

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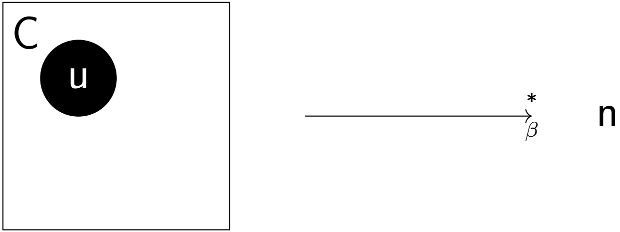
Given C a context, if for some u in $\mathcal{U}$



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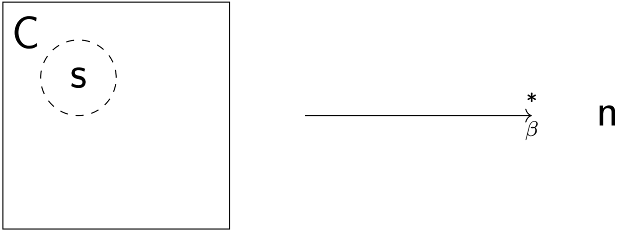

Barendregt's Genericity

Statement

Heavy Genericity: let u be head-diverging and C such that $C\langle u\rangle \to_{\beta}^{*} n$ where n is β-normal then $C\langle s\rangle \to_{\beta}^{*} n$ for all $s \in \Lambda$.

Consistency: $\exists \mathcal{T}$ such that for all u undefined we have that $\mathcal{T} \vdash u = \Omega$ and $\mathcal{T}$ is consistent

Maximality: $\exists \mathcal{T}$ maximal: if there exists $\mathcal{T}'$ such that $\mathcal{T} \subsetneq \mathcal{T}'$ then $\mathcal{T}'$ is inconsistent

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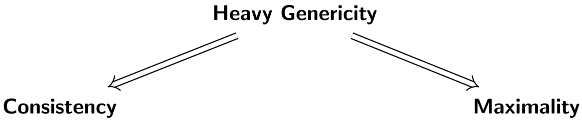

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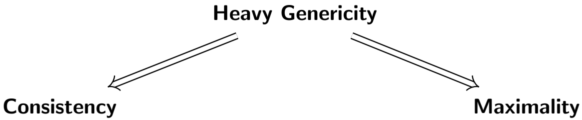

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Light Genericity

We want to consider a lighter genericity statement:

- Use a simpler reduction than $\to_\beta$
- Do not compare normal forms

Four reasons why:

- Powerful enough

- Modular

  It shall look the same for any reduction strategy–in particular CbN and CbV
- Connection with contextual equivalence
- Not looking at full normal forms will allow us to dualize the statement to *co-genericity*

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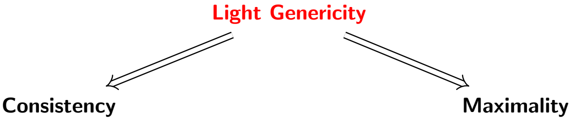

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Proving Light Genericity

Light Genericity is a good property, rather than a lemma.

- It is very close to what is called *sensible* theories
- Any good model should satisfy light genericity

So, how do we prove it?

- We present multiple proof techniques (denotational, bisimulations-al & operational).

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(Closed) Contextual Preorder

and induced equivalence

The (closed) **contextual preorder** associated to a reduction $\rightarrow_{\mathtt{s}}$ is defined as:

- $t \precsim_{\mathcal{C}}^{\mathtt{s}} u$ if for all closing[6] contexts C, $C\langle t\rangle$ is s-normalizing implies $C\langle u\rangle$ is s-normalizing.

(Closed) contextual equivalence $\simeq_{\mathcal{C}}^{\mathtt{s}}$ is defined as the symmetric closure of the preorder.

The two reductions we are interested in are the head CbN reduction and the weak CbV reduction and their associated (closed) contextual preorders.

