A directed homotopy type theory for 1-categories

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1 Introduction

1.1 Abstract

In Martin-Löf Type Theory, the identity types are defined as the family of types $\mathsf{Id}_A(x, y)$ for A a type and x and y, two elements of A, generated inductively by $\mathsf{refl}_a : \mathsf{Id}_A(a, a)$. From this definition, we can deduce the usual properties of the equality (reflexivity, symmetry and transitivity), and, in this context, these properties become groupoidal operation (identity, inverse and composition). In fact, Hofmann and Streicher [1] discovered that Martin-Löf Type Theory can be interpreted in the category of groupoids, such that identity types of a type A are interpreted by the morphisms between two objects of the groupoid interpretation of A. This discovery showed that Martin-Löf Type Theory can be used to prove and verify theorems in groupoid theory and homotopy theory and gave a new unifying perspective on these domains.

However, despite many attempts, no consensus has yet been reached on what should be used to reason synthetically on (higher) category theory and directed homotopy theory. One natural idea to solve this problem is to develop a Directed Type Theory, by replacing symmetric identity types with directed homomorphism types. Indeed, whereas groupoid theory and homotopy theory study symmetric paths, (higher) category theory and directed homotopy theory are linked by their study of directed paths.

In this report, we present one attempt at such a theory. Even though it has been designed for 1-category, we think of it as one more step towards a more general theory for higher category theory and directed homotopy theory.

1.2 Analytic and synthetic theories

Mathematical theories can in classified in two categories, either synthetic or analytic.

A synthetic theory uses the structures its interested in as building blocks and use axioms so the building blocks behave as intended. The most famous example is synthetic geometry, also called Euclid's geometry. Point and lines are supposed notion without a definition and there are axioms that ensures that they indeed corresponds to our idea of points and lines.

The opposite is an analytic theory – which is not to be confused with the subfield of mathematics that is analysis – a theory that breaks down structures in smaller pieces to make them fit in another theory, most of the times set theory. In analytic geometry, also know as Descartes' geometry, points are elements of $\mathbb{R}^n$ and lines are sets of points.

Synthetic and analytic theories are not meant to be in two entirely separate parts of mathematics, but to complement each other. For example, to prove the consistency of a logical theory, which is synthetic by nature, we construct an analytic example, i.e. a model, in set theory.

Almost all of mathematics is done analytically inside of ZFC. Building a synthetic theory of a given mathematical subject can be hard (and historically appear after the analytic theory) but it has many advantages. First, if we prove a theorem in a synthetic theory, it also proves the theorem in all of the models of the synthetic theory. Secondly, many lemmas relating to the structure can be internalized in the theory and thus will be automatically proved instead of having to prove it every time. For example, in synthetic computability, which use computable sets as building blocks, every map is computable.

1.3 Homotopy type theory (HoTT)

Martin-Löf Type Theory (which we will abbreviate to MLTT from now on) is an extension of simple type theory that adds dependent pair types, dependent function types and, arguably most interestingly, identity types, which are formed using the following rule :

$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma, x : A, y : A \vdash \mathsf{Id}_A(x, y) \ \mathsf{type}} \ \mathsf{Id}\text{-}\textsc{Form}$$

which should be read as "If A is a type in context Γ, then $\mathsf{Id}_A(x, y)$ is a type in the context $\Gamma, x : A, y : A$". The introduction rule states that, for every element x of A, there is an identity of x with itself called the reflection of x.

$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma,\ x : A \vdash \mathsf{refl}_x : \mathsf{Id}_A(x, x)} \ \mathsf{Id}\text{-}\textsc{Intro}$$

Finally, the elimination rule states that, when we want to use an identity to construct an element, we only have to consider the case where the identity is a reflection.

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, x : A, y : A, f : \mathsf{Id}_A(x, y) \vdash D(x, y, f) \ \mathsf{type} \qquad \Gamma, x : A \vdash d(x) : D(x, x, \mathsf{refl}_x)}{\Gamma, x : A, y : A, f : \mathsf{Id}_A(x, y) \vdash j_d(x, y, f) : D(x, y, f)} \ \mathsf{Id}\text{-}\textsc{Elim}$$

From the introduction rule, we have that every element has an identity with itself and from the elimination rule, we can deduce that identities can be composed and inversed. Thus, adding identity types gave types the structure of a groupoid, a category whose every morphisms are isomorphisms. Moreover, as identity types are also types, they have themselves the same structure, so it is actually an ∞-groupoid, a groupoid where the isomorphims between two elements form themselves an ∞-groupoid. Therefore, MLTT can be used as a synthetic theory of ∞-groupoids.

Moreover, MLTT has a lot of advantages, it has many models, including in ∞-groupoid, homotopy theory and higher toposes and thus proving a theorem in MLTT proves it in a lot of contexts. Also, many very useful notions when working on (∞-)groupoids are internalized in the theory. For example, every function is automatically functorial. Finally, there exists proof checkers of MLTT, such as Agda and Coq, that allow us to verify easily if a proof is correct or not.

All of this has motivated the rapid development of Homotopy Type Theory, which is MLTT with a few new axioms added to make it more expressive, during the last 15 years. Now, many mathematical works are being done inside of HoTT.

1.4 Motivations for a synthetic theory of higher category theory

With the still growing success of HoTT, one can only wonder what could be done with a powerful and simple enough to use synthetic theory of higher categories. Indeed, higher category theory has a lot of applications in other domains of mathematics or even in other sciences, for example, in logic and semantics. So a synthetic theory whose building blocks are higher categories and that comes with, ideally, core lemmas of category theory internalized (every function being a functor, every transformation being natural, etc) could be very useful and would allow mathematicians to have a new setting to do higher category theory.

Thus, many attempts have been and are still being done to design such a theory, but no consensus on which one should be used has emerged yet.

1.5 Our attempt at a synthetic category theory

Our attempt, like most other attempt, will be a type theory, as it will make it easier to implement a proof checker later on. More precisely, design-wise, we started from MLTT and made changes to add directedness, as, intuitively, the difference between groupoids and categories is the notion of direction.

For the moment, our attempt is focused on 1-categories but we hope that we will be able to generalize it to higher categories in the future.

1.6 Related works

Our work builds on the previous work of my advisor [2]. We keep the same goal of having a homomorphism type former with simple rules analogous to the identity type former of the Martin-Löf Type Theory, and we solve one of the main problems of the previous article by having identity types that don't collapse with the homomorphism types.

We also take inspirations from the work of Nuyts [3], especially how variances of assumptions and terms are marked in the judgments, and we improve on it by building an interpretation of our syntax.

Other works on directed type theory divert significantly from our project. Indeed, the works of Licata and Harper [4] and Ahrens, North and Van Der Weide [5] don't have a homomorphism type former and the work of Riehl and Shulman [6] builds on Cubical Type Theory rather than Martin-Löf Type Theory.

1.7 Organization

After the presentation of the syntax of the system (Section 2) and an interpretation of it with 1-categories (Section 3), we will look at its expressivity (Section 4). We will finish by considering an extension of the system (Section 5).

In the appendix, you will find meta-informations about the internship (Appendix A) and an unfinished attempt at a proof of the Yoneda lemma (Appendix B).

2 Syntax

In this section, we will present the syntax of our directed type theory. The idea of using orientations and general implementation of them comes from [3] even though many details were changed to work semantically. The rules for the homomorphism types comes from [2]. Everything else is original.

2.1 Judgments: orientations, contexts, terms and types

Our theory adds orientations to the standard Martin-Löf Type Theory (MLTT). They will mark the variance of assumptions and terms.

The first two orientations are $+$ and $-$, corresponding respectively to covariance and contravariance. For example, if we have a contravariant term t of type B depending covariantly on a variable x of type A (which, as we will see later, can be expressed by the judgment $x \overset{+}{:} A \vdash t(x) \overset{-}{:} B$) and a morphism $\varphi : a \to a'$ in A, then we obtain a morphism from $t(a')$ to $t(a)$ in B.

However, many mathematical notions that we want to express in our system are neither covariant nor contravariant. For example, the identity type of x and y can't depend covariantly nor contravariantly on x, nor on y. If it did, it would allow us to transport identity along morphisms and thus the two ends of every morphisms would be identified. We would want identities to only being transported along isomorphisms, so we add a new orientation for this case: $\circ$ which will correspond to isovariance.

With t a covariant term depending isovariantly on a variable x, a morphism $\phi : a \to a'$ will give us no information between $t(a)$ and $t(a')$. But if ϕ is an isomorphism, then we will have an isomorphism between $t(a)$ and $t(a')$. Conversely, if t is an isovariant term depending on x covariantly, any morphism from a to a' will induce an isomorphism between $t(a)$ and $t(a')$.

Our first type of judgment will be of the form ω ort, meaning that ω is an orientation.

$$\frac{}{\circ\ \mathsf{ort}} \qquad \frac{}{+\ \mathsf{ort}} \qquad \frac{}{-\ \mathsf{ort}}$$

Our second judgment will define a refinement order on the orientations. An isovariant term is also both co- and contravariant, so $\circ$ is the smallest orientation.

$$\frac{\omega\ \mathsf{ort}}{\omega \leq \omega\ \mathsf{ort}} \qquad \frac{\omega\ \mathsf{ort}}{\circ \leq \omega\ \mathsf{ort}}$$

$$\begin{tikzcd} & \circ \arrow[dl, dash] \arrow[dr, dash] & \\ + & & - \end{tikzcd}$$

We will now introduce contexts, terms and types. A context is a list of distinct variables, each associated with a type and an orientation. Contexts will be noted $x_1 \overset{\omega_1}{:} A_1, \ldots, x_n \overset{\omega_n}{:} A_n)$ in their expanded form and Γ^ℓ in their contracted form with $\ell := (\omega_1, \ldots, \omega_n)$. To simplify our notations, we will omit the orientation $+$ in contexts and write $x : A$ instead of $x \overset{+}{:} A$.

A term t of type A with orientation ω in a context Γ^ℓ will be expressed by the judgment

$$\Gamma^\ell \vdash t \overset{\omega}{:} A.$$

Types are themselves terms of universe types, that we will note $\mathcal{U}_k$ for $k \in \mathbb{N}$.

We also have a judgment for equalities between two terms of the same type and the same orientation. Equality of a and b, two terms of A with orientation ω in the context Γ^ℓ will be noted:

$$\Gamma^\ell \vdash a \equiv b \overset{\omega}{:} A.$$

As we will have another notion of equality, this one will be called judgmental equality. The judgmental equality must respect the usual axiom of the equality (reflexivity, symmetry and transitivity) and judgmentally equal types must have the same elements and the same judgmental equalities.

$$\frac{\Gamma^\ell \vdash a \overset{\omega}{:} A}{\Gamma^\ell \vdash a \equiv a \overset{\omega}{:} A} \qquad \frac{\Gamma^\ell \vdash a \equiv b \overset{\omega}{:} A}{\Gamma^\ell \vdash b \equiv a \overset{\omega}{:} A} \qquad \frac{\Gamma^\ell \vdash a \equiv b \overset{\omega}{:} A \qquad \Gamma^\ell \vdash b \equiv c \overset{\omega}{:} A}{\Gamma^\ell \vdash a \equiv c \overset{\omega}{:} A}$$

$$\frac{\Gamma^\ell \vdash A \equiv B : \mathcal{U}_k \qquad \Gamma^\ell \vdash a \overset{\omega}{:} A}{\Gamma^\ell \vdash a \overset{\omega}{:} B} \qquad \frac{\Gamma^\ell \vdash A \equiv B : \mathcal{U}_k \qquad \Gamma^\ell \vdash a \equiv b \overset{\omega}{:} A}{\Gamma^\ell \vdash a \equiv b \overset{\omega}{:} B}$$

A context is valid if the type of each of its variable can be constructed in the context formed by the previous variables and we will use the judgment Γ^ℓ ctx to express that Γ^ℓ is a valid context.

$$\frac{}{\Diamond^\Diamond\ \mathsf{ctx}}\ \textsc{Ctx-Emp} \qquad \frac{\Gamma^\ell\ \mathsf{ctx} \qquad \Gamma^\ell \vdash A \overset{\alpha}{:} \mathcal{U}_k \qquad \omega\ \mathsf{ort}}{\Gamma^\ell, x \overset{\omega}{:} A\,\mathsf{ctx}}\ \textsc{Ctx-Ext}$$

From now on, we will only consider valid contexts.

Finally, we introduce a judgment for an order on lists of orientations generalizing the order on the orientations.

$$\frac{}{\Diamond \leq \Diamond\ \ \Diamond\text{-}\mathsf{ort}}\ \leq\text{-}\textsc{Emp} \qquad \frac{\ell \leq m\ \ \Gamma\text{-}\mathsf{ort} \qquad \omega \leq \alpha\ \mathsf{ort} \qquad \Gamma^m \vdash A : \mathcal{U}_k}{\ell, \omega \leq m, \alpha\ \ (\Gamma, x : \mathrm{A})\text{-}\mathsf{ort}}\ \leq\text{-}\textsc{Ext}$$

With these two orders, we can define rules for weakening the orientations on both sides.

