Automata and FO$[\mathbb{R}, +, <]$

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Problem

Strong logic	Weak Logic	Weaker Logic
$\mathrm{FO}[\mathbb{R},+,<,X_b]$	$\mathrm{FO}[\mathbb{R},<,+]$	$\mathrm{FO}[\mathbb{R},<]$
$\|\|$	$\cap$	
Base b-Büchi automata	Weak-Büchi automata	

Question

Decide in *polynomial time*, given R definable in a strong logic, if it is also definable in a weaker logic.

Language

- $d \in \mathbb{N}$ the dimension
- $b \geq 2$ the basis

Definition (Langage)

Alphabet $[0, b-1]^d \cup \{\cdot\}$.

Example

$b = 2, d = 2$

$$\begin{pmatrix} 3.5 \\ 4.1 \end{pmatrix} = \begin{pmatrix} 011.0\overline{1111} \\ 100.0\overline{0011} \end{pmatrix} = \begin{pmatrix} 011.1\overline{0000} \\ 100.0\overline{0011} \end{pmatrix} \tag{1}$$

Automata reading d-tuples of integers

Definition (Non-deterministic Büchi Automata)

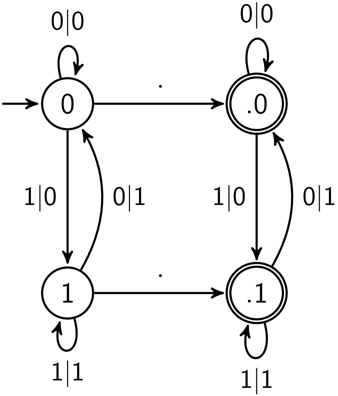
$A = (Q, [0, b-1]^d \cup \{\cdot\}, \delta, q_0, F)$
A reads a d-tuple of base b integers least digit first.

- $|A| = \{w \in ([0, b-1]^d)^*.([0, b-1]^d)^\omega \mid \exists q_0, \ldots, q_n, \ldots \text{a path}, \forall n.\exists m \geq n, q_m \in F\}$:
- $|\overline{A}| \subseteq \mathbb{R}^d$ its accepted set of reals.

Definition ((deterministic) Weak Büchi Automata)

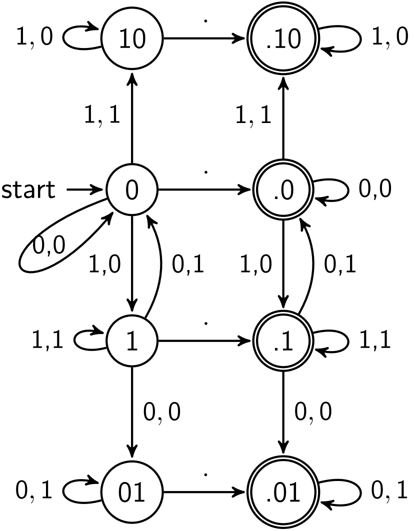
For any strongly connected component S, either $S \subseteq F$ or $S \cap F = \emptyset$.

Almost $\{(x, y) \in \mathbb{R}^2 \mid x = 2y\}$



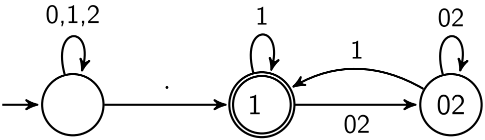
(Almost) Base 2 weak-Büchi automata. FO$[\mathbb{R}, +]$
$(w, 0w)$, $w \in ([0, b-1]^d)^\omega$.

$\{(x, y) \in \mathbb{R}^2 \mid x = 2y\}$



Base 2 weak-Büchi automaton. FO[$\mathbb{R}, +$]

Infinite number of 1



Base 3 Büchi automaton.
$\phi(x) = \forall z.\exists y < z.X_3(x, z, 1) \in \mathrm{FO}[\mathbb{R}, X_3, +]$

FO$[\mathbb{R}, +, <, X_b]$

Definition

$X_b(x, u, k)$ if

$$\begin{array}{rcccccccc} u = & 0 & \cdots & 0 & . & 0 & \cdots & 0 & 1 & \cdots \\ x = & & \cdots & & . & & \cdots & & k & \cdots \end{array}$$

Theorem (Boigelot, Rassart, Wolper)

For all $R \subseteq \mathbb{R}^d$, $b \geq 2$ the following are equivalent

1. *Accepted by a base b-Büchi automaton,*
2. *Definable in* FO$[+, <, X_b]$,
3. *Definable in* FO$[+, <, X_{b'}]$, *b and b' have the same prime factor.*

Same prime factor condition is not "multiplicativaly independant".

FO$[\mathbb{R}^+, <, +]$

Theorem (Boigelot, Brusten, Leroux -Boigelot Jodogne, Wolper)

1. *b- and b'−recognizable where p prime divides b and not b',*
2. *b-recognizable for all $b \in \mathbb{N}$*
3. *b-recognizable by weak Büchi automaton for all $b \in \mathbb{N}$,*
4. *Definable in* FO$[\mathbb{R}^+, <, \mathbb{N}, +, 1]$.

Theorem (Weispfenning)

FO$[\mathbb{R}^+, \mathbb{N}, +, <, \mathrm{mod}, \lfloor \cdot \rfloor]$ *has quantifier elimination.*

Main Tool: Residual

Example

In $[0,1]^2$

- $\phi = x_1 = 2x_2$

Main Tool: Residual

Example

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- $\phi = x_1 = 2x_2$
- $L = A^\omega + A^*(1,0)(1,1)^\omega$ with $A = (0,0) + (1,0)(1,1)^*(0,1)$.

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w	$([0,b-1]^d)^\omega$ $w^{-1}L$	$\Sigma_0[\mathbb{R},\mathbb{N},+,<]$ $w^{-1}\phi$

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$(1,0)$	$(1,1)^*(0,1)L + (1,1)^\omega$	$x_1 = 2x_2 - 1$

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$(1,1)$	$(1,0)^\omega$	$x_1 = 1 \wedge x_2 = 0$

Theorem

Residual of $\Sigma_0[\mathbb{R},\mathbb{Z},+,<]$ *is in* $\Sigma_0[\mathbb{R},\mathbb{Z},+,<]$.

$\mathrm{FO}[\mathbb{R}, +]$

- Well studied : e.g. Julien Brusten's Thesis 2011,
- From logic to automata only,
- No algorithm from automata to logic.

Theorem

The two facts are equivalent

- *Polynomial time algorithm* $\mathrm{FO}[\mathbb{R}, +, <]$
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  - *Polynomial time algorithm for* $\mathrm{FO}[\mathbb{N}, +, <]$ *highest-digit first*
  - *Polynomial time algorithm for* $\mathrm{FO}[[0, 1], +, <]$

$\mathrm{FO}[[0,1],+]$

Theorem

We can decide in time $O(n^4)$ if a deterministic weak-Büchi automaton accepts a set definable in $\mathrm{FO}[[0,1],+,<]$ *?*

Idea

1. Recognize affine sets
2. Recognnize polyhedra
3. Recognize boolean combinations of them

Theorem

We can decide in linear time if $R \subseteq \mathbb{R}$ is definable in $\mathrm{FO}[\mathbb{R},+,<]$.