Light Genericity

Beniamino Accattoli\textsuperscript{1}, Adrienne Lancelot\textsuperscript{12}

\textsuperscript{1}Inria & LIX, École Polytechnique
\textsuperscript{2}IRIF, Université Paris Cité & CNRS

February 22nd 2024 - séminaire PPS
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’sGenericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Lambda Calculus

In the beginning there was the Lambda Calculus

Then people asked ”What are equivalent lambda terms?”
In the beginning there was the Lambda Calculus

Then people asked "What are equivalent lambda terms?"
In the beginning there were also Partial Recursive Functions

And people knew what was the right preorder on partial functions

\[ f \leq_{\text{PRF}} g \text{ if } \forall n \in \mathbb{N}, \ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n) \]

\[ f_{\bot} : n \mapsto \bot \text{ is the minimum computable function for } \leq_{\text{PRF}} \]
Partial Recursive Functions

In the beginning there were also Partial Recursive Functions

And people knew what was the right preorder on partial functions

\[ f \leq_{\text{PRF}} g \text{ if } \forall n \in \mathbb{N}, \ f(n) = \bot \text{ or } f(n) =_\mathbb{N} g(n) \]

\[ f_\bot : n \mapsto \bot \text{ is the minimum computable function for } \leq_{\text{PRF}} \]
Partial Recursive Functions

In the beginning there were also Partial Recursive Functions

And people knew what was the right preorder on partial functions

\[ f \leq_{\text{PRF}} g \text{ if } \forall n \in \mathbb{N}, \ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n) \]

\[ f_{\bot} : n \mapsto \bot \text{ is the minimum computable function for } \leq_{\text{PRF}} \]
The Lambda-Calculus
Computable functions inside lambda

Q1: What is the lambda term that represents undefined (⊥) ?
Answer: possibly Ω := δδ

Q2: If one is concerned with program equivalence, what is the class of undefined lambda terms ?
Answer: any term equivalent with Ω
Q1: What is the lambda term that represents *undefined* (⊥) ?

Answer: possibly Ω := δδ

Q2: If one is concerned with program equivalence, what is the class of *undefined* lambda terms ?

Answer: any term equivalent with Ω
What should represent **undefined** in the lambda-calculus? The story of **solvability** is partially inspired by that question.

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta$-diverging</th>
<th>unsolvable</th>
<th>inscrutable$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent ☹</td>
<td>Consistent $^2$</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**Induced Eq. Theory:** the smallest equational theory equating undefined terms

**Consistency:** not equating everything

---

$^1$ synonym of non potentially valuable

$^2$ Why? Genericity Lemma
What should represent **undefined** in the lambda-calculus? The story of solvability is partially inspired by that question.

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta$-diverging</th>
<th>unsolvable</th>
<th>inscrutable$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent 😞</td>
<td>Consistent $^2$</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**Induced Eq. Theory:** the smallest equational theory equating undefined terms

**Consistency:** not equating everything

---

$^1$ synonym of non potentially valuable

$^2$ Why? Genericity Lemma
What should represent **undefined** in the lambda-calculus? The story of **solvability** is partially inspired by that question.

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta$-diverging</th>
<th>unsolvable</th>
<th>inscrutable$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent ☹</td>
<td>Consistent$^2$</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**Induced Eq. Theory:** the smallest equational theory equating undefined terms

**Consistency:** not equating everything

---

$^1$ synonym of non potentially valuable

$^2$ Why? Genericity Lemma
The Lambda-Calculus
Undefinedness, via Solvability

What should represent **undefined** in the lambda-calculus? The story of **solvability** is partially inspired by that question.

<table>
<thead>
<tr>
<th>Undefinded is..</th>
<th>$\beta$-diverging</th>
<th>unsolvable</th>
<th>inscrutable$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent $\implies$</td>
<td>Consistent $^2$</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**Induced Eq. Theory:** the smallest equational theory equating undefined terms

**Consistency:** not equating everything

$^1$ synonym of non potentially valuable
$^2$ Why? Genericity Lemma
What should represent **undefined** in the lambda-calculus? The story of **solvability** is partially inspired by that question.

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta$-diverging</th>
<th>unsolvable</th>
<th>inscrutable$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent $,$</td>
<td>Consistent</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**Induced Eq. Theory**: the smallest equational theory equating undefined terms

**Consistency**: not equating everything

---

$^1$ synonym of non potentially valuable

$^2$ Why? Genericity Lemma
What should represent **undefined** in the lambda-calculus?

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta$-diverging</th>
<th>head-diverging</th>
<th>whead-diverging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent 😞</td>
<td>Consistent $^3$</td>
<td>Consistent</td>
</tr>
</tbody>
</table>

**In Call-by-Name**, there is an operational characterization of solvability.

- $t$ is solvable $\iff$ $t$ is head-normalizing
- $t$ is scrutable $\iff$ $t$ is weakhead-normalizing

$^3$Why? Genericity Lemma
The Call-by-Value Lambda-Calculus

Undefinedness, a Mess

What should represent \textbf{undefined} in the Call-by-Value lambda-calculus?

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta_v$-diverging</th>
<th>unsolvable</th>
<th>inscrutable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent $\smile$</td>
<td>Inconsistent $\smile^4$</td>
<td>Consistent $^5$</td>
</tr>
</tbody>
</table>

No operational characterization. Open terms cause problems!