$\circ$ is the orientation giving the most information, so the weakening rule on the left, that is on assumptions, will transform $+$ and $-$ into $\circ$ and conversely, the weakening rule on terms will transform $\circ$ into $+$ or $-$.

$$\frac{\ell \leq m \qquad \Gamma^m \vdash \mathcal{J}}{\Gamma^\ell \vdash \mathcal{J}}\ \textsc{Ort-weak-L}$$

$$\frac{\omega \leq \alpha\ \mathsf{ort} \qquad \Gamma^\ell \vdash t \overset{\omega}{:} A}{\Gamma^\ell \vdash t \overset{\alpha}{:} A}\ \textsc{Ort-weak-R} \qquad \frac{\omega \leq \alpha\ \mathsf{ort} \qquad \Gamma^\ell \vdash t \equiv t' \overset{\omega}{:} A}{\Gamma^\ell \vdash t \equiv t' \overset{\alpha}{:} A}\ \textsc{Ort-weak-R-}\equiv$$

We also add the transmutation rule that changes the orientations on both sides. To give it, we need to introduce a monoidal structure on the orientations :

$\cdot$	$\circ$	$-$	$+$
$\circ$	$\circ$	$\circ$	$\circ$
$-$	$\circ$	$+$	$-$
$+$	$\circ$	$-$	$+$

$+$ is the neutral element, $\circ$ is an absorbing element and $-$ is a root of unity.

$$\frac{\Gamma^\ell \vdash t \overset{\omega_1}{:} A}{\Gamma^{\ell\cdot\omega_2} \vdash t \overset{\omega_1\cdot\omega_2}{:} A}\ \textsc{Transmut} \qquad \frac{\Gamma^\ell \vdash t \equiv t' \overset{\omega_1}{:} A}{\Gamma^{\ell\cdot\omega_2} \vdash t \equiv t' \overset{\omega_1\cdot\omega_2}{:} A}\ \textsc{Transmut-}\equiv$$

where:

$$\Diamond \cdot \omega := \Diamond \qquad (\ell, \alpha)\cdot\omega := \ell\cdot\omega, \alpha\cdot\omega$$

This rule translates the fact that a functor from A to B can be seen as a functor from A^{op} to B^{op} when applied with $\omega_2 = -$ and that the image of an isomorphism by a functor is also an isomorphism. Due to that rule, every term in the empty context – in fact in any context of the form Γ° – can be proved to be isovariant.

2.2 Structural rules

Now we introduce the usual structural rules for variables, weakening and substitution.

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \omega\ \mathsf{ort}}{\Gamma^\ell, x \overset{\omega}{:} A \vdash x \overset{\omega}{:} A}\ \textsc{Var} \qquad \frac{\Gamma^\ell, \Delta^m \vdash \mathcal{J} \qquad \Gamma^\ell \vdash A : \mathcal{U}_k \qquad \omega\ \mathsf{ort}}{\Gamma^\ell, x \overset{\omega}{:} A, \Delta^m \vdash \mathcal{J}}\ \textsc{Weak}$$

$$\frac{\Gamma^\ell, x \overset{\omega}{:} A, \Delta^m \vdash t \overset{\alpha}{:} B \qquad \Gamma^\ell \vdash a \overset{\omega}{:} A}{\Gamma^\ell, \Delta[a/x]^m \vdash t[a/x] \overset{\alpha}{:} B[a/x]}\ \textsc{Subst} \qquad \frac{\Gamma^\ell, x \overset{\omega}{:} A, \Delta^m \vdash t \overset{\alpha}{:} B \qquad \Gamma^\ell \vdash a \equiv b \overset{\omega}{:} A}{\Gamma^\ell, \Delta[a/x]^m \vdash t[a/x] \equiv t[b/x] \overset{\alpha}{:} B[a/x]}\ \textsc{Subst-eq}$$

We insist on the fact that substitution requires the variable substituted and the term to have the same variance.

2.3 Homomorphism types

We can now introduce the type former hom for homomorphism types. For A a type, a and b two elements of A respectively contravariant and covariant, $\hom_A(a, b)$ will be the type of homomorphism between a and b.

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{-}{:} A, y : A \vdash \hom_A(x, y) : \mathcal{U}_k}\ \text{hom-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{\circ}{:} A \vdash 1_x : \hom_A(x, x)}\ \text{hom-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{-}{:} A, y : A, \Delta^m \vdash t \overset{\omega}{:} \hom_A(x, y)}{\Gamma^\ell, x \overset{-}{:} A, y : A, \Delta^m \vdash t \overset{\circ}{:} \hom_A(x, y)}\ \text{hom-}\textsc{set}$$

$$\frac{\begin{array}{c}\Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{\circ}{:} A, y : A, z : \hom_A(x, y), \Delta^m \vdash D(x, y, z) : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) \overset{\omega}{:} D(x, x, 1_x)\end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, y : A, z : \hom_A(x, y), \Delta^m \vdash j_d^L(x, y, z) \overset{\omega}{:} D(x, y, z)}\ \text{hom-}\textsc{left-elim}$$

$$\frac{\begin{array}{c}\Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : \hom_A(x, y), \Delta^m \vdash D(x, y, z) : \mathcal{U}_k \qquad \Gamma^\ell, y \overset{\circ}{:} A, \Delta^m \vdash d(y) \overset{\omega}{:} D(y, y, 1_y)\end{array}}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : \hom_A(x, y), \Delta^m \vdash j_d^R(x, y, z) \overset{\omega}{:} D(x, y, z)}\ \text{hom-}\textsc{right-elim}$$

$$\frac{\begin{array}{c}\Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{\circ}{:} A, y : A, z : \hom_A(x, y), \Delta^m \vdash D(x, y, z) : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) : D(x, x, 1_x)\end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) \equiv j_d^L(x, x, 1_x) : D(x, x, 1_x)}\ \text{hom-}\textsc{left-comp}$$

$$\frac{\begin{array}{c}\Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : \hom_A(x, y), \Delta^m \vdash D(x, y, z) : \mathcal{U}_k \qquad \Gamma^\ell, y \overset{\circ}{:} A, \Delta^m \vdash d(y) \overset{\omega}{:} D(y, y, 1_y)\end{array}}{\Gamma^\ell, y \overset{\circ}{:} A, \Delta^m \vdash d(y) \equiv j_d^R(y, y, 1_y) \overset{\omega}{:} D(y, y, 1_y)}\ \text{hom-}\textsc{right-comp}$$

Due to the asymmetry between the two arguments of the constructor hom, we have two dual elimination rules (and two corresponding computation rules). The left rule should be understood as the elimination of a morphism in the original category, whereas the right rule corresponds to elimination in the opposite category.

Despite being directed, our homomorphism types are very similar in their construction to Martin-Löf's identity types. In fact, erasing the orientations in our rules give us the rules of the identity types of MLTT.

2.4 Universes

We will use a countable hierarchy of universes noted $(\mathcal{U}_k)_{k \in \mathbb{N}}$, such that, for every k, $\mathcal{U}_{k+1}$ contains $\mathcal{U}_k$ and all its terms.

$$\frac{\Gamma^\ell \ \mathsf{ctx}}{\Gamma^\ell \vdash \mathcal{U}_k \overset{\circ}{:} \mathcal{U}_{k+1}} \qquad \frac{\Gamma^\ell \vdash A \overset{\omega}{:} \mathcal{U}_k}{\Gamma^\ell \vdash A \overset{\omega}{:} \mathcal{U}_{k+1}}$$

2.5 Function types

We will introduce function types to our theory. We will first present the rules for simple function types, where the type of the output of the function doesn't depend on the the input, and then generalize it with dependent function types.

Simple function types

Function types will be annotated with the orientations of the input and the output, so $A^{\omega_1} \to B^{\omega_2}$ is the type of functions that take an element of the type A with orientation ω_1 and return an element of B with orientation ω_2. We stress the fact that A^ω is not a new type and is simply a notation. We will keep the same convention we use in orientation of contexts and note A^+ as simply A.

To be able to construct a function type, the domain type must be isovariant.

$$\frac{\Gamma^\ell \vdash A \overset{-}{:} \mathcal{U}_k \qquad \Gamma^\ell \vdash B : \mathcal{U}_k \qquad \omega_1\ \mathsf{ort} \qquad \omega_2\ \mathsf{ort}}{\Gamma^\ell \vdash A^{\omega_1} \to B^{\omega_2} : \mathcal{U}_k}\ \to\text{-}\textsc{form} \qquad \frac{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash t \overset{\omega_2}{:} B}{\Gamma^\ell \vdash \lambda x.t : A^{\omega_1} \to B^{\omega_2}}\ \to\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash f : A^{\omega_1} \to B^{\omega_2}}{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash f(x) \overset{\omega_2}{:} B}\ \to\text{-}\textsc{elim}$$

$$\frac{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash t \overset{\omega_2}{:} B}{\Gamma^\ell, y \overset{\omega_1}{:} A \vdash (\lambda x.t)(y) \equiv t[y/x] \overset{\omega_2}{:} B}\ \to\text{-}\beta \qquad \frac{\Gamma^\ell \vdash f : A^{\omega_1} \to B^{\omega_2}}{\Gamma^\ell \vdash \lambda x.f(x) \equiv f : A^{\omega_1} \to B^{\omega_2}}\ \to\text{-}\eta$$

Dependent function types

Dependent function types allow the codomain type to depend on the input. More specifically, to construct the type $\Pi_{x:A^{\omega_1}} B(x)^{\omega_2}$, we need $B(x)$ to depend ω_1 variantly on x.

$$\frac{\Gamma^\ell \vdash A \overset{-}{:} \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\omega_1}{:} A \vdash B(x) : \mathcal{U}_k}{\Gamma^\ell \vdash \Pi_{x:A^{\omega_1}} B(x)^{\omega_2} : \mathcal{U}_k}\ \Pi\text{-}\textsc{form} \qquad \frac{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash t : B(x)}{\Gamma^\ell \vdash \lambda x.t : \Pi_{x:A^{\omega_1}} B(x)^{\omega_2}}\ \Pi\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash f : \Pi_{x:A^{\omega_1}} B(x)^{\omega_2}}{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash f(x) \overset{\omega_2}{:} B(x)}\ \Pi\text{-}\textsc{elim}$$

$$\frac{\Gamma^\ell, x \overset{\omega_1}{:} A \vdash t \overset{\omega_2}{:} B(x)}{\Gamma^\ell, y \overset{\omega_1}{:} A \vdash (\lambda x.t)(y) \equiv t[y/x] : B(y)}\ \Pi\text{-}\beta \qquad \frac{\Gamma^\ell \vdash f : \Pi_{x:A^{\omega_1}} B(x)^{\omega_2}}{\Gamma^\ell \vdash \lambda x.f(x) \equiv f : \Pi_{x:A^{\omega_1}} B(x)^{\omega_2}}\ \Pi\text{-}\eta$$

Simple function types are a special case of dependent function types where x doesn't appear in B.

2.6 Product types

We present in this part the rules for the product types, first in the non-dependent case, then in the general case.

Cartesian product types

Just as function types, cartesian product types will be annotated with the orientations of their components. Thus, an element of $A^{\omega_1} \times B^{\omega_2}$ is a pair (a, b) where a is an element of A with orientation ω_1 and b is an element of B with orientation ω_2.

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell \vdash B : \mathcal{U}_k}{\Gamma^\ell \vdash A^{\omega_1} \times B^{\omega_2} : \mathcal{U}_k}\ \times\text{-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B}{\Gamma^\ell \vdash (a, b) : A^{\omega_1} \times B^{\omega_2}}\ \times\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash p : A^{\omega_1} \times B^{\omega_2}}{\Gamma^\ell \vdash \pi_1(p) \overset{\omega_1}{:} A}\ \times\text{-}\textsc{elim}\text{-L} \qquad \frac{\Gamma^\ell \vdash p : A^{\omega_1} \times B^{\omega_2}}{\Gamma^\ell \vdash \pi_2(p) \overset{\omega_2}{:} B}\ \times\text{-}\textsc{elim}\text{-R}$$

$$\frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B}{\Gamma^\ell \vdash \pi_1((a, b)) \equiv a \overset{\omega_1}{:} A}\ \times\text{-}\beta\text{-L} \qquad \frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B}{\Gamma^\ell \vdash \pi_2((a, b)) \equiv b \overset{\omega_2}{:} B}\ \times\text{-}\beta\text{-R}$$

$$\frac{\Gamma^\ell \vdash p : A^{\omega_1} \times B^{\omega_2}}{\Gamma^\ell \vdash (\pi_1(p), \pi_2(p)) \equiv p : A^{\omega_1} \times B^{\omega_2}}\ \times\text{-}\eta$$

Dependent pair types

Generalizing product types to the dependent case is the same as for function types.

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\omega_1}{:} A \vdash B(x) : \mathcal{U}_k}{\Gamma^\ell \vdash \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2} : \mathcal{U}_k}\ \Sigma\text{-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B(a)}{\Gamma^\ell \vdash (a,b) : \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2}}\ \Sigma\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash p : \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2}}{\Gamma^\ell \vdash \pi_1(p) \overset{\omega_1}{:} A}\ \Sigma\text{-}\textsc{elim}\text{-L} \qquad \frac{\Gamma^\ell \vdash p : \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2}}{\Gamma^\ell \vdash \pi_2(p) \overset{\omega_2}{:} B(\pi_1(p))}\ \Sigma\text{-}\textsc{elim}\text{-R}$$

$$\frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B(a)}{\Gamma^\ell \vdash \pi_1((a,b)) \equiv a \overset{\omega_1}{:} A}\ \Sigma\text{-}\beta\text{-L} \qquad \frac{\Gamma^\ell \vdash a \overset{\omega_1}{:} A \qquad \Gamma^\ell \vdash b \overset{\omega_2}{:} B(a)}{\Gamma^\ell \vdash \pi_2((a,b)) \equiv b \overset{\omega_2}{:} B(a)}\ \Sigma\text{-}\beta\text{-R}$$

$$\frac{\Gamma^\ell \vdash p : \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2}}{\Gamma^\ell \vdash (\pi_1(p), \pi_2(p)) \equiv p : \Sigma_{x:A^{\omega_1}} B(x)^{\omega_2}}\ \Sigma\text{-}\eta$$

2.7 Inductive types

In our study of inductive types, we were most interested in the ones that could not exist in MLTT, so the one where orientations matter. We found two types interesting : the opposite of a type and the core of a type. They may seem useless as they seem to be just a repeat of the orientations, but as we will see later, the core of a type is a useful tool to define identity types.

