Classical notions of computation and the Hasegawa-Thielecke theorem

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Introduction

Classical categorical proof theory : denotational and proof-relevant model for classical logic

Joyal's obstruction theorem

Any cartesian category with an involutive negation is a preorder.

In the early 90s, Griffin and Murthy discovered the link between classical logic and continuations.

Girard's polarised classical logic, letting go of the assumption of associativity.

Blass problem in game semantics

Introduction

*-autonomous categories are the categorical model for multiplicative linear logic, what would be the equivalent for polarised linear classical logic ?

Dialogue duploids

Due to the symmetrical nature of polarisation, we start from adjunction models of effects, that subsumes monadic and comonadic effects.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

Construct a deductive system and add more structures : a tensor product and an involutive negation.

The Hasegawa-Thielecke theorem : relating two notion of purity of morphisms in dialogue duploids, thunkability and centrality.

Collage of an adjunction

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathscr{A} & A \arrow[rr] & & B \arrow[d] & \mathscr{B} \\
& A' \arrow[u] & & B' &
\end{tikzcd}
$$

$$
\begin{array}{lcl}
\mathscr{E}(A', A) & := & \mathscr{A}(A', A) \\
\mathscr{E}(B, B') & := & \mathscr{B}(B, B') \\
\mathscr{E}(A, B) & := & \mathscr{A}(A, RB) \simeq \mathscr{B}(LA, B) \\
\mathscr{E}(B, A) & := & \emptyset
\end{array}
$$

Duploid construction from an adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A}_{RL} & A \arrow[rrr] & & & B \arrow[d] & \mathscr{B}_{LR} \\
& A' \arrow[u] & & & B' \arrow[lll] &
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{dupl}_{L,R}(A', A) &:= \mathscr{A}(A', RLA) \\
\mathrm{dupl}_{L,R}(B, B') &:= \mathscr{A}(RB, RB') \simeq \mathscr{B}(LRB, B') \\
\mathrm{dupl}_{L,R}(A, B) &:= \mathscr{A}(A, RB) \\
\mathrm{dupl}_{L,R}(B, A) &:= \mathscr{A}(RB, RLA)
\end{aligned}$$

Composition in the duploid construction

Positive (Kleisli) composition $\bullet$ use the composition in $\mathscr{B}$ and negative (co-Kleisli) composition $\circ$ the one in $\mathscr{A}$. We use $\odot$ when the polarity is unspecified.

$$\bullet : \mathrm{dupl}_{L,R}(A, Y) \times \mathrm{dupl}_{L,R}(X, A) \to \mathrm{dupl}_{L,R}(X, Y)$$

$$\circ : \mathrm{dupl}_{L,R}(B, Y) \times \mathrm{dupl}_{L,R}(X, B) \to \mathrm{dupl}_{L,R}(X, Y)$$

We have:

$$(f \bullet g) \odot h = f \bullet (g \odot h)$$
$$(f \odot g) \circ h = f \odot (g \circ h)$$

but, in general:

$$(f \circ g) \bullet h \quad \text{and} \quad f \circ (g \bullet h)$$

are not equal.

Magmoids

Definition

A **unital magmoid** (or **non-associative category**) $\mathcal{M}$ is defined just as a category without the usual associativity assumption that

$$h \odot (g \odot f) = (h \odot g) \odot f$$

for every path of length 3:

$$\begin{tikzcd} & B \arrow[r, "g"] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] & & & D \end{tikzcd}$$

For all unital magmoid $\mathcal{M}$, $\mathcal{M}^{\mathrm{op}}$ is defined to be the unital magmoid with the same objects as $\mathcal{M}$ but whose maps are reversed.

Linear and thunkable morphisms

Definition

- f **thunkable** $\Leftrightarrow \forall g, h,\ (h \odot g) \circ f = h \odot (g \circ f)$
- h **linear** $\Leftrightarrow \forall f, g,\ (h \odot g) \circ f = h \odot (g \odot f)$

$$\begin{tikzcd} & B \arrow[r, "g"] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] & & & D \end{tikzcd}$$

Polarity of objects

- B **negative** $\Leftrightarrow \forall A, \forall f \in \mathcal{M}(A, B)$, f is thunkable
- C **positive** $\Leftrightarrow \forall D, \forall h \in \mathcal{M}(C, D)$, h is linear

In $\mathcal{M}^{\mathrm{op}}$, thunkable morphisms become linear and reciprocally, and polarities are reversed.