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Open Contextual Preorder

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The **open contextual preorder** associated to a reduction $\to_{\mathtt{s}}$ is defined as:

- $t \precsim^{\mathtt{s}}_{\mathcal{CO}} u$ if **for all contexts** C, $C\langle t\rangle$ is $\mathtt{s}$-normalizing implies $C\langle u\rangle$ is $\mathtt{s}$-normalizing.

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In Call-by-Name, head open contextual equivalence is exactly the theory $\mathcal{H}*$.

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Minimum terms for the contextual preorder

For a reduction $\to_{\mathtt{s}}$, we can state light genericity:

Light Genericity:
let u be $\mathtt{s}$-diverging and C such that $C\langle u\rangle$ is $\mathtt{s}$-normalizing
then $C\langle t\rangle$ is $\mathtt{s}$-normalizing for all $t \in \Lambda$.

Or more concisely:

Light Genericity: $\mathtt{s}$-diverging terms are minimum terms for the open contextual preorder associated to $\mathtt{s}$.

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Equational theories

Or rather inequational theories

Definition (Inequational $\mathbb{s}$-theory)

Let $\mathbb{s}$ be a reduction. An inequational $\mathbb{s}$-theory $\leq^{\mathbb{s}}_{\mathcal{T}}$ is a compatible[7] pre-order on terms containing $\mathbb{s}$-conversion.

Closed/Open $\mathbb{s}$-contextual preorders are $\mathbb{s}$-inequational theories.

The non-trivial point is that they contain $\mathbb{s}$-conversion.

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Inequational theories

Generalization of sensible and semi-sensible

An inequational $\mathtt{s}$-theory $\leq^{\mathtt{s}}_{\mathcal{T}}$ is called:

- *Consistent*: whenever it does not relate all terms;
- $\mathtt{s}$-*ground*: if $\mathtt{s}$-diverging terms are minimum terms for $\leq^{\mathtt{s}}_{\mathcal{T}}$;
- $\mathtt{s}$-*adequate*: if $t \leq^{\mathtt{s}}_{\mathcal{T}} u$ and t is $\mathtt{s}$-normalizing entails u is $\mathtt{s}$-normalizing.

Groundness and Adequacy correspond (in CbN) with the order-variants of sensible and semi-sensible theories.

Adequacy implies: minimum terms for $\leq^{\mathtt{s}}_{\mathcal{T}}$ are $\mathtt{s}$-diverging.

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Maximality

For $\mathtt{s} \in \{\text{head CbN, weak CbV}\}$, we can state maximality uniformly.

The proof is not uniform as it relies on critical solvability/scrutability concepts.

Theorem

Maximality of $\precsim^{\mathtt{s}}_{\mathcal{CO}}$: $\precsim^{\mathtt{s}}_{\mathcal{CO}}$ is a maximal consistent inequational $\mathtt{s}$-theory, i.e.

$$\text{if } \precsim^{\mathtt{s}}_{\mathcal{CO}} \subsetneq \mathcal{R} \text{ then } \mathcal{R} \text{ is inconsistent.}$$

An elegant proof that closed and open contextual equivalence coincides follows: $\precsim^{\mathtt{s}}_{\mathcal{CO}} \subseteq \precsim^{\mathtt{s}}_{\mathcal{C}}$ and $\precsim^{\mathtt{s}}_{\mathcal{C}}$ is consistent, hence $\precsim^{\mathtt{s}}_{\mathcal{CO}} = \precsim^{\mathtt{s}}_{\mathcal{C}}$

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Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

CbN Light Genericity: head-diverging terms are minimum for the head open contextual preorder.

We can unfold the statement:

CbN Light Genericity: let u be head-diverging and C such that $C\langle u\rangle$ is head-normalizing then $C\langle t\rangle$ is head-normalizing for all $t \in \Lambda$.

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Direct proof of Light Genericity

Takahashi proves Barendregt's genericity with a very short proof [Tak94] and gives as a corollary light genericity.

Key idea/trick: Reason with substitutions instead of contexts!

We adapt **Takahashi's trick** to give a direct proof of light genericity!

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Takahashi's Trick

Takahashi's trick Light genericity as substitution implies light genericity!