The opposite of a type

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell \vdash A^{\mathsf{op}} : \mathcal{U}_k}\ \mathsf{op}\text{-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{-:\omega}{:} A \vdash \mathsf{flip}\, x \overset{\omega}{:} A^{\mathsf{op}}}\ \mathsf{op}\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\alpha}{:} A^{\mathsf{op}}, \Delta^m \vdash D(x) \overset{\kappa}{:} U_k \qquad \Gamma^\ell, x \overset{-:\alpha}{:} A, \Delta[\mathsf{flip}\, x/x]^m \vdash d(x) \overset{\omega}{:} D(\mathsf{flip}\, x)}{\Gamma^\ell, x \overset{\alpha}{:} A^{\mathsf{op}}, \Delta^m \vdash \mathsf{ind}_{\mathsf{op}}(d,x) \overset{\omega}{:} D(x)}\ \mathsf{op}\text{-}\textsc{elim}$$

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\alpha}{:} A^{\mathsf{op}}, \Delta^m \vdash D(x) \overset{\kappa}{:} U_k \qquad \Gamma^\ell, x \overset{-:\alpha}{:} A, \Delta[\mathsf{flip}\, x/x]^m \vdash d(x) \overset{\omega}{:} D(\mathsf{flip}\, x)}{\Gamma^\ell, x \overset{-:\alpha}{:} A, \Delta[\mathsf{flip}\, x/x]^m \vdash d(x) \equiv \mathsf{ind}_{\mathsf{op}}(d, \mathsf{flip}\, x) \overset{\omega}{:} D(\mathsf{flip}\, x)}\ \mathsf{op}\text{-}\textsc{comp}$$

The core of a type

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell \vdash A^{\mathsf{core}} : \mathcal{U}_k}\ \mathsf{core}\text{-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{\circ:\omega}{:} \vdash \mathsf{strip}\, x \overset{\omega}{:} A^{\mathsf{core}}}\ \mathsf{core}\text{-}\textsc{intro}$$

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\alpha}{:} A^{\mathsf{core}}, \Delta \vdash D \overset{\kappa}{:} U_k \qquad \Gamma^\ell, x \overset{\circ:\alpha}{:} A, \Delta[\mathsf{strip}\, x/x] \vdash d(x) \overset{\omega}{:} D(\mathsf{strip}\, x)}{\Gamma^\ell, x \overset{\alpha}{:} A, \Delta^m \vdash \mathsf{ind}_{\mathsf{core}}(d,x) \overset{\omega}{:} D(x)}\ \mathsf{core}\text{-}\textsc{elim}$$

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\alpha}{:} A^{\mathsf{core}}, \Delta^m \vdash D(x) \overset{\kappa}{:} U_k \qquad \Gamma^\ell, x \overset{\circ:\alpha}{:} A, \Delta[\mathsf{strip}\, x/x]^m \vdash d(x) \overset{\omega}{:} D(\mathsf{strip}\, x)}{\Gamma^\ell, x \overset{\circ:\alpha}{:} A, \Delta[\mathsf{strip}\, x/x]^m \vdash d(x) \equiv \mathsf{ind}_{\mathsf{core}}(d, \mathsf{strip}\, x) \overset{\omega}{:} D(\mathsf{strip}\, x)}\ \mathsf{core}\text{-}\textsc{comp}$$

For both of these types, the rules are interpreted semantically as the identity as a covariant term A^{core} and an isovariant variant term of A will be interpreted the same way.

3 Interpretation of the syntax in 1-categories

The model presented in this section is adapted from [2] but everything is written by me.

We will work in a set theory with a strictly increasing sequence of uncountable inaccessible cardinals $(\lambda_k)_{k\in\mathbb{N}}$ and, for any k, we will note Cat_k the category of λ_k-small categories. Except in the subsection about universes, we will always reason in a single given universe $\mathcal{U}_k$.

In this section, both covariant and contravariant Grothendieck constructions will be used for context extension, so we find it necessary to explain them here.

Let C be a category and F a functor from C to Cat_k. The covariant Grothendieck construction $C.F$ is the category such that :

- the objects of $C.F$ are pairs (X, Y) where $X \in Ob(C)$ and $Y \in Ob(F(X))$,
- the morphisms from (X, Y) to (X', Y') are pairs (f, g) where $f : X \to X'$ is a morphism in C and $g : F(f)(Y) \to Y'$ a morphism in $F(X')$.

We will also use the projection functor π_C associating X to each pair (X, Y) and for Z an object of $C.F$, we will note its second component Z_2.

The contravariant Grothendieck construction requires instead a contravariant functor F from C to Cat_k (i.e. a functor from C^{op} to Cat_k). $C.F$ is the category such that :

- the objects of $C.F$ are pairs (X, Y) where $X \in Ob(C)$ and $Y \in Ob(F(X))$, exactly as the covariant version,
- the morphisms from (X, Y) to (X', Y') are pairs (f, g) where $f : X \to X'$ is a morphism in C and $g : Y \to F(f)(Y')$ a morphism in $F(X)$.

Moreover, the projection functor for the contravariant Grothendieck construction will be noted π_C^-.

3.1 Interpretation of contexts, types and terms

A context Γ^ℓ is interpreted as an object of Cat_k. A type A in Γ^ℓ will be interpreted as

- a functor from Γ^ℓ to $\mathsf{Cat}_k^{\mathsf{core}}$ if it is isovariant,
- a functor from Γ^ℓ to Cat_k if it is covariant,
- a functor from $(\Gamma^\ell)^{\mathsf{op}}$ to Cat_k if it is contravariant.

It will sometimes be helpful to annotate the type with its context, so, in these situations, we will note the functor A_{Γ^ℓ}, but most of the time, we will simply write it A.

Orientations are interpreted as endofunctors of Cat_k, $+$ is the identity, $-$ associates a category to its opposite category and $\circ$ is the functor taking a category to its maximal sub-groupoid. For A a type, we will note A^ω for $\omega \circ A$.

We also have the natural transformations $i : \circ \Rightarrow +$ and $i^{\mathsf{op}} : \circ \Rightarrow -$ given by the identity on the objects and, respectively, the inclusion and the inversion on the morphisms. Therefore, for ω and α two orientations such that $\omega \leq \alpha$, we have a natural transformation from ω to α, and by whiskering it with a type A, we obtain a natural transformation from A^ω to A^α that we will note $A^{\omega \leq \alpha}$.

$$\begin{tikzcd} & \circ \arrow[dl, Rightarrow, "i"'] \arrow[dr, Rightarrow, "i^{\mathsf{op}}"] & \\ + & & - \end{tikzcd}$$

The empty context is interpreted as the terminal object $\star$ of Cat_k and we interpret the context $\Gamma^\ell, x \overset{\omega}{:} A$ as either the covariant Grothendieck construction $\Gamma^\ell.A^\omega$ if A is covariant or isovariant or the contravariant Grothendieck construction $\Gamma^\ell.A^\omega$ if A is contravariant.

As the order on orientations has been extended to contexts by induction, we can use the natural transformations between related orientations to construct functors between related contexts by induction, i.e. when $\ell \leq m$ Γ-ort, we can construct a functor $f^{\ell;m}$ from Γ^ℓ to Γ^m. $f^{\diamond;\diamond}$ is the identity functor and, for the hereditary cases, we use the following constructions for co-/isovariant and contravariant types respectively :

$$\begin{aligned} f^{\ell.\omega;m.\alpha} &: \Gamma^\ell.(A_{\Gamma^m} \circ f^{\ell;m})^\omega \to \Gamma^m.A^\alpha_{\Gamma^m} \\ f^{\ell.\omega;m.\alpha}(\gamma, x) &:= (f^{\ell;m}(\gamma), A^{\omega \leq \alpha}_{f^{\ell;m}(\gamma)}(x)) \\ f^{\ell.\omega;m.\alpha}(p_1, p_2) &:= (f^{\ell;m}(p_1), A^{\omega \leq \alpha}_{f^{\ell;m}(\gamma)}(p_2)) \end{aligned}$$

$$\begin{aligned} f^{\ell.\omega;m.\alpha} &: \Gamma^\ell.(B_{\Gamma^m} \circ (f^{\ell;m})^{\mathsf{op}})^\omega \to \Gamma^m.B^\alpha_{\Gamma^m} \\ f^{\ell.\omega;m.\alpha}(\gamma, x) &:= (f^{\ell;m}(\gamma), B^{\omega \leq \alpha}_{(f^{\ell;m}(\gamma))^{\mathsf{op}}}(x)) \\ f^{\ell.\omega;m.\alpha}(p_1, p_2) &:= (f^{\ell;m}(p_1), B^{\omega \leq \alpha}_{(f^{\ell;m}(\gamma))^{\mathsf{op}}}(p_2)) \end{aligned}$$

As we define f, we also need to impose the following coherence condition : if we have $\ell \leq m$ Γ-ort and for A and B respectively a co-/isovariant type and a contravariant type in Γ^m, then

$$A_{\Gamma^\ell} = A_{\Gamma^m} \circ f^{\ell;m} \qquad\qquad B_{\Gamma^\ell} = B_{\Gamma^m} \circ (f^{\ell;m})^{\mathsf{op}}.$$

In practice, we will construct the interpretation of each type for the finest list of orientations and use the formula of the coherence condition to generalize it.

Terms of co-/isovariant types A with orientation ω in a context Γ^ℓ will be interpreted as sections of $\pi_{A^\omega} : \Gamma^\ell.A^\omega \to \Gamma^\ell$ or, equivalently, as lax natural transformations $\star_{\Gamma^\ell} \Rightarrow A^\omega$. Indeed, a lax natural transmutation $\alpha : \star_{\Gamma^\ell} \Rightarrow A^\omega$ is characterized by the data of an object $\alpha_\gamma(\star)$ in $A^\omega(\gamma)$ for each $\gamma \in \Gamma$ and a morphism from $A^\omega(f)(\alpha_\gamma(\star))$ to $\alpha_{\gamma'}(\star)$ in $A^\omega(\gamma')$ for each $f : \gamma \to \gamma'$ (we will denote this morphism as α_f from now on) which is the exact data that characterizes a section of π_{Γ^ℓ}.

Dually, terms of contravariant types B will be interpreted as sections of $\pi^-_{\Gamma^\ell} : \Gamma^\ell.B^\omega \to \Gamma^\ell$ or, equivalently, oplax natural transformations $\star_{\Gamma^\ell} \Rightarrow B^\omega \circ (-)^{\mathsf{op}}$. Finally, judgmental equalities will be interpreted as equalities of the interpretation of the terms.

3.2 Interpretation of weakening of orientations and transmutation

Ort-Weak-L

Let ℓ and m be two lists of orientation for Γ such that $\ell \leq m$ Γ-ort, $A : \Gamma^m \to \mathsf{Cat}_k$ be a covariant functor, ω be an orientation and $t : \star_{\Gamma^m} \Rightarrow A^\omega$ be a lax natural transformation.

$$\begin{tikzcd} \Gamma^\ell \arrow[r, "f^{\ell;m}"] & \Gamma^m \arrow[rr, bend left, "\star_{\Gamma^\ell}"] \arrow[rr, bend right, "A^\omega"'] \arrow[rr, phantom, "\Downarrow t"] & & \mathsf{Cat}_k \end{tikzcd}$$

By whiskering t with $f^{\ell;m}$, we obtain a lax natural transformation $tf^{\ell;m} : \star_{\Gamma^\ell} \Rightarrow A^\omega : \Gamma^\ell \to \mathsf{Cat}_k$. The case of A being a contravariant functor and t an oplax natural transformation is treated dually.

Ort-Weak-R

Let Γ^ℓ be a category, $A : \Gamma^\ell \to \mathsf{Cat}_k$ be a covariant functor, ω and α be two orientations such that $\omega \leq \alpha$, and t be a lax natural transformation from $\star_{\Gamma^\ell}$ to A^ω.

$$\begin{tikzcd} \Gamma^\ell \arrow[rr, bend left=50, "\star_{\Gamma^\ell}"] \arrow[rr, "A^\omega"] \arrow[rr, bend right=50, "A^\alpha"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow t"] \arrow[rr, phantom, shift right=4ex, "\Downarrow A^{\omega\leq\alpha}"] & & \mathsf{Cat}_k \end{tikzcd}$$

Vertical composition gives us a natural transformation $A^{\omega\leq\alpha} \circ t : \star_{\Gamma^\ell} \Rightarrow A^\alpha$. The case of A being a contravariant functor and t an oplax natural transformation is treated dually.

Transmut

We will treat separately the transmutation by $\circ$ and by $-$.

For the isovariant transmutation, we will first need to prove that, for Γ^ℓ a context, $(\Gamma^\ell)^\circ$ and Γ° are isomorphic. We will prove it by induction on the length of Γ^ℓ.

In the case of the empty context, both object are the terminal object of Cat_k and thus are trivially isomorphic.