Subcategories

Notations

Given a non-associative category $\mathcal{M}$, we introduce the following notations for subcategories of $\mathcal{M}$:

- $\mathcal{M}_l$ is the subcategory of linear maps,
- $\mathcal{M}_t$ is the subcategory of thunkable maps,
- $\mathscr{P}$ is the full subcategory of positive objects,
- $\mathscr{N}$ is the full subcategory of negative objects,
- $\mathscr{P}_t$ is the subcategory of thunkable maps of $\mathcal{P}$,
- $\mathscr{N}_l$ is the subcategory of linear maps of $\mathcal{N}$.

$$\begin{tikzcd}
\mathscr{N}_l \arrow[rrr, hook] \arrow[dr, hook] & & & \mathcal{M}_l \arrow[dr, hook] & \\
& \mathscr{N} \arrow[drr, hook] & \mathscr{P} \arrow[ur, hook] & & \mathcal{M} \\
\mathscr{P}_t \arrow[rrr, hook] \arrow[urr, hook] & & & \mathcal{M}_t \arrow[ur, hook] &
\end{tikzcd}$$

Duploids

Shifts

A **positive shift** on a unital magmoid $\mathcal{M}$ consists of the data for every object A of a positive object $\Downarrow A$ equipped with a pair of thunkable maps

$$\omega_A : A \to \Downarrow A \qquad \overline{\omega}_A : \Downarrow A \to A$$

such that $\overline{\omega}_A \bullet \omega_A = \mathrm{id}_A$ and $\omega_A \odot \overline{\omega}_A = \mathrm{id}_{\Downarrow A}$.
Dually, a **negative shift** $\Uparrow$ on $\mathcal{M}$ is a positive shift on $\mathcal{M}^{\mathsf{op}}$.

Definition

A **duploid** is a non-associative category equipped with a positive and a negative shift, and where every object is either positive or negative (or both).

For $\mathcal{D}$ a duploid, $\mathcal{D}^{\mathsf{op}}$ is also a duploid.

Duploids and adjunctions

Theorem

Every non-associative category $\mathrm{dupl}_{L,R}$ associated to an adjunction $L \dashv R$ comes equipped with a duploid structure, where $\mathscr{P}$ is equivalent to the Kleisli category on the monad $T = R \circ L$, and $\mathscr{N}$ is equivalent to the co-Kleisli category on the comonad $K = L \circ R$.

Conversely, every duploid $\mathcal{D}$ induces an adjunction

$$\begin{tikzcd} \mathscr{P}_t \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathscr{N}_l \arrow[l, bend left] \end{tikzcd}$$

defined by restriction of the shifts, whose associated duploid is equivalent to $\mathcal{D}$.

Symmetric monoidal structures on adjunctions

We want to describe the duploid $\mathrm{dupl}_{L,R}$ associated to an adjunction of the form:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

where the category $\mathscr{A}$ is equipped with a symmetric monoidal structure $(\mathscr{A}, \owedge, \mathit{true})$ and where the monad $T = R \circ L$ is strong.
The symmetric monoidal structure on $\mathscr{A}$ induces a symmetric premonoidal structure on the Kleisli category on the monad T.

Central morphisms

A morphism f of a premonoidal category is said to be central for $\otimes$ if, for all g, the two following squares commute:

$$\begin{tikzcd} A \otimes B \arrow[r, "A \rtimes g"] \arrow[d, "f \ltimes B"] & A \otimes B' \arrow[d, "f \ltimes B'"] \\ A' \otimes B \arrow[r, "A' \rtimes g"] & A' \otimes B' \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \arrow[r, "B \rtimes f"] \arrow[d, "g \ltimes A"] & B \otimes A' \arrow[d, "g \ltimes A'"] \\ B' \otimes A \arrow[r, "B' \rtimes f"] & B' \otimes A' \end{tikzcd}$$

Symmetric monoidal structures on duploids

Symmetric monoidal Freyd category

A **symmetric monoidal Freyd category** consists of:

- a symmetric monoidal category $(\mathscr{M}, \otimes, 1)$,
- a symmetric premonoidal category $\mathscr{P}$,
- an identity-on-object functor $\iota : \mathscr{M} \to \mathscr{P}$ transporting the monoidal structure of $\mathscr{M}$ to the premonoidal structure of $\mathscr{P}$ and such that every morphism $\iota(f)$ is central in $\mathscr{P}$.