$\Omega_{nf} = (\lambda x.\delta)(yy)\delta$ is an inscrutable $\beta_v$-normal form.

We can recover operational characterizations in refinements of Plotkin’s CbV lambda-calculus.

---

$^4$Why? Short answer: all values should be \textit{defined}. Long answer: [AG22]

$^5$But no Genericity Lemma!
What should represent \textbf{undefined} in the Call-by-Value lambda-calculus?

<table>
<thead>
<tr>
<th>Undefined is..</th>
<th>$\beta_v$-diverging</th>
<th>unsolvable</th>
<th>inscrutable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. Theory</td>
<td>Inconsistent ☹</td>
<td>Inconsistent ☹</td>
<td>Consistent 5</td>
</tr>
</tbody>
</table>

No operational characterization. Open terms cause problems!

$$\Omega_{nf} = (\lambda x.\delta)(yy)\delta$$ is an inscrutable $\beta_v$-normal form.

We can recover operational characterizations in refinements of Plotkin’s CbV lambda-calculus.

---

4 Why? Short answer: all values should be \textit{defined}. Long answer: [AG22]  
5 But no Genericity Lemma!
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Barendregt’s Genericity

Intuitively

Given $t$ a term, if for some $u$ in $\mathcal{U}$ (the set of undefined terms)

\[
t \rightarrow_{\beta}^* n
\]

then for all $s$

\[
t \rightarrow_{\beta}^* n
\]
Given $t$ a term, if for some $u$ in $\mathcal{U}$ (the set of undefined terms) $t \xrightarrow{\beta} u^*$, then for all $s$: $t \xrightarrow{\beta} s \xrightarrow{\beta} u^*$. 

Barendregt’s Genericity
Barendregt’s Genericity

Intuitively

Given a term, if for some in (the set of undefined terms)

then for all

\[ \beta \]
Barendregt’s Genericity

Intuitively

Given t a term, if for some u in \( \mathcal{U} \) (the set of undefined terms)

\[
t \xrightarrow{\beta^*} u
\]

then for all s

\[
t \xrightarrow{\beta^*} s
\]
Barendregt’s Genericity

Intuitively

Given $C$ a context, if for some $u$ in $\mathcal{U}$

$$
\begin{array}{c}
C \\
\text{u}
\end{array}
\xrightarrow{\ast}^\beta n
$$

then for all $s$

$$
\begin{array}{c}
C \\
\text{s}
\end{array}
\xrightarrow{\ast}^\beta n
$$
Barendregt’s Genericity

Statement

**Heavy Genericity:** let \( u \) be head-diverging and \( C \) such that \( C\langle u\rangle \rightarrow^* \beta n \) where \( n \) is \( \beta \)-normal then \( C\langle s\rangle \rightarrow^* \beta n \) for all \( s \in \Lambda \).

\[
\begin{align*}
\text{Heavy Genericity} & \quad \text{Consistency} & \quad \text{Maximality} \\
\text{Consistency: } & \exists T \text{ such that for all } u \text{ undefined we have that } & T \vdash u = \Omega \\
& \text{and } T \text{ is consistent} & \\
\text{Maximality: } & \exists T \text{ maximal: if there exists } T' \text{ such that } & T \subsetneq T' \text{ then } \\
& \text{ } & T' \text{ is inconsistent}
\end{align*}
\]
Barendregt’s Genericity

**Statement**

**Heavy Genericity**: let $u$ be head-diverging and $C$ such that $C\langle u \rangle \rightarrow^* \beta n$ where $n$ is $\beta$-normal then $C\langle s \rangle \rightarrow^* \beta n$ for all $s \in \Lambda$.

---

**Consistency**: $\exists T$ such that for all $u$ undefined we have that $T \vdash u = \Omega$ and $T$ is consistent

**Maximality**: $\exists T$ maximal: if there exists $T'$ such that $T \subset T'$ then $T'$ is inconsistent
Barendregt’s Genericity
Statement

**Heavy Genericity:** let \( u \) be head-diverging and \( C \) such that \( C\langle u \rangle \rightarrow^* \beta n \) where \( n \) is \( \beta \)-normal then \( C\langle s \rangle \rightarrow^* \beta n \) for all \( s \in \Lambda \).

**Consistency:** \( \exists T \) such that for all \( u \) undefined we have that
\[ T \vdash u = \Omega \]
and \( T \) is consistent

**Maximality:** \( \exists T \) maximal:
if there exists \( T' \) such that
\[ T \subsetneq T' \] then
\( T' \) is inconsistent
Light Genericity

We want to consider a lighter genericity statement:

- Use a simpler reduction than $\rightarrow_\beta$
- Do not compare normal forms

Four reasons why:

- Powerful enough
- Modular
  It shall look the same for any reduction strategy—in particular CbN and CbV
- Connection with contextual equivalence
- Not looking at full normal forms will allow us to dualize the statement to co-genericity
Light Genericity

We want to consider a lighter genericity statement:

▶ Use a simpler reduction than $\rightarrow_\beta$
▶ Do not compare normal forms

Four reasons why:

▶ Powerful enough

▶ Modular
It shall look the same for any reduction strategy—in particular CbN and CbV

▶ Connection with contextual equivalence

▶ Not looking at full normal forms will allow us to dualize the statement to co-genericity
Proving Light Genericity

Light Genericity is a **good property**, rather than a lemma.
- It is very close to what is called *sensible* theories
- Any good model should satisfy light genericity

So, how do we prove it?
- We present **multiple proof techniques** (denotational, bisimulations-AL & operational).
Proving Light Genericity

Light Genericity is a good property, rather than a lemma.