Let Γ^ℓ be a context such that $(\Gamma^\ell)^\circ$ and Γ° are isomorphic and let $A : \Gamma^\ell \to \mathsf{Cat}_k$ be a covariant functor and ω be an orientation. We have :

$$\begin{array}{rcl} Ob(\Gamma^\circ.A^\circ) & = & \{(\gamma, a) \mid \gamma \in \Gamma^\circ, a \in A^\circ(\gamma)\} \\ Mor(\Gamma^\circ.A^\circ)((\gamma, a), (\gamma', a')) & = & \{(f, g) \mid f \in Mor(\Gamma^\circ)(\gamma, \gamma'), g \in Mor(A^\circ(\gamma'))(A^\circ(f)(a), a')\} \end{array}$$

$$\begin{array}{rcl} Ob((\Gamma^\ell.A^\omega)^\circ) & = & \{(\gamma, a) \mid \gamma \in (\Gamma^\ell)^\circ, a \in A^\circ(\gamma)\} \\ Mor((\Gamma^\ell.A^\omega)^\circ) & = & IsoMor(\Gamma^\ell.A^\omega) \end{array}$$

An isomorphism of $(\Gamma^\ell.A^\omega)$ from (γ, a) to (γ', a') is a pair (f, g) such that f is an isomorphism of Γ^ℓ (i.e. a morphism of $(\Gamma^\ell)^\circ$) and $g : A^\omega(f)(a) \to a'$ is a morphism of $A^\omega(\gamma')$ such that there exists $g' : A^\omega(f^{-1})(a') \to a$ in $A^\omega(\gamma)$ and $g \circ A^\omega(f)(g') = id_{a'}$ and $g' \circ A^\omega(f^{-1})(g) = id_a$. So g is an isomorphism of $A^\omega(\gamma')$ (and therefore a morphism of $A^\circ(\gamma')$) whose inverse is $A^\omega(f)(g')$.

Thus, we have

$$Mor((\Gamma^\ell.A^\omega)^\circ)((\gamma, a), (\gamma', a')) = \{(f, g) \mid f \in Mor((\Gamma^\ell)^\circ)(\gamma, \gamma'), g \in Mor(A^\circ(\gamma'))(A^\circ(f)(a), a')\}$$

and we can construct an isomorphism between $(\Gamma^\ell.A^\omega)^\circ$ and $\Gamma^\circ.A^\circ$ by using the isomorphism between $(\Gamma^\ell)^\circ$ and Γ°.

For the case of an extension of the context with a contravariant type, let Γ^ℓ be a context such that $(\Gamma^\ell)^\circ$ and Γ° are isomorphic and let $B : (\Gamma^\ell)^{\mathsf{op}} \to \mathsf{Cat}_k$ be a functor and ω be an orientation.

Morphisms of $\Gamma^\circ.B^\circ$ from (γ, b) to (γ', b') are pairs (f, g) with $f : \gamma \to \gamma'$ and $g : a \to B^\circ(f)(a')$ two isomorphisms.

Morphisms of $(\Gamma^\ell.B^\omega)^\circ$ from (γ, b) to (γ', b') are pairs (f, g) with f an isomorphism in Γ^ℓ and $g : b \to B^\omega(f)(b')$ a morphism in $B^\omega(\gamma)$ such that, for a given $g' : b' \to B^\omega(f^{-1})(b)$, we have $B^\omega(f)(g') \circ g = id_b$ and $B^\omega(f^{-1})(g) \circ g' = id_{b'}$. Thus, g is an isomorphism whose inverse is $B^\omega(f)(g')$ and $(\Gamma^\ell.B^\omega)^\circ$ and $\Gamma^\circ.B^\circ$ are isomorphic.

Let $A : \Gamma^\ell \to \mathsf{Cat}_k$ be a covariant functor, ω be an orientation and $t : \Gamma^\ell \to \Gamma^\ell A^\omega$ be a section of π_{Γ^ℓ}. By applying $(-)^\circ$ to t, we obtain a functor $t^\circ : (\Gamma^\ell)^\circ \to (\Gamma^\ell.A^\omega)^\circ$. By pre- and post-composing by the isomorphism we constructed just before, we obtain a functor from Γ° to $\Gamma^\circ.A^\circ$ and, as we know the behavior of each functor that has been composed, we can easily verify that it is a section of π_{Γ°. For $B : (\Gamma^\ell)^{\mathsf{op}} \to \mathsf{Cat}_k$ and $t : \Gamma^\ell \to \Gamma^\ell.B^\omega$ a section of $\pi^-_{\Gamma^\ell}$, the same process gives us a section of $\pi^-_{\Gamma^\circ} : \Gamma^\circ.B^\circ \to \Gamma^\circ$.

For the contravariant transmutation, we will first prove that, for any context Γ^ℓ, $(\Gamma^\ell)^-$ and $\Gamma^{\ell\cdot-}$ are isomorphic. We will prove it by induction on Γ^ℓ.

The case of the empty context is trivial, as they both are the empty context. For the case of context extension by a covariant type, let Γ^ℓ be a context such that $(\Gamma^\ell)^-$ and $\Gamma^{\ell\cdot-}$ are isomorphic, $A : \Gamma^\ell \to \mathsf{Cat}_k$ a functor and ω on orientation. We'll note that, to construct $\Gamma^\ell.A^\omega$, we use the covariant Grothendieck construction on A^ω, whereas, for $\Gamma^{\ell\cdot-}.A^{\omega\cdot-}$, we use the contravariant Grothendieck construction on $A^{\omega\cdot-}$ seen as a functor from $(\Gamma^{\ell\cdot-})^{\mathsf{op}}$ to Cat_k. We have :

$$Mor((\Gamma^\ell.A^\omega)^-)((\gamma, a), (\gamma', a')) = \{(f, g) \mid f \in Mor((\Gamma^\ell)^-)(\gamma, \gamma'), g \in Mor(A^\omega(\gamma))(A^\omega(f)(a'), a)\}$$

$$Mor(\Gamma^{\ell\cdot-}.A^{\omega\cdot-})((\gamma, a), (\gamma', a')) = \{(f, g) \mid f \in Mor(\Gamma^{\ell\cdot-})(\gamma, \gamma'), g \in Mor(A^\omega(\gamma))(A^\omega(f)(a'), a)\}$$

By using the isomorphism between $(\Gamma^\ell)^-$ and $\Gamma^{\ell\cdot-}$, we can construct an isomorphism between $(\Gamma^\ell.A^\omega)^-$ and $\Gamma^{\ell\cdot-}.A^{\omega\cdot-}$. The case of context extension by a contravariant type is symmetric.

By applying the same process as we did for the isovariant transmutation, we can turn any section of π_{Γ^ℓ} t into a section of $\pi^-_{\Gamma^{\ell\cdot-}}$ and reciprocally.

3.3 Interpretation of structural rules

Var

For Γ^ℓ a category, A a covariant functor from Γ^ℓ to Cat_k and ω an orientation, we can construct $Var : \star_{\Gamma^\ell.A^\omega} \Rightarrow A^\omega$ as the (lax) natural transformation such that, for any $\gamma \in \Gamma^\ell$ and any $a \in A^\omega(\gamma)$, $Var_{(\gamma,a)}(\star)$ is equal to a.

Weak

Let Γ^ℓ be a category, n a natural number, $(\omega_k)_{k\in[1,n]}$ a list of orientations and $(\Delta_k)_{k\in[1,n]}$ a list of functors such that, for all $k \in [1, n]$, Δ_k is a functor from either $\Gamma^\ell.\Delta_1^{\omega_1}.\dots.\Delta_{k-1}^{\omega_{k-1}}$ or $(\Gamma^\ell.\Delta_1^{\omega_1}.\dots.\Delta_{k-1}^{\omega_{k-1}})^{\mathsf{op}}$ to Cat_k, and let A be a covariant functor from Γ^ℓ to Cat_k and α an orientation. We construct the functor $Weak : \Gamma^\ell.A^\alpha.\Delta_1^{\omega_1}.\dots.\Delta_n^{\omega_n} \to \Gamma^\ell.\Delta_1^{\omega_1}.\dots.\Delta_n^{\omega_n}$ that takes the $n+2$-uplet $(\gamma, a, \delta_1, \dots, \delta_n)$ to the $n+1$-uplet $(\gamma, \delta_1, \dots, \delta_n)$. $Weak$ is the functor that "erases" A^α from the context.

Let B be a covariant (respectively contravariant) functor from $\Gamma^\ell.\Delta_1^{\omega_1}.\dots.\Delta_n^{\omega_n}$ to Cat_k and α an orientation. By whiskering any lax (respectively oplax) natural transformation from $\Gamma^\ell.\Delta_1^{\omega_1}.\dots.\Delta_n^{\omega_n}$ to B^α with $Weak$, we obtain a lax (respectively oplax) natural transformation from $\Gamma^\ell.A^\omega.\Delta_1^{\omega_1}.\dots.\Delta_n^{\omega_n}$ to B^α.

Subst

We will only treat the case where Δ^m contains only one variable. The generalization for any Δ^m is easily done by induction on its length.

Let A be a covariant functor from Γ^ℓ to Cat_k, ω an orientation, $a : \Gamma^\ell \Rightarrow A^\omega$ a lax natural transformation and B a covariant functor from $\Gamma^\ell.A^\omega$ to Cat_k. We construct the functor $Subst$ from $\Gamma^\ell.B[a/x]^\alpha$ (where $B[a/x]$ is the functor $B \circ a$) to $\Gamma^\ell.A^\omega.B^\alpha$ that associates to a pair (γ, b) the triple $(a(\gamma), b)$. The cases where A or B are contravariant are treated similarly.

Let C be a covariant (respectively contravariant) functor from $\Gamma^\ell.A^\omega.B^\alpha$ to Cat_k and β an orientation. We can turn any lax (respectively oplax) natural transformation from $\Gamma^\ell.A^\omega.B^\alpha$ to C^β into a lax (respectively oplax) natural transformation from $\Gamma^\ell.B[a/x]^\alpha$ to C^β by whiskering it with $Subst$.

3.4 Universes

For k a natural number, the universe $\mathcal{U}_k$ as a type in the empty context is interpreted as the category Cat_k. The two rules related to the universes corresponds to the fact that Cat_k and all its elements are included in Cat_{k+1}.

A isovariant type, i.e. a term of a universe $\mathcal{U}_k$ with orientation $\circ$, in a context Γ^ℓ is a functor from Γ^ℓ to Cat_k°. Thus, isovariant types are exactly the types that, for a morphism in the context, induce an isomorphism on Cat_k.

3.5 Homomorphism types

Hom formation

Let A be a functor $\Gamma^\ell \to \mathsf{Cat}_k$. We want to construct a functor $\hom_A(a,b)$ from $\Gamma^\ell.A^-.(A \circ \pi_{\Gamma^\ell})$ (which will be noted $\Gamma^\ell.A^-.A$ from now on) to Cat_k. The objects of $\Gamma.A^-.A$ are triples (γ, a, b) with γ in Γ and a and b two objects of $A(\gamma)$. A morphism from (γ, a, b) to (γ', a', b') is a triple (f, g, h) with $f : \gamma \to \gamma'$ a morphism in Γ and $g : a' \to A(f)(a)$ and $h : A(f)(b) \to b'$ two morphisms in $A(\gamma')$.

$$\begin{array}{llcll}
\hom_A & : & \Gamma^\ell.A^-.A & \to & \mathsf{Cat}_k \\
\hom_A & := & (\gamma, a, b) & \mapsto & \mathsf{hom}\mathcal{S}\mathsf{et}_{A(\gamma)}(a,b) \\
\hom_A & := & (f,g,h) : (\gamma,a,b) \to (\gamma',a',b') & \mapsto & (\varphi \mapsto h \circ A(f)(\varphi) \circ g)
\end{array}$$

where $\mathsf{hom}\mathcal{S}\mathsf{et}_{A(\gamma)}(a,b)$ is the set of homomorphisms in $A(\gamma)$ from a to b seen as a (discrete) category.

Hom introduction

Let A be a functor $\Gamma^\ell \to \mathsf{Cat}_k$. We will note $i^{\mathsf{op}} \times i$ the functor from $\Gamma^\ell.A^\circ$ to $\Gamma^\ell.A^-.A$ that duplicates the element of A° and injects in A^- and A. We want to construct a lax natural transformation $1 : \star_{\Gamma^\ell.A^\circ} \Rightarrow (\hom_A \circ (i^{\mathsf{op}} \times i))^\circ$.

Let γ be an object in Γ^ℓ and a one in A°. Defining a functor $1_{\gamma,a} : \star \to \hom_A(\gamma, a, b)^\circ$ is picking an object in $\hom_A(\gamma, a, a)$. This category being equal to $\mathsf{hom}\mathcal{S}\mathsf{et}_{A(\gamma)}(a,a)$, we can pick id_a, the identity of a.

For the (lax) naturality of 1, we need to check that, for each morphism $(f,g) : (\gamma, a) \to (\gamma', a')$ in $\Gamma^\ell.A^\circ$, $(\hom_A \circ (i^{\mathsf{op}} \times i))^\circ(f,g)(1_{\gamma,a}(\star))$ is equal to $1_{\gamma',a'}(\star)$. By unfolding the definitions, we have that $(\hom_A \circ (i^{\mathsf{op}} \times i))^\circ(f,g)(1_{\gamma,a})$ is equal to $g \circ A(f)(id_a) \circ g^{-1}$ and thus, equal to $id_{a'}$. So 1 is (lax) natural.

Hom left elimination and computation

As we did for the substitution rule, we will only consider the case where Δ^m contains only one variable. Proving the general case can be easily done by induction on the length of Δ^m.

Let A be a functor from Γ^ℓ to Cat_k, B a functor from $\Gamma^\ell.A^\circ$ to Cat_k, ω_1 and ω_2 two orientations, D a functor $\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1} \to \mathsf{Cat}_k$. We will note $i^{\mathsf{op}} \times i \times 1.B^{\omega_1} : \Gamma^\ell.A^\circ.B^{\omega_1} \to \Gamma^\ell.A^-.A.\hom_A.B^{\omega_1}$ the functor associating to a triple (γ, a, b) the quintuple $(\gamma, a, a, 1_a, b)$. Finally, let d be a lax natural transformation from $\star_{\Gamma^\ell.A^\circ.B^{\omega_1}}$ to $(D \circ (i^{\mathsf{op}} \times i \times 1.B^{\omega_1}))^{\omega_2}$.