(Positive) symmetric monoidal duploids (M, Melliès, Munch-Maccagnoni 2025)

A (positive) **symmetric monoidal duploid** $(\mathcal{D}, \otimes, 1)$ is a duploid whose inclusion functor $\mathscr{P}_t \hookrightarrow \mathscr{P}$ is equipped with the structure of a symmetric monoidal Freyd category $(\mathscr{P}_t, \otimes, 1) \to (\mathscr{P}, \otimes, 1)$.

Symmetric monoidal structures on duploids

With the positive shift, we can generalize the premonoidal structure on $\mathscr{P}$ to $\mathcal{D}$:

$$
\begin{array}{lrcl}
\forall X, Y & X \otimes Y & := & \Downarrow X \otimes \Downarrow Y \\
\forall f : X \to X', Y & f \ltimes Y & := & \Downarrow f \ltimes \Downarrow Y \\
\forall X, g : Y \to Y' & X \rtimes g & := & \Downarrow X \rtimes \Downarrow g
\end{array}
$$

However, this construction does not preserve the composition, meaning that the following diagram does not commute in general.

$$
\begin{tikzcd}
 & X' \otimes Y \arrow[dr, "f' \ltimes Y"] & \\
X \otimes Y \arrow[ur, "f \ltimes Y"] \arrow[rr, "(f' \circ f) \ltimes Y"] & & X'' \otimes Y
\end{tikzcd}
$$

Lemma

For this generalized structure, every thunkable maps is central. The converse property is not true in general.

Symmetric monoidal duploids and adjunctions

Theorem (M, Melliès, Munch-Maccagnoni 2025)

Every non-associative category $\mathsf{dupl}_{L,R}$ associated to an adjunction $L \dashv R : \mathscr{A} \to \mathscr{B}$ where $\mathscr{A}$ is symmetric monoidal and the monad $T = R \circ L$ is strong comes equipped with a symmetric monoidal duploid structure.

Conversely, every symmetric monoidal duploid $\mathcal{D}$ induces an adjunction

$$\begin{tikzcd} \mathscr{P}_t \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathscr{N}_l \arrow[l, bend left] \end{tikzcd}$$

where $\mathscr{P}_t$ is equipped with a symmetric monoidal structure $(\mathscr{P}_t, \otimes, 1)$ and the associated monad on $\mathscr{P}_t$ is strong.

Negative symmetric monoidal duploids

The De Morgan duality of classical logic implies to consider the dual of positive symmetric monoidal duploid.

Definition (M, Melliès, Munch-Maccagnoni 2025)

A **negative symmetric monoidal duploid** $(\mathcal{D}, \parr, \bot)$ is a duploid $\mathcal{D}$ such that $\mathcal{D}^{\mathsf{op}}$ comes equipped with a positive symmetric monoidal structure.

Dialogue categories

Definition (Melliès 2009)

A **dialogue category** is a symmetric monoidal category $\mathscr{C}$ equipped with an object $\perp$, called return object, such that, for all object A, the presheaf:

$$B \mapsto \mathscr{C}(A \otimes B, \perp)$$

has a representation:

$$\varphi_{A,B} : \mathscr{C}(A \otimes B, \perp) \simeq \mathscr{C}(B, \neg A)$$

$\neg$ is a functor from $\mathscr{C}$ to $\mathscr{C}^{\mathrm{op}}$ and defines an adjunction with itself:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "\neg"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{\mathrm{op}} \arrow[l, bend left, "\neg"] \end{tikzcd}$$

Dialogue chiralities

Dialogue categories can be reformulated (up to equivalence) symmetrically as dialogue chiralities:

Definition (Melliès 2016)

A **dialogue chirality** is a pair of symmetric monoidal categories $(\mathscr{A}, \varowedge, \mathit{true})$ and $(\mathscr{B}, \varovee, \mathit{false})$ equipped with an adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$ together with a symmetric monoidal equivalence:

$$\begin{tikzcd} (\mathscr{A}, \varowedge, \mathit{true}) \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\simeq"] & (\mathscr{B}, \varovee, \mathit{false})^{\mathrm{op}} \arrow[l, bend left, "(-)^*"] \end{tikzcd}$$

and a family of bijections

$$\chi_{A_1, A_2, B} : \mathscr{A}(A_1 \varowedge A_2, RB) \longrightarrow \mathscr{A}(A_1, R(A_2^* \varovee B))$$

natural in A_1, A_2 and B and satisfying a coherence diagram.