- It is very close to what is called sensible theories
- Any good model should satisfy light genericity

So, how do we prove it?

- We present multiple proof techniques (denotational, bisimulations-al & operational).
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
(Closed) Contextual Preorder
and induced equivalence

The (closed) contextual preorder associated to a reduction $\rightarrow_s$ is defined as:

$\triangleright t \preceq_C u$ if for all closing$^6$ contexts $C$, $C\langle t \rangle$ is $s$-normalizing implies $C\langle u \rangle$ is $s$-normalizing.

(Closed) contextual equivalence $\simeq_C$ is defined as the symmetric closure of the preorder.

The two reductions we are interested in are the head CbN reduction and the weak CbV reduction and their associated (closed) contextual preorders.

---

$^6$ i.e. $C\langle t \rangle$ and $C\langle u \rangle$ are closed terms
The (closed) contextual preorder associated to a reduction $\rightarrow_s$ is defined as:

$\triangleright \; t \preceq^s_C u \text{ if for all closing}^6 \text{ contexts } C, \; C\langle t \rangle \text{ is } s\text{-normalizing implies } C\langle u \rangle \text{ is } s\text{-normalizing.}$

(Closed) contextual equivalence $\simeq^s_C$ is defined as the symmetric closure of the preorder.

The two reductions we are interested in are the head CbN reduction and the weak CbV reduction and their associated (closed) contextual preorders.

---

$^6$ i.e. $C\langle t \rangle$ and $C\langle u \rangle$ are closed terms
(Closed) Contextual Preorder
and induced equivalence

The (closed) contextual preorder associated to a reduction $\rightarrow_s$ is defined as:

$\triangleright t \precsim_C^s u$ if for all closing$^6$ contexts $C$, $C\langle t \rangle$ is $s$-normalizing implies $C\langle u \rangle$ is $s$-normalizing.

(Closed) contextual equivalence $\simeq_C^s$ is defined as the symmetric closure of the preorder.

The two reductions we are interested in are the head CbN reduction and the weak CbV reduction and their associated (closed) contextual preorders.

$^6$ i.e. $C\langle t \rangle$ and $C\langle u \rangle$ are closed terms
Open Contextual Preorder
and induced equivalence

The **open contextual preorder** associated to a reduction $\to_s$ is defined as:

$\triangleright t \prec^s_{\mathcal{CO}} u$ if **for all contexts** $C$, $C\langle t \rangle$ is $s$-normalizing implies $C\langle u \rangle$ is $s$-normalizing.

Open contextual equivalence $\simeq^s_{\mathcal{CO}}$ is defined as the symmetric closure of the open preorder.

In Call-by-Name, **head open contextual equivalence** is exactly the theory $\mathcal{H}^\bullet$. 
The **open contextual preorder** associated to a reduction $\rightarrow_s$ is defined as:

$$t \prec^s_{CO} u \text{ if for all contexts } C, C\langle t \rangle \text{ is } s\text{-normalizing implies } C\langle u \rangle \text{ is } s\text{-normalizing.}$$

Open contextual equivalence $\simeq^s_{CO}$ is defined as the symmetric closure of the open preorder.

In Call-by-Name, head open contextual equivalence is exactly the theory $\mathcal{H}^\ast$. 
The **open contextual preorder** associated to a reduction $\rightarrow_s$ is defined as:

- $t \preceq^s_{\text{CO}} u$ if for all contexts $C$, $C\langle t \rangle$ is $s$-normalizing implies $C\langle u \rangle$ is $s$-normalizing.

Open contextual equivalence $\simeq^s_{\text{CO}}$ is defined as the symmetric closure of the open preorder.

In Call-by-Name, **head open contextual equivalence** is exactly the theory $\mathcal{H}^\ast$.  

Light Genericity
Minimum terms for the contextual preorder

For a reduction $\to_s$, we can state light genericity:

**Light Genericity:**
let $u$ be $s$-diverging and $C$ such that $C\langle u \rangle$ is $s$-normalizing
then $C\langle t \rangle$ is $s$-normalizing for all $t \in \Lambda$.

Or more concisely:

**Light Genericity:** $s$-diverging terms are *minimum terms* for the open contextual preorder associated to $s$. 
Light Genericity
Minimum terms for the contextual preorder

For a reduction $\rightarrow_s$, we can state light genericity:

**Light Genericity:**

let $u$ be $s$-diverging and $C$ such that $C\langle u \rangle$ is $s$-normalizing
then $C\langle t \rangle$ is $s$-normalizing for all $t \in \Lambda$.

Or more concisely:

**Light Genericity:** $s$-diverging terms are **minimum terms** for the open contextual preorder associated to $s$. 
Equational theories
Or rather inequational theories

Definition (Inequational s-theory)

Let \( s \) be a reduction. An inequational \( s \)-theory \( \leq^s_T \) is a compatible\(^7\) pre-order on terms containing \( s \)-conversion.