We want to construct a natural transformation :

$$j_d^L : \star_{\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}} \Rightarrow D^{\omega_2}$$

$\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}$ has for objects quintuples $(\gamma, a_1, a_2, \varphi, b)$ where γ is an object of Γ^ℓ, a_1 and a_2 are two objects of $A(\gamma)$, $\varphi : a_1 \to a_2$ a morphism in $A(\gamma)$ and b an object in $B^{\omega_1}(\gamma, a_1)$. Morphisms in $\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}$ are quadruples :

$$(f, g_1, g_2, h) : (\gamma, a_1, a_2, \varphi, b) \to (\gamma', a_1', a_2', g_2 \circ A(f)(\varphi) \circ g_1^{-1}, b')$$

with f a morphism in Γ^ℓ from γ to γ', $g_1 : A(f)(a_1) \to a_1'$ an isomorphism in $A(\gamma')$, $g_2 : A^\circ(f)(a_2) \to a_2'$ a morphism in $A(\gamma')$ and $h : B^{\omega_1}(f, g_1)(b) \to b'$ in $B^{\omega_1}(\gamma', a_1')$.

To construct the lax natural transformation $j_d^L : \star_{\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}} \Rightarrow D^{\omega_2}$, we need to specify its component for each object of $\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}$. That is, for each $(\gamma, a_1, a_2, \varphi, b)$, we have to pick an object of $D^{\omega_2}(\gamma, a_1, a_2, \varphi, b)$.

Let $(\gamma, a_1, a_2, \varphi, b)$ be an object of $\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}$. We consider the morphism:

$$(id_\gamma, id_{a_1}, \varphi, id_b) : (\gamma, a_1, a_1, id_{a_1}, b) \to (\gamma, a_1, a_2, \varphi, b).$$

Applying D^{ω_2} gives us the functor:

$$D^{\omega_2}(id_\gamma, id_{a_1}, \varphi, id_b) : D^{\omega_2}(\gamma, a_1, a_1, id_{a_1}, b) \to D^{\omega_2}(\gamma, a_1, a_2, \varphi, b)$$

Moreover, $d_{(\gamma, a_1, b)}(\star)$ is an element of $D^{\omega_2}(\gamma, a_1, a_1, id_{a_1}, b)$. So we define:

$$j_d^L(\gamma, a_1, a_2, \varphi, b)(\star) := D^{\omega_2}(id_\gamma, id_{a_1}, \varphi, id_b) d_{(\gamma, a_1, b)}(\star)$$

Now, to check the lax naturality of j_d^L, we need to verify that for any morphism $(f, g_1, g_2, h) : (\gamma, a_1, a_2, \varphi, b) \to (\gamma', a_1', a_2', g_2 \circ A^\circ(f)(\varphi) \circ g_1^{-1}, b')$ in $\Gamma^\ell.A^\circ.A.\hom_A.B^{\omega_1}$, we have a morphism

$$D^{\omega_2}(f, g_1, g_2, h) j_d^L(\gamma, a_1, a_2, \varphi, b)(\star) \to j_d^L(\gamma', a_1', a_2', g_2 \circ A^\circ(f)(\varphi) \circ g_1^{-1}, b')(\star)$$

in $D^{\omega_2}(\gamma', a'_1, a'_2, b')$

$$\begin{aligned} D^{\omega_2}(f, g_1, g_2, h) j^L_d(\gamma, a_1, a_2, \varphi, b) &= D^{\omega_2}(f, g_1, g_2, h) D^{\omega_2}(id_\gamma, id_{a_1}, \varphi, id_b) d_{(\gamma, a_1, b)} \\ &= D^{\omega_2}(f, g_1, g_2 \circ A^\circ(f)(\varphi), h) d_{(\gamma, a_1, b)} \\ &= D^{\omega_2}(id_{\gamma'}, id_{a'_1}, g_2 \circ A^\circ(f)(\varphi) \circ g_1^{-1}, id_{b'}) D^{\omega_2}(f, g_1, g_1, h) d_{(\gamma, a_1, b)} \\ &\to D^{\omega_2}(id_{\gamma'}, id_{a'_1}, g_2 \circ A^\circ(f)(\varphi) \circ g_1^{-1}, id_{b'}) d_{(\gamma', a'_1, b')} \\ &= j^L_d(\gamma', a'_1, a'_2, g_2 \circ A^\circ(f)(\varphi) \circ g_1^{-1}, b') \end{aligned}$$

The first and last equalities are the definition of j°_d, the second and third ones are due to the composition in $\Gamma^\ell.A^\circ.A^\circ.\mathsf{Id}^\circ_A.B^{\omega_1}$ and the morphism of the fourth step is the lax naturality of d.

Finally, we have to check that our construction validates the computation rule, i.e. $j^L_d \circ (i^{\mathsf{op}} \times i \times 1.B^{\omega_1}) = d$.

Let (γ, a, b) be an object of $\Gamma^\ell.A^\circ.B^{\omega_1}$. We have that :

$$\begin{aligned} (j^L_d \circ (i \times i^{\mathsf{op}} \times 1.B^{\omega_1}))(\gamma, a, b) &= j^L_d(\gamma, a, a, id_a, b) \\ &= D^{\omega_2}(id_\gamma, id_a, id_a, id_b) d_{(\gamma, a, b)} \\ &= d_{(\gamma, a, b)} \end{aligned}$$

So $j^L_d \circ (i^{\mathsf{op}} \times i \times 1.B^{\omega_1}) = d$ holds and the computation rule is verified.

3.6 Dependent function types

This construction is taken from Hofmann and Streicher [1] and adapted to the directed case.

We start by defining $\mathsf{Tm}_C(F)$ the category of terms of a functor $F : C \to \mathsf{Cat}_k$.

$$\begin{array}{rcl} Ob(\mathsf{Tm}_C(F)) & := & \{t : C \to C.F \mid \pi_C \circ t = id_C\} \\ Mor(\mathsf{Tm}_C(F))(t, t') & := & \{\tau : t \Rightarrow t' \mid \forall x \in C, \pi_C(\tau_x) = id_x\} \end{array}$$

Let Γ^ℓ be a category, A a contravariant functor from Γ^ℓ to Cat_k, ω_1 and ω_2 two orientations and $B : \Gamma^\ell.A^\omega_1 \to \mathsf{Cat}_k$ a functor.

For each $\gamma \in \Gamma^\ell$, we construct the functor $B^{\omega_2}_\gamma : A^{\omega_1}(\gamma) \to \mathsf{Cat}_k$.

$$\begin{array}{rcl} B^{\omega_2}_\gamma & : & A^{\omega_1}(\gamma) \to \mathsf{Cat}_k \\ B^{\omega_2}_\gamma(a) & := & B(\gamma, a) \\ B^{\omega_2}_\gamma(p) & := & B(id_\gamma, p) \end{array}$$

We define $\Pi_{A^{\omega_1}}(B^{\omega_2})$ as the functor associating to each γ the category of terms of B_γ.

$$\begin{array}{rcl} \Pi_{A^{\omega_1}}(B^{\omega_2}) & : & \Gamma^\ell \to \mathsf{Cat}_k \\ \Pi_{A^{\omega_1}}(B^{\omega_2})(\gamma) & := & Tm_{A^{\omega_1}(\gamma)}(B^{\omega_2}_\gamma) \end{array}$$

We now define the action on morphism of $\Pi_{A^{\omega_1}}(B^{\omega_2})$. For $f : \gamma \to \gamma'$ and $t \in Tm_{A^{\omega_1}(\gamma)}(B^{\omega_2}_\gamma)$,

$$\begin{array}{rcl} \Pi_{A^{\omega_1}}(B^{\omega_2})(f, t) & : & A^{\omega_1}(\gamma') \to A^{\omega_1}(\gamma').B^{\omega_2}_{\gamma'} \\ \Pi_{A^{\omega_1}}(B^{\omega_2})(f, t)(a) & := & (a, B^{\omega_2}(f, id_a)(t(A^{\omega_1}(f)(a))_2)) \\ \Pi_{A^{\omega_1}}(B^{\omega_2})(f, t)(a \xrightarrow{p} a') & := & (p, B^{\omega_2}(f, id_{a'})(t(A^{\omega_1}(f)(p))_2)) \end{array}$$

In this construction, we are using the fact that A is contravariant to obtain the functor $A^{\omega_1}(f)$ from $A^{\omega_1}(\gamma')$ to $A^{\omega_1}(\gamma)$ and transport a and p to $A^{\omega_1}(\gamma)$. Thus, we have to restrict the syntax of function types to ones where the domain type is contravariant.

To interpret the introduction, elimination and computation rules, we want to construct a bijection between the lax natural transformations from $\star_{\Gamma^\ell.A^{\omega_1}}$ to B^{ω_2} and the lax natural transformations from $\star_{\Gamma^\ell}$ to $\Pi_{A^{\omega_1}} B^{\omega_2}$.

Let $t : \star_{\Gamma^\ell.A^{\omega_1}} \Rightarrow B^{\omega_2}$ be a lax natural transformation. We construct a lax natural transformation λt from $\star_{\Gamma^\ell}$ to $\Pi_{A^{\omega_1}} B^{\omega_2}$ by currying t.

$$\begin{array}{rcl} \lambda t & : & \star_{\Gamma^\ell} \Rightarrow \Pi_{A^{\omega_1}} B^{\omega_2} \\ \lambda t_\gamma(\star)(a) & := & (a, t_{(\gamma, a)}(\star)) \\ \lambda t_\gamma(\star)(a \xrightarrow{p} a') & := & (p, t_{(id_\gamma, p)}) \end{array}$$

For the lax naturality of λt, we need to construct, for all $f : \gamma \to \gamma'$ in Γ^ℓ, a morphism from $\Pi_{A^{\omega_1}} B^{\omega_2}(f)(\lambda t_\gamma(\star))$ to $\lambda t'_\gamma(\star)$ in $\Pi_{A^{\omega_1}} B^{\omega_2}(\gamma')$, i.e. a natural transformation equal to the identity on the first component. For $a \in A^{\omega_1}(\gamma')$, we pick $t_{(f, id_a)}$.

Dually, for $t : \star_{\Gamma^\ell} \Rightarrow \Pi_{A^{\omega_1}} B^{\omega_2}$ a lax natural transformation, we construct a lax natural transformation $\lambda^{-1}t : \star_{\Gamma^\ell.A^{\omega_1}} \Rightarrow B^{\omega_2}$ by decurrying.

$$\begin{array}{rcc}
\lambda^{-1}t & : & \star_{\Gamma^\ell.A^{\omega_1}} \Rightarrow B^{\omega_2} \\
\lambda^{-1}t_{(\gamma,a)}(\star) & := & t_\gamma(\star)(a) \\
\lambda^{-1}t_{(f,g)} & := & t_f(a')_2 \circ B^{\omega_2}(f, id_{a'})(t_\gamma(\star)(g)_2)
\end{array}$$

The bijection and its inverse give us respectively the introduction and elimination rules and the fact that it is a bijection gives us both computation rules.

3.7 Dependent pair types

This construction is also an adaptation of [1] to the directed case.

We'll fix Γ^ℓ a category, ω_1 and ω_2 two orientations, $A : \Gamma^\ell \to \mathsf{Cat}_k$ and $B : \Gamma^\ell.A^{\omega_1} \to \mathsf{Cat}_k$ two covariant functors.

To construct the interpretation of the dependent pair type $\Sigma_{A^{\omega_1}}(B^{\omega_2})$, we'll use again the construction $B^{\omega_2}_\gamma$ defined in the proof for dependent function types.

$$\begin{array}{rcl}
\Sigma_{A^{\omega_1}}(B^{\omega_2}) & : & \Gamma^\ell \to \mathsf{Cat}_k \\
\Sigma_{A^{\omega_1}}(B^{\omega_2})(\gamma) & := & A^{\omega_1}(\gamma).B^{\omega_2}_\gamma
\end{array}$$

The action on morphism of $\Sigma_{A^{\omega_1}}(B^{\omega_2})$ is defined as follows :

For $p : \gamma \to \gamma'$ and $(a, b) \in A^{\omega_1}(\gamma).B^{\omega_2}_\gamma$, $\quad \Sigma_{A^{\omega_1}}(B^{\omega_2})(p)(a, b) := (A^{\omega_1}(p)(a), B^{\omega_2}(p, id_{A^{\omega_1}(p)(a)})(b))$

For the introduction rule, let a be a section of $\pi_{\Gamma^\ell} : \Gamma^\ell.A^{\omega_1} \to \Gamma^\ell$ and b be a section of $\pi_{\Gamma^\ell} : \Gamma^\ell.(B \circ a)^{\omega_2} \to \Gamma^\ell$. We want a section (a, b) of $\pi_{\Gamma^\ell} : \Gamma^\ell.\Sigma_{A^{\omega_1}} B^{\omega_2} \to \Gamma^\ell$.

$$\begin{array}{rcl}
(a, b) & : & \Gamma^\ell \to \Gamma^\ell.\Sigma_{A^{\omega_1}} B^{\omega_2} \\
(a, b)(\gamma) & := & (\gamma, (a(\gamma)_2, b(\gamma)_2)) \\
(a, b)(f) & := & (f, (a(f)_2, b(f)_2))
\end{array}$$

Now for the elimination rules, let $p : \Gamma^\ell \to \Gamma^\ell.\Sigma_{A^{\omega_1}} B^{\omega_2}$ be a section of π_{Γ^ℓ}. We need to construct a section $\pi_1(p)$ of $\pi_{\Gamma^\ell} : \Gamma^\ell.A^{\omega_1} \to \Gamma^\ell$ and a section $\pi_2(p)$ of $\pi_{\Gamma^\ell} : \Gamma^\ell \to \Gamma^\ell.(B \circ \pi_1(p))^{\omega_1}$.

$$\begin{array}{rcl}
\pi_1(p) & : & \Gamma^\ell \to \Gamma^\ell.A^{\omega_1} \\
\pi_1(p)(\gamma) & := & (\gamma, \pi_{A^{\omega_1}(\gamma)}(p(\gamma)_2)) \\
\pi_1(p)(\gamma \xrightarrow{f} \gamma') & := & (f, \pi_{A^{\omega_1}(\gamma')}(p(f)_2))
\end{array}$$

$$\begin{array}{rcl}
\pi_2(p) & : & \Gamma^\ell \to \Gamma^\ell.(B \circ \pi_1(p))^{\omega_1} \\
\pi_2(p)(\gamma) & := & (\gamma, (p(\gamma)_2)_2) \\
\pi_2(p)(f) & := & (f, (p(f)_2)_2)
\end{array}$$

We can easily verify that these constructions behave the intended way together and that the computation rules holds.