Dialogue duploids

Definition (M, Melliès, Munch-Maccagnoni 2025)

A **dialogue duploid** (or *-autonomous duploid) is a duploid $\mathcal{D}$ equipped with a positive and negative symmetric monoidal duploid structure $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{D}, ⅋, \perp)$ related by an appropriate notion of monoidal equivalence:

$$\begin{tikzcd} (\mathcal{D}, \otimes, 1) \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\simeq"] & (\mathcal{D}, ⅋, \perp)^{op} \arrow[l, bend left, "(-)^*"] \end{tikzcd}$$

together with a family of bijections

$$\chi_{X,Y,Z} \quad : \quad \mathcal{D}(X \otimes Y, Z) \quad \simeq \quad \mathcal{D}(X, Y^* ⅋ Z)$$

natural component-wise in X, Y and Z and satisfying a coherence diagram.

Since we don't always have associativity of composition, $\mathcal{D}(-, =)$ is only a functor of graph component by component on $\mathcal{D}$.

Dialogue duploids and adjunctions

Theorem (M, Melliès, Munch-Maccagnoni 2025)

Every duploid $\mathrm{dupl}_{L,R}$ associated to a dialogue chirality $L \dashv R : \mathscr{A} \to \mathscr{B}$ comes equipped with a dialogue duploid structure.

Conversely, every dialogue duploid $(\mathcal{D}, \otimes, \parr)$ induces a dialogue chirality structure on the adjunction

$$\begin{tikzcd} \mathscr{P}_t \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathscr{N}_l \arrow[l, bend left] \end{tikzcd}$$

whose associated dialogue duploid is equivalent to $\mathcal{D}$.

Hasegawa-Thielecke theorem

Theorem (M, Melliès, Munch-Maccagnoni 2025)

In dialogue duploids, central morphisms for $\otimes$ are exactly the thunkable morphisms.

Idea of the proof

The proof relies on two elements :

- Thunkability and centrality are properties of commutation, of what is evaluated first.
- An involutive negation connects them.

More concretely, the theorem is proved by purely equational reasoning, by using the following observation:

$$\forall f : A \to B, g : B \to C, \quad g \odot f = \chi_{A,C^*,\perp}(\chi^{-1}_{A,B^*,\perp}(f) \bullet (A \rtimes g^*)) \quad : \quad A \to C$$

(up to structural morphisms).

Hasegawa-Thielecke theorem

Idempotent monads

A monad (T, μ, η) is **idempotent** if $\mu : TT \to T$ is an isomorphism.

For an adjunction $L \dashv R$, $\mathrm{dupl}_{L,R}$ is associative if and only if the monad $T = R \circ L$ is idempotent.

Commutative monads

A strong monad T on a monoidal category is **commutative** if the two canonical morphisms $TA \otimes TB \rightrightarrows T(A \otimes B)$ coincide for all A and B.

For a strong adjunction $L \dashv R$, every morphism of $\mathrm{dupl}_{L,R}$ is central if and only if the monad $T = R \circ L$ is commutative.

Corollary of the Hasegawa-Thielecke theorem

The continuation monad in a dialogue category is idempotent if and only if it is commutative.

Conclusion

We see the framework based on *duploids with structure* as solid foundation for the study of non-associative and effectful logical systems and term calculi for classical logic, integrating the lessons of linear logic and continuation models.

We also have developped an internal language to dialogue duploids called the classical L-calculus, an abstract-machine-like calculus inspired by Curien and Herbelin's $\mu\tilde{\mu}$.

Further works :

- Adding more structure to duploids (internal homs, exponential, sums, etc)
- Integrating Miquey's work on delimited continuations to have dependent duploids

Thank you for your attention !