Closed/Open \( s \)-contextual preorders are \( s \)-inequational theories.

The non-trivial point is that they contain \( s \)-conversion.

\(^7\)Stable by contextual closure: \( t \leq^s_T u \implies \forall C, \ C\langle t \rangle \leq^s_T C\langle u \rangle \)
Equational theories
Or rather inequational theories

Definition (Inequational $s$-theory)
Let $s$ be a reduction. An inequational $s$-theory $\leq^s_T$ is a compatible\(^7\) pre-order on terms containing $s$-conversion.

Closed/Open $s$-contextual preorders are $s$-inequational theories.
The non-trivial point is that they contain $s$-conversion.

\(^7\)Stable by contextual closure: $t \leq^s_T u \implies \forall C,\ C\langle t \rangle \leq^s_T C\langle u \rangle$
Inequational theories
Generalization of sensible and semi-sensible

An inequational $s$-theory $\leq^s_T$ is called:
- **Consistent**: whenever it does not relate all terms;
- **$s$-ground**: if $s$-diverging terms are minimum terms for $\leq^s_T$;
- **$s$-adequate**: if $t \leq^s_T u$ and $t$ is $s$-normalizing entails $u$ is $s$-normalizing.

Groundness and Adequacy correspond (in CbN) with the order-variants of sensible and semi-sensible theories.

Adequacy implies: minimum terms for $\leq^s_T$ are $s$-diverging.
Inequational theories
Generalization of sensible and semi-sensible

An inequational $s$-theory $\leq^s_T$ is called:

- **Consistent**: whenever it does not relate all terms;
- **$s$-ground**: if $s$-diverging terms are minimum terms for $\leq^s_T$;
- **$s$-adequate**: if $t \leq^s_T u$ and $t$ is $s$-normalizing entails $u$ is $s$-normalizing.

Groundness and Adequacy correspond (in CbN) with the order-variants of sensible and semi-sensible theories.

Adequacy implies: minimum terms for $\leq^s_T$ are $s$-diverging.
Inequational theories
Generalization of sensible and semi-sensible

An inequational $s$-theory $\leq^s_T$ is called:

- **Consistent**: whenever it does not relate all terms;
- **$s$-ground**: if $s$-diverging terms are minimum terms for $\leq^s_T$;
- **$s$-adequate**: if $t \leq^s_T u$ and $t$ is $s$-normalizing entails $u$ is $s$-normalizing.

**Groundness** and **Adequacy** correspond (in CbN) with the order-variants of **sensible** and **semi-sensible** theories.

**Adequacy** implies: minimum terms for $\leq^s_T$ are $s$-diverging.
Maximality

For $s \in \{\text{head CbN, weak CbV}\}$, we can state maximality uniformly.

The proof is not uniform as it relies on critical solvability/scrutability concepts.

**Theorem**

**Maximality of $\simeq^s_{CO}$:** $\simeq^s_{CO}$ is a maximal consistent inequational $s$-theory, i.e.

$$\text{if } \simeq^s_{CO} \subsetneq \mathcal{R} \text{ then } \mathcal{R} \text{ is inconsistent.}$$

An elegant proof that closed and open contextual equivalence coincides follows: $\simeq^s_{CO} \subseteq \simeq^s_{C}$ and $\simeq^s_{C}$ is consistent, hence $\simeq^s_{CO} = \simeq^s_{C}$.
Maximality

For $s \in \{\text{head CbN, weak CbV}\}$, we can state maximality uniformly. The proof is not uniform as it relies on critical solvability/scrutability concepts.

**Theorem**

*Maximality of $\bowtie^s_{CO}$: $\bowtie^s_{CO}$ is a maximal consistent inequational $s$-theory, i.e.*

$$\text{if } \bowtie^s_{CO} \subsetneq \mathcal{R} \text{ then } \mathcal{R} \text{ is inconsistent.}$$

An elegant proof that closed and open contextual equivalence coincides follows: $\bowtie^s_{CO} \subseteq \bowtie^s_{C}$ and $\bowtie^s_{C}$ is consistent, hence $\bowtie^s_{CO} = \bowtie^s_{C}$
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

**CbN Light Genericity:** head-diverging terms are minimum for the head open contextual preorder.

We can unfold the statement:

**CbN Light Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u \rangle$ is head-normalizing then $C\langle t \rangle$ is head-normalizing for all $t \in \Lambda$.

Main difficulty: reasoning with contexts and reduction.
Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

**CbN Light Genericity:** head-diverging terms are minimum for the head open contextual preorder.

We can unfold the statement:

**CbN Light Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u \rangle$ is head-normalizing then $C\langle t \rangle$ is head-normalizing for all $t \in \Lambda$.

Main difficulty: reasoning with contexts and reduction.
Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

**CbN Light Genericity:** head-diverging terms are minimum for the head open contextual preorder.

We can unfold the statement:

**CbN Light Genericity:** let $u$ be head-diverging and $C$ such that $C\langle u \rangle$ is head-normalizing then $C\langle t \rangle$ is head-normalizing for all $t \in \Lambda$.

**Main difficulty:** reasoning with contexts and reduction.
Direct proof of Light Genericity

Takahashi proves Barendregt’s genericity with a very short proof [Tak94] and gives as a corollary light genericity.