4 Expressivity of the syntax

In this part, we will present multiple usual constructions of both category theory and homotopy type theory in our theory.

4.1 Identity types

In HoTT, the identity type between a and b is meant to be thought as the type of isomorphisms between a and b. The solution we chose to have identity types in our system is to look at the homomorphisms in the core of a type and we can deduce formation, introduction and elimination rules just as if we had defined them as independent types. We will prove the elimination rule in detail and give the formation and introduction rules.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell \vdash A^{\mathsf{core}} : \mathcal{U}_k} \qquad \begin{array}{c} \dfrac{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, z : \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y), \Delta^m \vdash D(x,y,z) : \mathcal{U}_k}{\Gamma^\ell, x' \overset{\circ}{:} A^{\mathsf{core}}, y' : A^{\mathsf{core}}, z : \hom_{A^{\mathsf{core}}}(x', y'), \Delta^m \vdash \mathsf{ind}_{\mathsf{core}}(\mathsf{ind}_{\mathsf{core}}(D(-,=,z),x'),y') : \mathcal{U}_k} \\ \dfrac{\Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) \overset{\omega}{:} D(x,x,1_{\mathsf{strip}\, x})}{\Gamma^\ell, x' \overset{\circ}{:} A^{\mathsf{core}}, \Delta^m \vdash \mathsf{ind}_{\mathsf{core}}(d,x') \overset{\circ}{:} \mathsf{ind}_{\mathsf{core}}(\mathsf{ind}_{\mathsf{core}}(D(-,-,1_{\mathsf{strip}\, -})),x'),x')} \end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, x' \overset{\circ}{:} A^{\mathsf{core}}, y' : A^{\mathsf{core}}, z : \hom_{A^{\mathsf{core}}}(x',y'), \Delta^m \vdash^L_j {}_{\mathsf{ind}_{\mathsf{core}}(d,-)}(x',y',z) \overset{\omega}{:} \mathsf{ind}_{\mathsf{core}}(D(-,=,z),x'),y')} \quad \Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, x' \overset{\circ}{:} A^{\mathsf{core}} \vdash \mathsf{strip}\, y : A^{\mathsf{core}}}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, x' \overset{\circ}{:} A^{\mathsf{core}}, z : \hom_{A^{\mathsf{core}}}(x', \mathsf{strip}\, y), \Delta^m \vdash j^L_{\mathsf{ind}_{\mathsf{core}}(d,-)}(x', \mathsf{strip}\, y, z) \overset{\omega}{:} \mathsf{ind}_{\mathsf{core}}(D(-,y,z),x') \quad \Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A \vdash \mathsf{strip}\, x \overset{\circ}{:} A^{\mathsf{core}}}}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, z : \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y), \Delta^m \vdash j^L_{\mathsf{ind}_{\mathsf{core}}(d,-)}(\mathsf{strip}\, x, \mathsf{strip}\, y, z) : D(x,y,z)}
$$

$$
\dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A \vdash \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y) : \mathcal{U}_k}\ \hom_{-\mathsf{core}}\text{-FORM} \qquad \dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{\circ}{:} A \vdash 1_{\mathsf{strip}\, x} : \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, x)}\ \hom_{-\mathsf{core}}\text{-INTRO}
$$

$$
\dfrac{\begin{array}{c} \Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, z : \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y), \Delta^m \vdash D(x,y,z) : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) : D(x,x,1_{\mathsf{strip}\, x}) \end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, z : \hom_{A^{\mathsf{core}}}(x,y), \Delta^m \vdash j^L_{\mathsf{ind}_{\mathsf{core}}(d,-)}(\mathsf{strip}\, x, \mathsf{strip}\, y, z) \overset{\omega}{:} D(x,y,z)}\ \hom_{-\mathsf{core}}\text{-ELIM}
$$

$$
\dfrac{\begin{array}{c} \Gamma^\ell \vdash A : \mathcal{U}_k \\ \Gamma^\ell, x \overset{\circ}{:} A, y \overset{\circ}{:} A, z : \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y), \Delta^m \vdash D(x,y,z) : \mathcal{U}_k \qquad \Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) \overset{\omega}{:} D(x,x,1_{\mathsf{strip}\, x}) \end{array}}{\Gamma^\ell, x \overset{\circ}{:} A, \Delta^m \vdash d(x) \equiv j^L_{\mathsf{ind}_{\mathsf{core}}(d,-)}(\mathsf{strip}\, x, \mathsf{strip}\, y, z) \overset{\omega}{:} D(x,y,z)}\ \hom_{-\mathsf{core}}\text{-COMP}
$$

We will use the notation $\mathsf{Id}_A(x,y) := \hom_{A^{\mathsf{core}}}(\mathsf{strip}\, x, \mathsf{strip}\, y)$, $\mathsf{refl}_x := 1_{\mathsf{strip}\, x}$ and $j^\circ_d(x,y,z) := j^L_{\mathsf{ind}_{\mathsf{core}}(d,-)}(\mathsf{strip}\, x, \mathsf{strip}\, y, z)$.

We could have used a new type former for identity types (and we did initially) but constructing them as the homomorphism of the core of a type is simpler as we don't have to make a new proof for their interpretation and we can deduce their interpretation from the interpretation of homomorphism types and the one of cores of types.

4.2 Compositions

As we want types to be synthetic categories, with terms as objects and terms of the homomorphism types as morphisms, a composition of homomorphisms is necessary. Our syntax allows us to define two constructions of the compositions, both by homomorphism induction, the first one using the right elimination rule and the second one the left.

$$
\dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \begin{array}{c} \dfrac{\cdots}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, g : \hom_A(x,y), f : \hom_A(y,z) \vdash \hom_A(x,z) : \mathcal{U}_k} \\ \dfrac{}{\Gamma^\ell, y \overset{\circ}{:} A, z : A, f : \hom_A(y,z) \vdash f : \hom_A(y,z)} \end{array}}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, g : \hom_A(x,y), f : \hom_A(y,z) \vdash j^R_f(x,y,g) : \hom_A(x,z)}
$$

$$
\dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k \qquad \begin{array}{c} \dfrac{\cdots}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, g : \hom_A(x,y), f : \hom_A(y,z) \vdash \hom_A(x,z) : \mathcal{U}_k} \\ \dfrac{}{\Gamma^\ell, x \overset{-}{:} a, y \overset{\circ}{:} A, g : \hom(x,y) \vdash g : \hom(x,y)} \end{array}}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, g : \hom_A(x,y), f : \hom_A(y,z) \vdash j^L_g(y,z,f) : \hom_A(x,z)}
$$

We will note $f \circ^R g := j^R_f(x,y,g)$ and $f \circ^L g := j^L_g(y,z,f)$.

The two constructions are not judgmentally equal but we can construct an identification between the two by using elimination twice, once in each direction.

$$
\dfrac{\dfrac{\dfrac{\cdots}{\Gamma^\ell, y \overset{\circ}{:} A \vdash \mathsf{refl}_{1_y} : \mathsf{Id}_{\hom_A(y,y)}(1_y, 1_y)}}{\Gamma^\ell, y \overset{\circ}{:} A, z : A, f : \hom_A(y,z) \vdash j^L_{\mathsf{refl}_{1_y}}(y,z,f) : \mathsf{Id}_{\hom_A(y,z)}(f \circ^L 1_x, f)}}{\Gamma^\ell, x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, g : \hom_A(x,y), f : \hom_A(y,z) \vdash j^R_{j^L_{\mathsf{refl}_{1_x}}(x,z,f)}(x,y,g) : \mathsf{Id}_{\hom_A(x,z)}(f \circ^L g, f \circ^R g)}
$$

We remark that $j^L_{\mathsf{refl}_{1_y}}(y, z, f)$ is a proof of the right unitality of 1_x for $- \circ^L -$ and that the left unitality is true by judgmental equality. We can also prove the associativity of this composition, and thus that our types are categories.

In the following, we will note $f \circ g := f \circ^L g$.

4.3 Isomorphisms

We want to have a notion of inversible homomorphisms that we will call isomorphisms. For A a type, $x, y \overset{\circ}{:} A$ and $f : \hom_A(x, y)$,

$$\mathsf{leftInv}(f) := \Sigma_{g:\hom_A(y,x)} \mathsf{Id}_{\hom_A(x,x)}(g \circ f, 1_x) \qquad \mathsf{rightInv}(f) := \Sigma_{g:\hom_A(y,x)} \mathsf{Id}_{\hom_A(y,y)}(f \circ g, 1_y)$$

$$\mathsf{isIso}(f) := \mathsf{leftInv}(f) \times \mathsf{rightInv}(f)$$

If we have an element of $\mathsf{isIso}(f)$, we will say that f is an isomorphism. We will note $\mathsf{iso}_A(x, y)$ the type of isomorphisms between x and y.

$$\mathsf{iso}_A(x, y) := \Sigma_{f:\hom_A(x,y)} \mathsf{isIso}(f)$$

Only isovariant terms can be an end of an isomorphism, as they have to be both covariant and contravariant to construct the type.

5 A new orientation

One of the main problem of the system I've presented so far is that homomorphisms between a contravariant and a covariant objects are near useless. Even really simple things such as applying a function to a morphism where neither ends are isovariant are not possible. One notable example of that issue is that we are unable to construct an equivalent to what is called ap_f in HoTT and which apply a function to an homomorphism.

$$F \overset{\circ}{:} A \to B, x \overset{-}{:} A, y : A, f : \hom_A(x, y) \vdash \hom_B(F(x), F(y)),$$

Indeed, we can only apply hom-elimination if at least one end is isovariant.

The first idea that comes to mind is to add rules about how morphisms and functions work together that could look like this :

$$\frac{}{\Gamma, x \overset{-}{:} A, y : A, f : \hom_A(x, y), F : A \to B \vdash F(f) : \hom_B(F(a), F(b))} .$$

However, this solution seems ad-hoc and if we want to generalize to higher dimensions, we would need to add new rules for each dimension.

The solution I chose is to add a new orientation noted 1 (as in 1-cell). A 1-variant element of the type A is a morphism of A and we will write it in the form $x \xrightarrow{f} y \overset{1}{:} A$. Semantically, it has the same interpretation as $x \overset{-}{:} A, y : A, f : \hom_A(x, y)$. The endofunctors of Cat_k associated with it is the following :

$$\begin{array}{cccc} 1 & : & \mathsf{Cat}_k & \to & \mathsf{Cat}_k \\ & & A & \mapsto & A^- \times A. \hom_A \\ & & A \xrightarrow{F} B & \mapsto & \begin{cases} (a, b, f) & \mapsto & (F(a), F(b), F(f)) \\ (g, h) & \mapsto & (F(g), F(h)) \end{cases} \end{array}$$

Thus we can add the following rule to introduce a new 1-variant term :

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{-}{:} A, y : A, f : \hom_A(x, y) \vdash x \xrightarrow{f} y \overset{1}{:} A} \text{ 1-INTRO}$$

which semantically is simply the identity.

A 1-variant term can be weakened to a covariant or a contravariant term and it would give us respectively the end point and the beginning point of the morphism. Also, an isovariant term can be weakened to 1-variant and it would give the identity of this term.

$$\begin{tikzcd} & \circ \arrow[d, dash] & \\ & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ + & & - \end{tikzcd}$$

We will sometimes use notation a bit arbitrary when we do weakening, etc, with the intent of making what is happening in the semantics the clearest. For example, when a term $x \xrightarrow{f} y \overset{1}{:} A$ is weakened to $-$, we will note it $x \overset{-}{:} A$ instead of just keeping the same variable name like we did for other weakenings.

This new orientation allows a new introduction rule for hom :

$$\frac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \xrightarrow{f} y \overset{1}{:} A \vdash f : \hom_A(x,y)}\ \text{hom-}\textsc{intro}$$

The semantic of this new rule is simply a projection on the hom part. The previous one can still be deduced by applying this rule to a weakened isovariant term.

$$\frac{\dfrac{\Gamma^\ell \vdash A : \mathcal{U}_k}{\Gamma^\ell, x \overset{\circ}{:} A, y \xrightarrow{f} z \overset{1}{:} A \vdash f : \hom_A(y,z)} \qquad \dfrac{\Gamma^\ell, x \overset{\circ}{:} A \vdash x \overset{\circ}{:} A}{\Gamma^\ell, x \overset{\circ}{:} A \vdash x \xrightarrow{id_x} x \overset{1}{:} A}}{\Gamma^\ell, x \overset{\circ}{:} A \vdash id_x : \hom_A(x,x)}$$

A 1-variant type $A \xrightarrow{F} B$ in the universe $\mathcal{U}_k$ in the context Γ^ℓ is interpreted as a functor $\Gamma^\ell \to \mathsf{Cat}_k^{\mathsf{op}} \times \mathsf{Cat}_k.\hom_{\mathsf{Cat}_k}$, that is, for every object $\gamma \in \Gamma$, $F(\gamma)$ is a functor between two categories of Cat_k.

To be able to extend a context with a 1-variant type, we will use the following construction inspired by the Grothendieck construction.

Let C be a category and $F : C \to \mathsf{Cat}_k^1$ a functor.