Light genericity as substitution: let u be s-diverging and t such that $t\{x{\leftarrow}u\}$ is s-normalizing then $t\{x{\leftarrow}s\}$ is s-normalizing for all $s \in \Lambda$.

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Takahashi's Trick in CbN

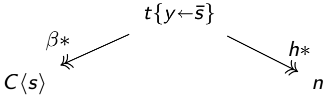
Proof: [Hyp: $C\langle u\rangle$ is h-normalizing]

Let $\mathtt{fv}(u) \cup \mathtt{fv}(s) = \{x_1, \ldots, x_k\}$, and y a fresh variable.

- $\bar{u} := \lambda x_1. \ldots .\lambda x_k.u$ is a closed term.
- Consider $t := C\langle y x_1 \ldots x_k\rangle$, and note that:
  $$t\{y{\leftarrow}\bar{u}\} = C\langle \bar{u} x_1 \ldots x_k\rangle = C\langle(\lambda x_1. \ldots .\lambda x_k.u) x_1 \ldots x_k\rangle \to_\beta^k C\langle u\rangle.$$
- u is h-diverging implies that $\bar{u}$ is also h-diverging.
- (Head Normalization Theorem) $C\langle u\rangle$ is h-normalizing then so is $t\{y{\leftarrow}\bar{u}\}$

By *light genericity as substitution*, $t\{y{\leftarrow}s'\}$ is h-normalizing for every s'.

In particular, take $s' := \bar{s} = \lambda x_1. \ldots .\lambda x_k.s$:



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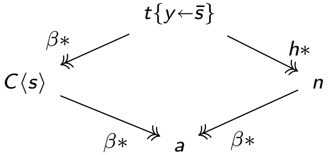

Confluence of β

Takahashi's Trick in CbN

Proof: [Hyp: $C\langle u\rangle$ is h-normalizing]

Let $\mathtt{fv}(u) \cup \mathtt{fv}(s) = \{x_1, \ldots, x_k\}$, and y a fresh variable.

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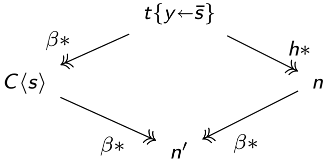

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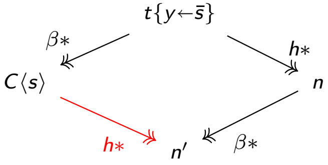

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Head normal forms are stable by β

Head Normalization Theorem

Light Genericity in CbN

Light Genericity

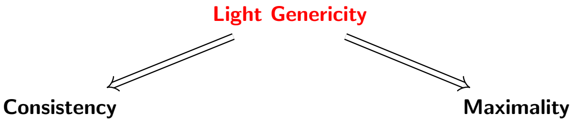
$\swarrow$ $\searrow$

Consistency **Maximality**

We use the head open contextual preorder $\precsim^h_{\mathcal{CO}}$ to prove both.

- It is consistent to collapse unsolvable terms:
  (by light genericity) $\precsim^h_{\mathcal{CO}}$ equates unsolvable terms and $\precsim^h_{\mathcal{CO}}$ is consistent ($\mathtt{I} \not\precsim^h_{\mathcal{CO}} \Omega$)
- $\precsim^h_{\mathcal{CO}}$ is maximal:
  (by light genericity) any larger theory is inconsistent
- $\precsim^h_{\mathcal{CO}}$ coincides with $\precsim^h_{\mathcal{C}}$ (by maximality)

Light Genericity in CbN



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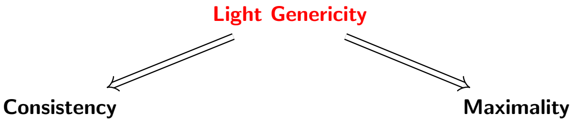

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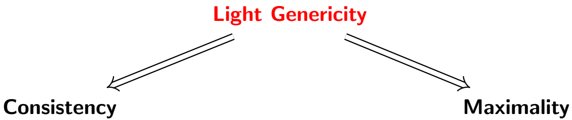

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Outline

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At least not using Plotkin's calculus

$\Omega_{nf} = ((\lambda x.\delta)(yz))\delta$ is meaningless!