**Key idea/trick:** Reason with substitutions instead of contexts!

We adapt Takahashi’s trick to give a direct proof of light genericity!
Direct proof of Light Genericity

Takahashi proves Barendregt’s genericity with a very short proof [Tak94] and gives as a corollary light genericity.

**Key idea/trick:** Reason with substitutions instead of contexts!

We adapt Takahashi’s trick to give a direct proof of light genericity!
Direct proof of Light Genericity

Takahashi proves Barendregt’s genericity with a very short proof [Tak94] and gives as a corollary light genericity.

**Key idea/trick:** Reason with substitutions instead of contexts!

We adapt Takahashi’s trick to give a direct proof of light genericity!
Takahashi’s Trick

**Takahashi’s trick** Light genericity as substitution implies light genericity!

Light genericity as substitution: let \( u \) be \( s \)-diverging and \( t \) such that \( t\{x\leftarrow u\} \) is \( s \)-normalizing then \( t\{x\leftarrow s\} \) is \( s \)-normalizing for all \( s \in \Lambda \).
Takahashi’s Trick

**Takahashi’s trick** Light genericity as substitution implies light genericity!

**Light genericity as substitution:** let $u$ be $s$-diverging and $t$ such that $t\{x\leftarrow u\}$ is $s$-normalizing then $t\{x\leftarrow s\}$ is $s$-normalizing for all $s \in \Lambda$. 
Takahashi’s Trick in CbN

Proof: [Hyp: $C\langle u \rangle$ is $h$-normalizing]

Let $fv(u) \cup fv(s) = \{x_1, \ldots, x_k\}$, and $y$ a fresh variable.

- $\bar{u} := \lambda x_1 \ldots \lambda x_k. u$ is a closed term.
- Consider $t := C\langle yx_1 \ldots x_k \rangle$, and note that:
  
  $$t\{y \leftarrow \bar{u}\} = C\langle \bar{u}x_1 \ldots x_k \rangle = C\langle (\lambda x_1 \ldots \lambda x_k. u)x_1 \ldots x_k \rangle \rightarrow^k_{\beta} C\langle u \rangle.$$  

- $u$ is $h$-diverging implies that $\bar{u}$ is also $h$-diverging.
- (Head Normalization Theorem) $C\langle u \rangle$ is $h$-normalizing then so is $t\{y \leftarrow \bar{u}\}$

By light genericity as substitution, $t\{y \leftarrow s'\}$ is $h$-normalizing for every $s'$.

In particular, take $s' := \bar{s} = \lambda x_1 \ldots \lambda x_k. s$:

$$t\{y \leftarrow \bar{s}\} \beta^* \downarrow\downarrow \downarrow h^* \downarrow\downarrow n$$

26 / 40
Takahashi’s Trick in CbN

Proof: [Hyp: $C\langle u \rangle$ is $h$-normalizing]
Let $fv(u) \cup fv(s) = \{x_1, \ldots, x_k\}$, and $y$ a fresh variable.

- $\bar{u} := \lambda x_1 \ldots \lambda x_k . u$ is a closed term.
- Consider $t := C\langle y x_1 \ldots x_k \rangle$, and note that:

$$
t\{y \leftarrow \bar{u}\} = C\langle \bar{u} x_1 \ldots x_k \rangle = C\langle (\lambda x_1 \ldots \lambda x_k . u) x_1 \ldots x_k \rangle \rightarrow^k_{\beta} C\langle u \rangle.
$$

- $u$ is $h$-diverging implies that $\bar{u}$ is also $h$-diverging.
- (Head Normalization Theorem) $C\langle u \rangle$ is $h$-normalizing then so is $t\{y \leftarrow \bar{u}\}$

By **light genericity as substitution**, $t\{y \leftarrow s'\}$ is $h$-normalizing for every $s'$.

In particular, take $s' := \bar{s} = \lambda x_1 \ldots \lambda x_k . s$:
Takahashi’s Trick in CbN

Proof: [Hyp: \( C\langle u \rangle \) is \( h \)-normalizing]
Let \( \text{fv}(u) \cup \text{fv}(s) = \{x_1, \ldots, x_k\} \), and \( y \) a fresh variable.

- \( \bar{u} := \lambda x_1 \ldots \lambda x_k . u \) is a closed term.
- Consider \( t := C\langle yx_1 \ldots x_k \rangle \), and note that:
  \[
  t\{y \leftarrow \bar{u}\} = C\langle \bar{u}x_1 \ldots x_k \rangle = C\langle (\lambda x_1 \ldots \lambda x_k . u)x_1 \ldots x_k \rangle \to^k \beta C\langle u \rangle.
  \]

- \( u \) is \( h \)-diverging implies that \( \bar{u} \) is also \( h \)-diverging.
- (Head Normalization Theorem) \( C\langle u \rangle \) is \( h \)-normalizing then so is \( t\{y \leftarrow \bar{u}\} \)

By light genericity as substitution, \( t\{y \leftarrow s'\} \) is \( h \)-normalizing for every \( s' \).
In particular, take \( s' := \bar{s} = \lambda x_1 \ldots \lambda x_k . s \):

\[
\begin{array}{c}
C\langle s \rangle \\
\xrightarrow{\beta^*} t\{y \leftarrow \bar{s}\} \\
\xrightarrow{\beta^*} \beta^* \\
\xrightarrow{\beta^*} n' \\
\xrightarrow{\beta^*} n \\
\xrightarrow{h^*} h^* \\
\xrightarrow{h^*} n \\
\xrightarrow{\beta^*} \beta^* \\
\end{array}
\]

Confluence of \( \beta \)

Head normal forms are stable by \( \beta \)
Takahashi’s Trick in CbN

Proof: [Hyp: $C\langle u \rangle$ is $h$-normalizing]
Let $fv(u) \cup fv(s) = \{x_1, \ldots, x_k\}$, and $y$ a fresh variable.