For $X \in C$, we'll note $F^+(X)$ the projection on the first component of $F(X)$, $F^-(x)$ the second component and $F^\to(X)$ the third. For f a morphism in C, we note $F^l(f)$ the first component of $F(f)$ and $F^r(f)$ the second.

We construct $C.^1F$ as the category :

- whose objects are quadruplets (X, Y_1, Y_2, φ) where $X \in Ob(C)$, $Y_1 \in Ob(F^-(X))$, $Y_2 \in Ob(F^+(X))$ and $\varphi : F^\to(X)(Y_1) \to Y_2$ is a morphism in $F^+(X)$,
- whose morphisms from (X, Y_1, Y_2, φ) to $(X', Y_1', Y_2', \varphi')$ are triplets (f, g_1, g_2) where $f : X \to X'$, $g_1 : F^l(f)(Y_1') \to Y_1$ in $F^-(X)$, $g_2 : F^r(f)(Y_2) \to Y_2'$ in $F^+(X')$ such that :

$$\begin{tikzcd}
F^r(f)(F^\to(X)(Y_1)) \arrow[r, "F^r(f)(\varphi)"] & F^r(f)(Y_2) \arrow[d, "g_2"] \\
F^\to(X')(Y_1') \arrow[u, "F^r(f)(F^\to(X)(g_1))"] \arrow[r, "\varphi'"'] & Y_2'
\end{tikzcd}$$

1-variant types behave differently than the other kinds of types. This is caused by the fact that, in the empty context, 1-variant types are not interpreted as categories. Let $A \xrightarrow{F} B$ be a 1-variant type. Contravariant terms of F are contravariant terms of A, covariant terms are covariant terms in B and 1-variant types are morphisms in B where the source has been transported from A to B through F. There are isovariant terms only if F is a weakened isovariant type. We add the following rules to translate these facts :

$$\frac{\Gamma^\ell \vdash A \xrightarrow{F} B \overset{1}{:} \mathcal{U}_k}{\Gamma^\ell, x \overset{-}{:} A \xrightarrow{F} B \vdash x \overset{-}{:} A} \qquad \frac{\Gamma^\ell \vdash A \xrightarrow{F} B \overset{1}{:} \mathcal{U}_k}{\Gamma^\ell, x \overset{-}{:} A \vdash x \overset{-}{:} A \xrightarrow{F} B}$$

such that composing the two rules is judgmentally equal to the identity. We also add similar rules for the covariant terms.

We also add a new type former $\overrightarrow{\hom}$ for homomorphism over a 1-variant type.

$$\frac{\Gamma^\ell \vdash A \xrightarrow{F} B \overset{1}{:} \mathcal{U}_k}{\Gamma^\ell, x \overset{-}{:} A, y : B \vdash \overrightarrow{\hom}_F(x,y) : \mathcal{U}_k}\ \overrightarrow{\hom}\text{-}\textsc{form} \qquad \frac{\Gamma^\ell \vdash A \xrightarrow{F} B \overset{1}{:} \mathcal{U}_k}{\Gamma^\ell, x \xrightarrow{f} y \overset{1}{:} A \xrightarrow{F} B \vdash f : \overrightarrow{\hom}_F(x,y)}\ \overrightarrow{\hom}\text{-}\textsc{intro}$$

We can still define a multiplication on orientations but it requires to add an infinity of new orientations. Indeed, for each natural number n, 1^n is a distinct orientation. Conceptually, 1 is a line, 1^2 is a square, 1^3 is a cube, etc. These compositions of orientations are not so interesting when working with 1-categories but are promising for higher dimensions.

$\cdot$	$\circ$	$-$	$+$	1	$\dots$
$\circ$	$\circ$	$\circ$	$\circ$	$\circ \cdot 1$	$\dots$
$-$	$\circ$	$+$	$-$	1	$\dots$
$+$	$\circ$	$-$	$+$	1	$\dots$
1	$1 \cdot \circ$	$1 \cdot -$	1	1^2	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

We have not yet studied the rest of the orientations created by this multiplication, but we will use later on that $\circ$ can be weakened to $\circ \cdot 1$. Indeed, a $\circ \cdot 1$-variant term of A is an isomorphism and we can weaken any isovariant term to an isomorphism by taking its identity. We can now add the transmutation by 1 :

$$\frac{\Gamma^{\ell} \vdash t \overset{\omega_1}{:} A}{\Gamma^{\ell \cdot 1} \vdash t \overset{\omega_1 \cdot 1}{:} A} \textsc{Transmut}$$

To visualize what this rule means semantically, I will explain in details its interpretation when we apply it to $x : A \vdash t(x) : B(x)$ with B a covariant type.

We have a functor $t : A \to A.B$ which is a section of π_A and when we apply the endofunctor 1 to it, we obtain a functor from A^1 to $(A.B)^1$.

The category $(A.B)^1$ has for objects triples $((a, b), (a', b'), (f, g))$ with a, a' two objects of A, b and b' objects of $B(a)$ and $B(a')$ respectively, $f : a \to a'$ a morphism in A and $g : B(f)(b) \to b'$ a morphism in $B(a')$. A morphism from $((a, b), (a', b'), (f, g))$ to $((c, d), (c', d'), (f', g'))$ is a pair $((\alpha, \beta), (\alpha', \beta'))$ such that the following two diagrams commute :

$$\begin{tikzcd} a \arrow[r, "f"] & a' \arrow[d, "\alpha'"] \\ c \arrow[u, "\alpha"] \arrow[r, "f'"'] & c' \end{tikzcd} \qquad \begin{tikzcd} B(\alpha' \circ f)(c) \arrow[r, "B(\alpha')(f)"] & B(\alpha')(c') \arrow[d, "\beta'"] \\ B(f')(d) \arrow[u, "B(\alpha' \circ f)(\beta)"] \arrow[r, "g'"'] & d' \end{tikzcd}$$

We can easily verify that $(A.B)^1$ is isomorphic to $A^1.^1B^1$ and thus we obtain a functor $\tilde{t}$ from A^1 to $A^1.^1B^1$.

With all this, we can have a far greater control on homomorphisms between contravariant and covariant terms. For example, we can apply a function to a homomorphism and therefore prove that elements of a function type are functors.

$$\dfrac{\dfrac{\dfrac{\Gamma^\circ, F \xrightarrow{\varphi} F' \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A, a \xrightarrow{f'} b \overset{1}{:} B \vdash f' : \hom_B(a, b) \qquad \dfrac{\Gamma^\circ, F : A \to B \vdash F : A \to B}{\Gamma^\circ, F : A \to B, x : A \vdash F(x) : B}}{\Gamma^{\circ \cdot 1}, F \xrightarrow{\varphi} F' \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A \vdash F(x) \xrightarrow{F(f)} F(y) \overset{1}{:} B}}{\Gamma^\circ, F \xrightarrow{\varphi} F' \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A \vdash F(f) : \hom_B(F(x), F(y))}}{\Gamma^\circ, F \overset{\circ}{:} A \to B, x \overset{-}{:} A, y : A, f : \hom_A(x, y) \vdash F(f) : \hom_B(F(x), F(y))}$$

The limitation to isovariant contexts is necessary because of our use of the transmutation rule. In the rest of this section, many construction will have the same limitation on the context for the same reason.

This new orientation allows us to define natural transformations, which is an essential tool for category theory. In the rest of the section, we will define natural transformations and other constructions derived from them.

5.1 Natural transformations

Let A be a contravariant type in Γ and B a covariant type family on A. For F and G two functions of $\Pi_{x:A} B(x)$, respectively contravariant and covariant, the type of homomorphisms from F to G will be called the type of natural transformations.

$$\mathsf{Transf}(F, G) := \hom_{\Pi_{x:A} B(x)}(F, G)$$

There are no rules to construct a non-trivial (i.e. except the identity) term of this type, which makes it really unpractical to use. We will prefer to use a more useful relation between functions. In the non-dependent case, it will be :

$$\mathsf{Nat}(F, G) := \Pi_{x \xrightarrow{f} y \overset{1}{:} A} \hom_B(F(x), G(y))$$

This is the type of functions that associates to each morphism $f : x \to y$ in A a morphism from $F(x)$ to $G(y)$.

We can generalize it to the dependent case by using $\overrightarrow{\hom}$.

$$\mathsf{Nat}_d(F,G) := \Pi_{x \xrightarrow{f} y \overset{1}{:} A} \overrightarrow{\hom}_{B(f)}(F(x), G(y))$$

It may not seem clear why this is the type of natural transformations as it doesn't resemble the usual definition, so we will prove that every term of this type is indeed a natural transformation from F to G. To do that, we first recall that an equivalent definition of a natural transformation from F to G is a function α that associates to each morphism $f : x \to y$ a morphism $\alpha(f) : F(x) \to G(y)$ such that, for all f and g such that $f \circ g$ exists, we have $\alpha(f) \circ F(g) = G(f) \circ \alpha(g)$. So we need to need to verify that every term of $\mathsf{Nat}(F,G)$ verifies this last condition.

$$\dfrac{\dfrac{\Gamma^\circ, \alpha \overset{\circ}{:} \mathsf{Nat}(F,G), y \mathrel{:\!\overset{\circ}{:}} A \vdash 1_{\alpha(1_y)} : \mathsf{Id}_{\hom_B(F(y),G(y))}(\alpha(1_y), \alpha(1_y))}{\Gamma^\circ, \alpha \overset{\circ}{:} \mathsf{Nat}(F,G), x \overset{-}{:} A, y \overset{\circ}{:} A, g : \hom_A(x,y) \vdash j^R_{1_{\alpha(1_y)}}(x,y,g) : \mathsf{Id}_B(\alpha(1_y) \circ F(g), 1_{G(y)} \circ \alpha(g))}}{\Gamma^\circ, \alpha \overset{\circ}{:} \mathsf{Nat}(F,G), x \overset{-}{:} A, y \overset{\circ}{:} A, z : A, f : \hom_A(y,z), g : \hom_A(x,y) \vdash j^L_{j^R_{1_{\alpha(1_y)}}(x,y,g)}(y,z,f) : \mathsf{Id}_B(\alpha(f) \circ F(g), G(f) \circ \alpha(g))}$$

So every term of $\mathsf{Nat}(F,G)$ is a natural transformation. We can also prove semantically that the interpretation of this type corresponds exactly to the natural transformations between F and G.

We can construct a function from $\mathsf{Trans}(F,G)$ to $\mathsf{Nat}(F,G)$.

$$\dfrac{\dfrac{\dfrac{\Gamma^\circ, F \xrightarrow{\varphi} G \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A, a \xrightarrow{f'} b : B \vdash f' : \hom_B(a,b) \qquad \dfrac{\dfrac{\dfrac{\Gamma^\circ, F : A \to B \vdash F : A \to B}{\Gamma^\circ, F : A \to B, x : A \vdash F(x) : B}}{\Gamma^{\circ\cdot 1}, F \xrightarrow{\varphi} G \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A \vdash F(x) \xrightarrow{\varphi(f)} G(y) \overset{1}{:} B}}{\Gamma^\circ, F \xrightarrow{\varphi} G \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A \vdash F(x) \xrightarrow{\varphi(f)} G(y) \overset{1}{:} B}}{\Gamma^\circ, F \xrightarrow{\varphi} G \overset{1}{:} A \to B, x \xrightarrow{f} y \overset{1}{:} A \vdash \varphi(f) : \hom_B(F(x), G(y))}}{\Gamma^\circ, F \xrightarrow{\varphi} G \overset{1}{:} A \to B \vdash \lambda f.\varphi(f) : \mathsf{Nat}(F,G)}}{\Gamma^\circ, F \overset{-}{:} A \to B, G : A \to B, \varphi : \hom_{A \to B}(F,G) \vdash \lambda f.\varphi(f) : \mathsf{Nat}(F,G)}$$

We will note this function napply for "natural apply" and we will come back to it later.

5.2 Natural isomorphisms

Now that we have considered homomorphisms between functions, we are also interested in identities between functions. Unfortunately, we run into the same problem and we can't construct identities between functions besides the reflection.

In HoTT, the solution is to use the type of homotopies between two functions, which is defined as $\Pi_{x:A}\mathsf{Id}_B(F(x), G(x))$. If we try to directly translate this type in our system, we would obtain the following type :

$$\Pi_{x:A^\circ}\mathsf{Id}_A(F(x), G(x)).$$

However, terms of this type only acts on isomorphisms which is not satisfactory.

The solution we chose instead is to use natural isomorphisms, that is, natural transformations which gives an isomorphisms when applied to the identity morphism.

Let A and B be two types in a context Γ° and F and G be two isovariant functions from A to B. We define the type of natural isomorphisms between F and G as :

$$\mathsf{NatIso}(F,G) := \Sigma_{\alpha:\mathsf{Nat}(F,G)}\Pi_{x \overset{\circ}{:} A}\mathsf{isIso}(\alpha(x \xrightarrow{id_x} x))$$

From an identity between two functions, we can construct an natural isomorphism by first transforming the identity into an isomorphism, and then using napply on the isomorphism. We will note this function $\mathsf{niapply}$.

5.3 Equivalences

In HoTT, equivalences are functions that are inversible up to homotopy. It's an important concept as it replaces the notion of bijections from set theory. Now that we've defined natural isomorphisms, we have all the tools to define them is our system.