Open and closed CbV contextual equivalences do not coincide:

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Call-by-Value Light Genericity

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The good call-by-value contextual equivalence is Plotkin's closed.

$$\simeq_{\mathcal{C}}^{v} := \simeq_{\mathcal{C}}^{p_v} = \simeq_{\mathcal{C}}^{vsc} = \simeq_{\mathcal{CO}}^{vsc}$$

We use a nicer calculus (the Value Substitution Calculus [AP12]) that knows how to deal with open terms, but retains the same closed contextual equivalence.

Undefined terms are exactly *vsc*-diverging terms.[8]

There, we can show:

- Light Genericity: *vsc*-diverging terms are minimum for $\precsim_{\mathcal{CO}}^{vsc}$
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Proofs of Call-by-Value Light Genericity

How to prove light genericity?

- Direct proof: *Takahashi's trick* adapts, but not very smoothly.
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Characterization of minimum terms

For $\mathbb{s} \in \{\mathit{head}, \mathit{vsc}\}$:

Light genericity says:

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Adequacy ($t \mathrel{\mathcal{R}} u$ and t is $\mathbb{s}$-normalizing then u is $\mathbb{s}$-normalizing) implies the converse implication.

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Super terms

A term t is *$\mathtt{s}$-super* if, coinductively, $t \to_{\mathtt{s}}^{*} \lambda x.u$ and u is $\mathtt{s}$-super.

Intuitively, t infinitely normalizes to $\lambda x_1. \lambda x_2. \dots \lambda x_k. \dots$

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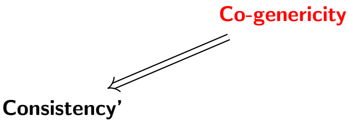
Co-genericity

In call-by-value:



Again, we use the open call-by-value contextual preorder $\precsim^{v}_{\mathcal{CO}}$ to prove it.

- It is consistent to equate super terms, as $\precsim^{v}_{\mathcal{CO}}$ does it and is consistent.
- It is consistent to equate diverging terms and to equate super terms, as $\precsim^{v}_{\mathcal{CO}}$ does it and is consistent.

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- Direct proof: *Takahashi's trick* adapts, and the proof is easier than for light genericity.
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Outline

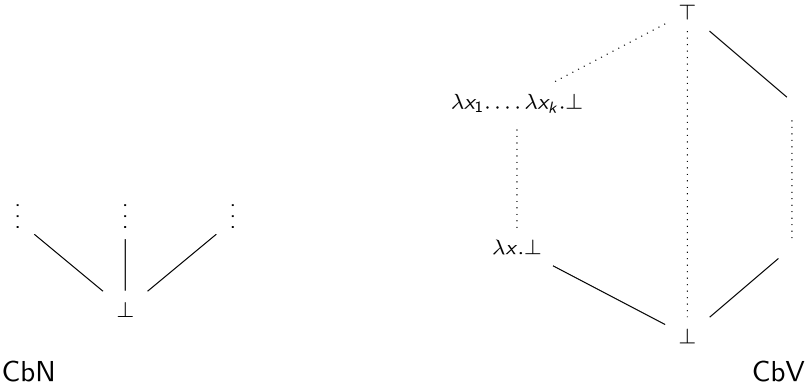
Conclusion

- **Light genericity** is a modular concept that is strong enough to imply the two main consequences of Barendregt's genericity.
- It is naturally dualizable as **co-genericity**. Both concepts are inspired and tied with contextual preorders.
- An application of light genericity and maximality is an elegant proof of the fact that closed and open contextual equivalences coincide.

A question remains: we named the two genericity statements Heavy and Light, but we don't know whether one implies the other or not.

Thank you!

To appear in FoSSaCS24 & technical report: https://hal.science/hal-04406343



Contextual preorder for lambda terms

$\bot :=$ equivalence class of Ω

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