- $\bar{u} := \lambda x_1 \ldots \lambda x_k . u$ is a closed term.
- Consider $t := C\langle yx_1 \ldots x_k \rangle$, and note that:
  \[ t\{y \leftarrow \bar{u}\} = C\langle \bar{ux}_1 \ldots x_k \rangle = C\langle (\lambda x_1 \ldots \lambda x_k . u)x_1 \ldots x_k \rangle \rightarrow^k_{\beta} C\langle u \rangle. \]

- $u$ is $h$-diverging implies that $\bar{u}$ is also $h$-diverging.
- (Head Normalization Theorem) $C\langle u \rangle$ is $h$-normalizing then so is $t\{y \leftarrow \bar{u}\}$

By light genericity as substitution, $t\{y \leftarrow s'\}$ is $h$-normalizing for every $s'$.

In particular, take $s' := \bar{s} = \lambda x_1 \ldots \lambda x_k . s$:

\[
\begin{array}{c}
C\langle s \rangle \\
\downarrow h^* \\
n'
\end{array}
\quad
\begin{array}{c}
t\{y \leftarrow \bar{s}\} \\
\downarrow h^* \\
n
\end{array}
\quad
\begin{array}{c}
\downarrow h^* \\
\downarrow \beta^* \\
n'
\end{array}
\]

Confluence of $\beta$

Head normal forms are stable by $\beta$

Head Normalization Theorem
Light Genericity in CbN

We use the head open contextual preorder \( \preceq^h_{CO} \) to prove both.

- It is consistent to collapse unsolvable terms:
  (by light genericity) \( \preceq^h_{CO} \) equates unsolvable terms and \( \preceq^h_{CO} \) is consistent (I \( \preceq^h_{CO} \Omega \))

- \( \preceq^h_{CO} \) is maximal:
  (by light genericity) any larger theory is inconsistent

- \( \preceq^h_{CO} \) coincides with \( \preceq^h_C \) (by maximality)
Light Genericity in CbN

We use the **head open contextual preorder** $\prec^h_{\mathcal{C}O}$ to prove both.

- It is consistent to collapse unsolvable terms:
  (by light genericity) $\prec^h_{\mathcal{C}O}$ equates unsolvable terms and $\prec^h_{\mathcal{C}O}$ is consistent (I $\prec^h_{\mathcal{C}O} \Omega$)

- $\prec^h_{\mathcal{C}O}$ is maximal:
  (by light genericity) any larger theory is inconsistent

- $\prec^h_{\mathcal{C}O}$ coincides with $\prec^h_{\mathcal{C}}$ (by maximality)
We use the head open contextual preorder $\bowtieh_{CO}$ to prove both.

- **It is consistent** to collapse unsolvable terms:
  (by light genericity) $\bowtieh_{CO}$ equates unsolvable terms and $\bowtieh_{CO}$ is consistent ($I \bowtieh_{CO} \Omega$)

- $\bowtieh_{CO}$ is maximal:
  (by light genericity) any larger theory is inconsistent

- $\bowtieh_{CO}$ coincides with $\bowtieh_{C}$ (by maximality)
We use the head open contextual preorder $\simCO^h$ to prove both.

- It is consistent to collapse unsolvable terms:
  (by light genericity) $\simCO^h$ equates unsolvable terms and $\simCO^h$ is consistent (I $\not\simCO^h \Omega$)

- $\simCO^h$ is maximal:
  (by light genericity) any larger theory is inconsistent

- $\simCO^h$ coincides with $\simC^h$ (by maximality)
Light Genericity in CbN

We use the head open contextual preorder $\prec_{\mathcal{CO}}^h$ to prove both.

- It is consistent to collapse unsolvable terms:
  (by light genericity) $\prec_{\mathcal{CO}}^h$ equates unsolvable terms and $\prec_{\mathcal{CO}}^h$ is consistent (I $\not\prec_{\mathcal{CO}}^h \Omega$)

- $\prec_{\mathcal{CO}}^h$ is maximal:
  (by light genericity) any larger theory is inconsistent

- $\prec_{\mathcal{CO}}^h$ coincides with $\prec_{\mathcal{C}}^h$ (by maximality)
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Call-by-Value Light Genericity

Call-by-Value problems

Spoiler: it won’t work

At least not using Plotkin’s calculus

\[ \Omega_{nf} = ((\lambda x. \delta)(yz))\delta \] is meaningless!