Let A and B be two types in a context Γ° and F an isovariant function from A to B.

$$\mathsf{leftInv}(F) := \Sigma_{G:(B\to A)^\circ}\mathsf{NatIso}(G \circ F, \lambda x.x) \qquad \mathsf{rightInv}(F) := \Sigma_{H:(B\to A)^\circ}\mathsf{NatIso}(F \circ H, \lambda x.x)$$

$$\mathsf{isEquiv}(F) := \mathsf{leftInv}(F) \times \mathsf{rightInv}(F)$$

If we have an element of $\mathsf{isEquiv}(F)$, we will say that F is an equivalence. We will note $A \simeq B$ the type of equivalences between A and B.

$$A \simeq B := \Sigma_{F:(A \to B)^{\circ}} \mathsf{isEquiv}(F)$$

5.4 Function extensionality and directed function extensionality

To be able to construct homomorphisms and identities between functions, we can add two axioms stating that the canonical functions we constructed earlier are equivalences.

Let A and B be two types in a groupoidal context and F and G two functions from A to B. The directed function extensionality states that napply is an equivalence between $\mathsf{Trans}(F, G)$ and $\mathsf{Nat}(F, G)$. As for function extensionnality, it states that $\mathsf{niapply}$ is an equivalence.

These axioms are really useful as they allow us to construct natural transformations or natural isomorphisms, which are easy to construct, and then use homomorphism or identity induction on them.

Both of these axioms are not provable in our system. In HoTT, function extensionality can be derived from the Univalence Axiom. We hope to be able to prove function extensionality and directed function extensionality with an adequate notion of univalence.

We note that in the model described in Section 3, both of these axioms holds.

5.5 Univalence in a directed setting

In HoTT, the Univalence axiom states that the canonical function from the identity type between two types to the type of equivalences between the two types is an equivalence. It's a central part of HoTT and formalizes the mathematical intuition that isomorphic objects are equal.

In directed homotopy type theory, we would like to have a directed Univalence axiom, a directed version of the Univalence axiom, that would state that some canonical function from $\hom_{\mathcal{U}_k}(A, B)$ to $A \to B$ is an equivalence. However, there exists two such canonical functions, depending on which elimination rule we use.

$$\dfrac{\dfrac{\Gamma^{\ell}, A \overset{\circ}{:} \mathcal{U}_k, x : A \vdash x : A}{\Gamma^{\ell}, A \overset{\circ}{:} \mathcal{U}_k \vdash \lambda x.x : A \to A}}{\Gamma^{\ell}, A \overset{\circ}{:} \mathcal{U}_k, B \overset{\circ}{:} \mathcal{U}_k, \phi : \hom_{\mathcal{U}_k}(A, B) \vdash j^{L}_{\lambda x.x}(A, B, \phi) : A \to B}$$

$$\dfrac{\dfrac{\Gamma^{\ell}, B \overset{\circ}{:} \mathcal{U}_k, x : B \vdash x : B}{\Gamma^{\ell}, B \overset{\circ}{:} \mathcal{U}_k \vdash \lambda x.x : B \to B}}{\Gamma^{\ell}, A \overset{\circ}{:} \mathcal{U}_k, B \overset{\circ}{:} \mathcal{U}_k, \phi : \hom_{\mathcal{U}_k}(A, B) \vdash j^{R}_{\lambda x.x}(A, B, \phi) : A \to B}$$

These two functions are semantically equal, but not syntactically. One solution that seems reasonable is to have the directed Univalence axiom stating that both functions are equivalences, but more work is necessary to understand what this axiom would entail.

The link between identities and isomorphisms also need to be studied. From an identity, we can construct an isomorphism between the same two endpoints by reasoning by induction on the identity.

$$\dfrac{\dfrac{\Gamma^{\ell} \vdash A : \mathcal{U}_k}{\Gamma^{\ell}, x \overset{\circ}{:} A \vdash 1_x : \hom_A(x, x)}}{\Gamma^{\ell}, x \overset{\circ}{:} A, y \overset{\circ}{:} A, p : \mathsf{Id}_A(x, y) \vdash j^{\circ}_{1_x}(x, y, p) : \hom_A(x, y)}$$

The inverse homomorphism is obtained by using induction on the inverse identity, that is $j^{\circ}_{1_y}(y, x, p^{-1})$ and we can prove by induction that they are indeed inverses.

$$\dfrac{\Gamma^{\ell}, x \overset{\circ}{:} A \vdash \mathsf{refl}_{1_x} : \mathsf{Id}_{\hom_A(x,x)}(1_x, 1_x)}{\Gamma^{\ell}, x \overset{\circ}{:} A, y \overset{\circ}{:} A, p : \mathsf{Id}_A(x, y) \vdash \mathsf{Id}_{\hom_A(x,x)}(j^{\circ}_{1_y}(y, x, p^{-1}) \circ j^{\circ}_{1_x}(x, y, p), 1_x)}$$

The Categorical Univalence axiom states that the function that constructs an isomorphism from an identity is an equivalence. Further research is necessary to know the impact of adding it as an axiom but we can note that from the categorical Univalence and the directed Univalence axioms, we can deduce that the types of equivalences and of identities between two types are equivalent, i.e. the usual Univalence axiom.

6 Conclusion

During this internship, I have defined a synthetic 1-category theory, constructed an interpretation of it in 1-categories and studied the mathematics we can do in the theory.

There is still a lot of work left to do. Proving the Yoneda lemma in our system was one of our goals since almost the beginning of the internship, but we're not there yet and many improvements would probably have to be made to make it possible. We would also like to have a greater control on the orientation in the context, so we would not have to restrict ourselves to groupoidal contexts in many proofs. Finally, it would be very interesting to study what higher inductive types would add to the system.

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A Meta-informations about the internship

October - November	I started to work on the flaws of the previous attempt at a directed type theory of my advisor, mainly the lack of identity types. I added an identity type former to the theory and I started looking at the mathematics we could develop in this system.
December	As the type formers `op` and `core` became a liability, I chose to replace them by orientations and adapt the work already done to orientations. I also added dependent pair and dependent function types.
January - February	Due to housing issues, I had to go back to France and work from there for 2 months. During this time, I read a lot about (higher) inductive types in HoTT with the goal of adding them in our system and I worked on adapting the model used in the previous attempt to the new system.
04.03.2024	Fernando Chu started a PhD with Paige North and started working with me on Directed Homotopy Type Theory. It was really good to have an exterior eye on the project at this point and many notations were changed to make more sense thanks to him. He also noticed many errors I had make and help to correct them.
March	I finished the first report of the internship.
April	I worked on inductive types and defined the core of a type and the opposite of a type, the first allowing me to remove the identity type former and instead having the identity types be defined as the homomorphism type of the core of a type. I had hoped to be able to define higher inductive types, but it would require a lot more work.
May	I added contravariant types to the syntax and worked on changing the interpretation to accommodate them.
14.06.2024	Our work was presented at TYPES 2024 by Fernando Chu. Due to a lack of funds, I couldn't go myself.
June	Fernando noticed an issue with the natural transformations, which revealed that many constructions and proofs in the syntax we had done so far were not quite right. It led to the addition of 1-variance to be able to have a type of natural transformations that is indeed interpreted as natural transformations in the model. As it was very late in the internship, I didn't have the time to add it to the interpretation.

B Incomplete try at a proof of the Yoneda Lemma

One of the main goals we had when we started this project was to prove the Yoneda Lemma, as it is a central result of Category Theory, but unfortunately, we didn't get there during the internship. In this section, we will show an unfinished try at a proof of the Yoneda Lemma in a groupoidal context (i.e. a context Γ°). I believe it's possible to finish it but we doesn't quite have all the tools necessary.

The Yoneda Lemma states that, for A a type, a an isovariant element of A and P an isovariant presheaf on A (that is a term of the type $A^- \to \mathcal{U}_k^\circ$), we have an equivalence :

$$\Pi_{x \xrightarrow{f} y : A^1} (\hom_A(y, a) \to P(x)) \simeq P(a)$$

To prove it, assuming function extensionality is necessary.

We first need to make sure that both types in the equivalences exist. Because of the presheaf, we have to reason in the universe $\mathcal{U}_{k+1}$.

$$\begin{array}{c}
\Gamma^\circ, a \overset{-}{:} A \vdash A \overset{-}{:} \mathcal{U}_{k+1} \qquad \Gamma^\circ, a \overset{-}{:} A \vdash \mathcal{U}_k : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{-}{:} A \vdash A^- \to \mathcal{U}_k^\circ : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{-}{:} A, P : A^- \to \mathcal{U}_k^\circ \vdash P : A^- \to \mathcal{U}_k^\circ \\ \hline
\Gamma^\circ, a \overset{-}{:} A, P : A^- \to \mathcal{U}_k^\circ, x \overset{-}{:} A \vdash P(x) : \mathcal{U}_k^\circ \qquad \Gamma^\circ, a \overset{-}{:} A, P : A^- \to \mathcal{U}_k^\circ \vdash a \overset{-}{:} A \\ \hline
\Gamma^\circ, a \overset{-}{:} A, P : A^- \to \mathcal{U}_k^\circ \vdash P(a) \overset{\circ}{:} \mathcal{U}_k \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash P(a) \overset{\circ}{:} \mathcal{U}_k \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash P(a) \overset{\circ}{:} \mathcal{U}_{k+1}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, y \overset{-}{:} A \vdash \hom_A(y, a) : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, y : A \vdash \hom_A(y, a) \overset{-}{:} \mathcal{U}_{k+1}
\end{array}
\qquad
\begin{array}{c}
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash P : A^- \to \mathcal{U}_k^\circ \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, x \overset{-}{:} A \vdash P(x) \overset{\circ}{:} \mathcal{U}_k
\end{array} \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, x \overset{-}{:} A, y : A, f : \hom_A(x, y) \vdash \hom_A(y, a) \to P(x) : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, x \xrightarrow{f} y \overset{1}{:} A \vdash \hom_A(y, a) \to P(x) : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) : \mathcal{U}_{k+1} \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \overset{\circ}{:} \mathcal{U}_{k+1}
\end{array}$$

Now, we construct both functions. The first function consists of applying the function of $\Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x)$ to the identity of a.

$$\begin{array}{c}
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \vdash F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x), x \xrightarrow{f} y \overset{1}{:} A \vdash F(f) : \hom_A(y, a) \to P(x) \\
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \vdash a \xrightarrow{1_a} a \overset{1}{:} A \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \vdash F(a \xrightarrow{1_a} a) : \hom_A(a, a) \to P(a) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x), p : \hom_A(a, a) \vdash F(a \xrightarrow{1_a} a, p) : P(a) \\
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \vdash 1_a : \hom_A(a, a) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, F : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x) \vdash F(a \xrightarrow{1_a} a, 1_a) : P(a) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \lambda F. F(a \xrightarrow{1_a} a, 1_a) : (\Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x)) \to P(a) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \lambda F. F(a \xrightarrow{1_a} a, 1_a) \overset{\circ}{:} (\Pi_{x \xrightarrow{f} y : A^1} \hom_A(y, a) \to P(x)) \to P(a)
\end{array}$$

The second function is constructed by two homomorphism inductions. We have to be very careful about the variance of the types we are using.

$$\begin{array}{c}
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a) \vdash z : P(a) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a), x \overset{-}{:} A, f : \hom_A(x,a) \vdash j_z^R(x,a,f) : P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z \overset{-}{:} P(a), x : A, f : \hom_A(x,a) \vdash j_z^R(x,a,f) \overset{-}{:} P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z \overset{-}{:} P(a), x : A \vdash \lambda f. j_z^R(x,a,f) : \hom_A(x,a) \to P(x)^- \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z \overset{-}{:} P(a), x : A, y \overset{-}{:} A, p : \hom_A(y,a) \vdash j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p) : \hom_A(x,y) \to P(x)^- \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z \overset{-}{:} P(a), x : A, y \overset{-}{:} A, f : \hom_A(x,y), p : \hom_A(y,a) \vdash j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) \overset{-}{:} P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a), x \overset{-}{:} A, y : A, f : \hom_A(x,y), p : \hom_A(y,a) \vdash j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) : P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a), x \overset{-}{:} A, y : A, f : \hom_A(x,y) \vdash \lambda p. j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) : \hom_A(y,a) \to P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a), x \xrightarrow{f} y \overset{1}{:} A \vdash \lambda p. j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) : \hom_A(y,a) \to P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ, z : P(a) \vdash \lambda fp. j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) : \Pi_{x \xrightarrow{f} y : A^1} \hom_A(y,a) \to P(x) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \lambda zfp. j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) : P(a) \to (\Pi_{x \xrightarrow{f} y : A^1} \hom_A(y,a) \to P(x)) \\ \hline
\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \lambda zfp. j^R_{\lambda f. j_z^R(x,a,f)}(y,a,p)(f) \overset{\circ}{:} P(a) \to (\Pi_{x \xrightarrow{f} y : A^1} \hom_A(y,a) \to P(x))
\end{array}$$

Now, we prove that the compositions of the functions are homotope to the identity. For the first direction, we use the fact that, for $z : P(a)$, we have that $j^R_{\lambda f. j_z^R(a,a,f)}(a,a,1_a)(1_a)$ is judgmentally equal to z.

$$\frac{\dots}{\Gamma^\circ, a \overset{\circ}{:} A, P \overset{\circ}{:} A^- \to \mathcal{U}_k^\circ \vdash \lambda z \xrightarrow{\varphi} z'.\varphi : \Pi_{z \xrightarrow{\varphi} z' : P(a)^1} \hom_{P(a)}(j^R_{\lambda f. j_z^R(a,a,f)}(a,a,1_a)(1_a), z')}$$

Unfortunately, proving the other direction is not possible yet. We would have to construct an element of the following type :

$$\Pi_{F \xrightarrow{\varphi} F' : (\Pi_{x \xrightarrow{f} y : A^1} (\hom_A(y,a) \to P(x)))^1} \hom_{\Pi_{x \xrightarrow{f} y : A^1} (\hom_A(y,a) \to P(x))}(\lambda fp. j^R_{\lambda f. j^R_{F(1_a, 1_a)}(x,a,f)}(y,a,p)(f), F'),$$

which is not feasible, as we can't use directed function extensionality on non-covariant functions.