Open and closed CbV contextual equivalences do not coincide:

\[ \Omega_{nf} \not\simeq_{CO}^{pv} \Omega \]
Call-by-Value Light Genericity

Call-by-Value problems

Spoiler: it won’t work

At least not using Plotkin’s calculus

\[ \Omega_{nf} = ((\lambda x. \delta)(yz))\delta \text{ is meaningless!} \]

Open and closed CbV contextual equivalences do not coincide:

\[ \Omega_{nf} \approx_{C}^{p_v} \Omega \quad \text{but} \quad \Omega_{nf} \not\approx_{C\Omega}^{p_v} \Omega \]
Call-by-Value Light Genericity
Call-by-Value problems

Spoiler: it won’t work

At least not using Plotkin’s calculus

\[ \Omega_{nf} = ((\lambda x.\delta)(yz))\delta \] is meaningless!

Open and closed CbV contextual equivalences do not coincide:

\[ \Omega_{nf} \vdash_{C^p} \Omega \quad \text{but} \quad \Omega_{nf} \not\vdash_{C^p} \Omega \]
Call-by-Value Light Genericity

Call-by-Value problems

Spoiler: it won’t work

At least not using Plotkin’s calculus

$$\Omega_{nf} = (((\lambda x. \delta)(yz)) \delta$$ is meaningless!

Open and closed CbV contextual equivalences do not coincide:

$$\Omega_{nf} \simeq^p_C \Omega \quad \text{but} \quad \Omega_{nf} \not\simeq^p_C CO \Omega$$
Change Call-by-Value

The good call-by-value contextual equivalence is Plotkin’s closed.

\[ \sim^V_C := \sim^{P^V}_C = \sim^{VSC}_C = \sim^{VSC}_C \]

We use a nicer calculus (the Value Substitution Calculus [AP12]) that knows how to deal with open terms, but retains the same closed contextual equivalence.

**Undefined** terms are exactly \( vsc \)-diverging terms.\(^8\)

There, we can show:

- **Light Genericity**: \( vsc \)-diverging terms are minimum for \( \sim^{VSC}_{CO} \)
- **Consistency**: \( \sim^{VSC}_{CO} \) equates diverging terms and is consistent
- **Maximality**: \( \sim^{VSC}_{CO} \) is a maximal inequational theory
- **Closed and Open coincide**: \( \sim^{VSC}_C = \sim^{VSC}_{CO} \)

\(^8\)weak \( vsc \)-diverging
Change Call-by-Value

The good call-by-value contextual equivalence is Plotkin’s closed.

\[
\simeq^v_C := \simeq^{pv}_C = \simeq^{vsc}_C \equiv \simeq^{vsc}_{CO}
\]

We use a nicer calculus (the Value Substitution Calculus [AP12]) that knows how to deal with open terms, but retains the same closed contextual equivalence.

**Undefined** terms are exactly \(vsc\)-diverging terms.\(^8\)

There, we can show:

- **Light Genericity:** \(vsc\)-diverging terms are minimum for \(\simeq^{vsc}_{CO}\)
- **Consistency:** \(\simeq^{vsc}_{CO}\) equates diverging terms and is consistent
- **Maximality:** \(\simeq^{vsc}_{CO}\) is a maximal inequational theory
- **Closed and Open coincide:** \(\simeq^{vsc}_C = \simeq^{vsc}_{CO}\)

\(^8\text{weak }vsc\text{-diverging}
Change Call-by-Value

The good call-by-value contextual equivalence is Plotkin’s closed.

\[
\simeq^v_c := \simeq^p_c = \simeq^{vsc}_c = \simeq^{vsc}_c
\]

We use a nicer calculus (the Value Substitution Calculus [AP12]) that knows how to deal with open terms, but retains the same closed contextual equivalence.

**Undefined** terms are exactly \(vsc\)-diverging terms.\(^8\)

There, we can show:

- **Light Genericity:** \(vsc\)-diverging terms are minimum for \(\simeq^{vsc}_{ CO} \)
- **Consistency:** \(\simeq^{vsc}_{ CO} \) equates diverging terms and is consistent
- **Maximality:** \(\simeq^{vsc}_{ CO} \) is a maximal inequational theory
- **Closed and Open coincide:** \(\simeq^{vsc}_c = \simeq^{vsc}_{ CO} \)

\(^8\)weak \(vsc\)-diverging
Proofs of Call-by-Value Light Genericity

How to prove light genericity?

- Direct proof: Takahashi’s trick adapts, but not very smoothly.
- Using a good model of CbV: relational semantics [Ehr12]
- Applicative similarity or any program preorder that is included in $\simeq^s_{CO}$ and has diverging terms as minimums.
Proofs of Call-by-Value Light Genericity

How to prove light genericity?

- Direct proof: Takahashi’s trick adapts, but not very smoothly.

- Using a good model of CbV: relational semantics [Ehr12]

- Applicative similarity or any program preorder that is included in $\sim^s_{CO}$ and has diverging terms as minimums.
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name Light Genericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Characterization of minimum terms

For \( s \in \{ \text{head, vsc} \} \):

**Light genericity** says:

\[
 \text{\( t \) is \( s \)-diverging} \implies \text{\( t \) is a minimum for } \prec^{s}_{\text{CO}}
\]

**Adequacy** (\( t \mathcal{R} u \) and \( t \) is \( s \)-normalizing then \( u \) is \( s \)-normalizing) implies the converse implication.

\[
 \text{\( t \) is \( s \)-diverging} \iff \text{\( t \) is a minimum for } \prec^{s}_{\text{CO}}
\]
Characterization of minimum terms

For $s \in \{head, vsc\}$:

Light genericity says:

\[ t \text{ is } s\text{-diverging} \implies t \text{ is a minimum for } \prec^s \]

Adequacy ($t \not\rightarrow u$ and $t$ is $s$-normalizing then $u$ is $s$-normalizing) implies the converse implication.

\[ t \text{ is } s\text{-diverging} \iff t \text{ is a minimum for } \prec^s \]
Well, what about maximums?

\[ t \text{ is } ?? \iff t \text{ is a maximum for } \preceq^s \]

- Call-by-Name: no maximum elements
- Call-by-Value: super terms!

**Co-genericity** states that super terms are maximum for \( \preceq^s \).
Well, what about maximums?

\[ t \text{ is } ?? \iff t \text{ is a maximum for } \sim^s_{\text{CO}} \]

- **Call-by-Name**: no maximum elements
- **Call-by-Value**: *super terms*!

**Co-genericity** states that super terms are maximum for \( \sim^s_{\text{CO}} \)
Well, what about maximums?

\[ t \text{ is ??? } \iff t \text{ is a maximum for } \simeq_{CO} \]

- **Call-by-Name**: no maximum elements
- **Call-by-Value**: *super terms!*

**Co-genericity** states that super terms are maximum for \( \simeq_{CO} \)
A term $t$ is \textit{s-super} if, coinductively, $t \rightarrow_s^* \lambda x. u$ and $u$ is s-super.

Intuitively, $t$ infinitely normalizes to $\lambda x_1. \lambda x_2. \ldots \lambda x_k. \ldots$

$\Omega_\lambda := (\lambda x. \lambda y. xx)(\lambda x. \lambda y. xx)$ is a call-by-value super term.
Super terms

A term $t$ is \textit{s-super} if, coinductively, $t \rightarrow^* \lambda x. u$ and $u$ is s-super.

Intuitively, $t$ infinitely normalizes to $\lambda x_1. \lambda x_2. \ldots \lambda x_k. \ldots$

$\Omega_{\lambda} := (\lambda x. \lambda y. xx)(\lambda x. \lambda y. xx)$ is a call-by-value super term
A term $t$ is \textit{s-super} if, coinductively, $t \rightarrow^{*}_s \lambda x. u$ and $u$ is s-super.

Intuitively, $t$ infinitely normalizes to $\lambda x_1. \lambda x_2. \ldots \lambda x_k. \ldots$

$\Omega_\lambda := (\lambda x. \lambda y.xx)(\lambda x. \lambda y.xx)$ is a call-by-value super term
Co-genericity

In call-by-value:

\[ \vdash \]

Consistency

Again, we use the open call-by-value contextual preorder \( \simCO_v \) to prove it.

- It is consistent to equate super terms, as \( \simCO_v \) does it and is consistent.
- It is consistent to equate diverging terms and to equate super terms, as \( \simCO_v \) does it and is consistent.
Proofs of co-genericity

How to prove co-genericity?

▶ Direct proof: Takahashi’s trick adapts, and the proof is easier than for light genericity.

▶ Using a good model of CbV? relational semantics [Ehr12] do not work, as s-super terms are not maximum elements!

▶ Applicative similarity or any program preorder that is included in \( \lesssim_{sCO} \) and has super terms as maximums\(^9\).

---

\(^9\)I don’t think I know of any other one
Proofs of co-genericity

How to prove co-genericity?

- Direct proof: *Takahashi’s trick* adapts, and the proof is easier than for light genericity.

- *Using a good model of CbV*? relational semantics [Ehr12] do not work, as s-super terms are not maximum elements!

- *Applicative similarity* or any program preorder that is included in $\preceq^s CO$ and has super terms as maximums$^9$.

---

$^9$I don’t think I know of any other one
Outline

The Lambda-Calculus & Computable Functions

From Barendregt’s Genericity to Light Genericity

Light Genericity & Contextual Preorders

Call-by-Name LightGenericity

Call-by-Value Light Genericity

Co-Genericity

Conclusion
Conclusion

- **Light genericity** is a modular concept that is strong enough to imply the two main consequences of Barendregt’s genericity.

- It is naturally dualizable as **co-genericity**. Both concepts are inspired and tied with contextual preorders.

- An **application of light genericity and maximality** is an elegant proof of the fact that closed and open contextual equivalences coincide.

**A question remains:** we named the two genericity statements **Heavy** and **Light**, but we don’t know whether one implies the other or not.
Thank you!

To appear in FoSSaCS24 & technical report: https://hal.science/hal-04406343

Contextual preorder for lambda terms

⊥ := equivalence class of Ω
Beniamino Accattoli and Giulio Guerrieri.
The theory of call-by-value solvability (long version).

Beniamino Accattoli and Luca Paolini.
Call-by-value solvability, revisited.

Thomas Ehrhard.
Collapsing non-idempotent intersection types.
In Patrick Cégielski and Arnaud Durand, editors, Computer Science Logic (CSL’12) - 26th International Workshop/21st Annual Conference of the EACSL, CSL 2012, September 3-6, 2012, Fontainebleau, France, volume 16 of LIPIcs, pages
Masako Takahashi.