Undecidability of satisfiability of expansions of FO[<] over words with a FO[+]-definable set

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Example - Logic over words

Example (Words containing aa)

$$\exists x.\exists y.(x + 1 = y) \wedge P_a(x) \wedge P_a(y)$$

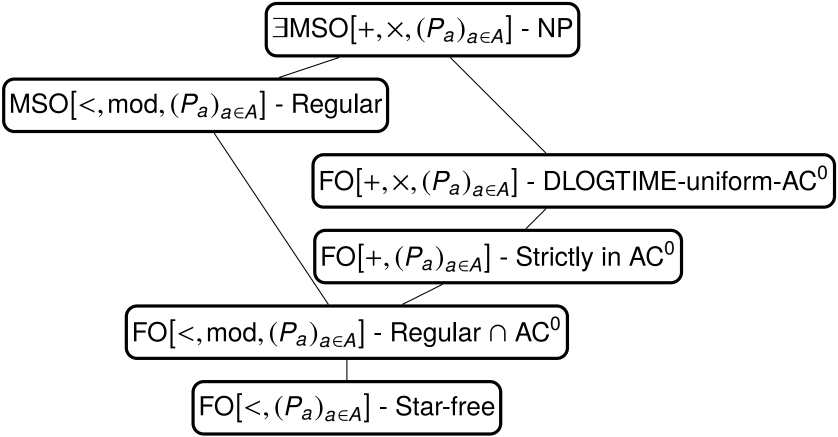
Accepted:

a	*a*	*b*	*a*
0	1	2	3
x	*y*		

Rejected:

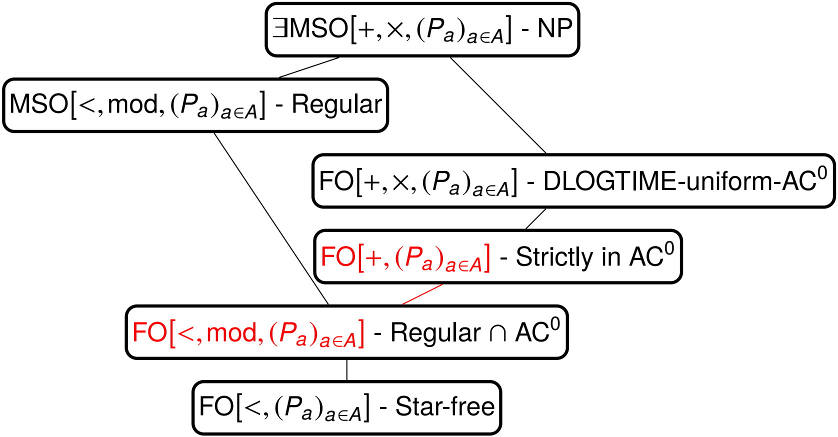
a	*b*	*a*
0	1	2

Logic and set of languages



Logic and set of languages

Logics considered in this presentation



Characterization of FO$[<, \text{mod}, (P_a)_{a \in A}]$

Theorem (Büchi 60)

The regular languages are exactly the MSO$[<, \text{mod}, (P_a)_{a \in A}]$*-definable languages.*

Theorem (Péladeau 1992)

The class $\mathcal{R}$ *of* FO$[<, \text{mod}]$*-definable sets is the greatest class such that* FO$[\mathcal{R}, (P_a)_{a \in A}]$ *only define regular language.*

Problem: Satisfiability

Theorem

The satisfiability of	*is*	
$\text{FO}[+, (P_a)_{a \in A}]$	*undecidable*	*(Lange 2004)*
$\text{FO}[<, \times 2, (P_a)_{a \in A}]$	*?*	
$\text{FO}[<, \text{mod}, (P_a)_{a \in A}]$	*decidable*	*(Büchi 1960)*

Example

1) Satisfiable formula: $\exists x. \exists y. (x + 1 = y) \wedge P_a(x) \wedge P_a(y)$
2) Unsatisfiable formula: $\forall x. \exists y. x + 1 = y$, over finite words,
3) and $\forall x. \exists y. \forall z. P_a(x) \wedge b(y) \vee (x + 2y = z \Longrightarrow P_a(z))$?

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Main result

Theorem

The class $\mathcal{R}$ of $\mathrm{FO}[<, \mathrm{mod}]$*-definable sets is a maximal fragment of* $\mathrm{FO}[+]$ *such that* $\mathrm{FO}[\mathcal{R}, (P_a)_{a \in A}]$*'s satisfiability is decidable.*

Proof sketch

Let R be a $\mathrm{FO}[+]$-definable set which is not $\mathrm{FO}[<, mod, (P_a)_{a \in A}]$-definable.

1) $\mathrm{FO}[R, <]$-define a function $g : \mathbb{N} \to \mathbb{N}$ increasing such that $g(n) - n$ is unbounded.
2) Create a $\mathrm{FO}[<, g, (P_a)_{a \in A}]$-formula which defines encoding of simulation of a halting computation of a 2-counter automaton.

Quantifier elimination

Theorem

The following logics admit quantifier elimination:

$FO[+,<,\mathrm{mod}]$, *(Presburger 29)*

$FO[\{n \mapsto n+c \mid c \in \mathbb{Z}\},<,\mathrm{mod}]$. *(Consequence of Cooper 72)*

Example

$$\exists x.(x \doteq y \wedge x \doteq 0) \vee (x+4 \leq y \wedge x \equiv 0 \mod 2).$$

is equivalent to

$$\bigvee_{i=-2}^{2}\{(\ i \doteq y \wedge\ i \doteq 0) \vee (\ i+4 \leq y \wedge\ i \equiv 0 \mod 2)\} \vee$$
$$\bigvee_{i=-6}^{2}\{(y+i \doteq y \wedge y+i \doteq 0) \vee (y+i+4 \leq y \wedge y+i \equiv 0 \mod 2)\}.$$

Section and diagonal

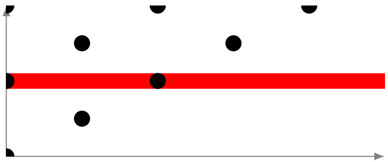


Figure : Section $y = 5$

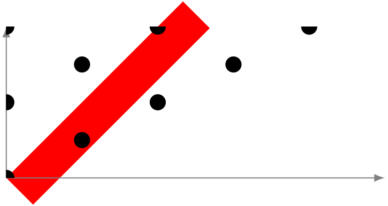


Figure : diagonal $x = y$

Recursive and local characterization - $R \subseteq \mathbb{N}^d$

Theorem (Muchnik 91)

1) *R is* FO$[+]$*-definable if and only if*
2) - *each section of R are* FO$[+]$*-definable,*
   - *each bounded subsets of R, far enough from 0, is equal to another bounded subset of R, and the distance between two such sets is bounded.*

Theorem

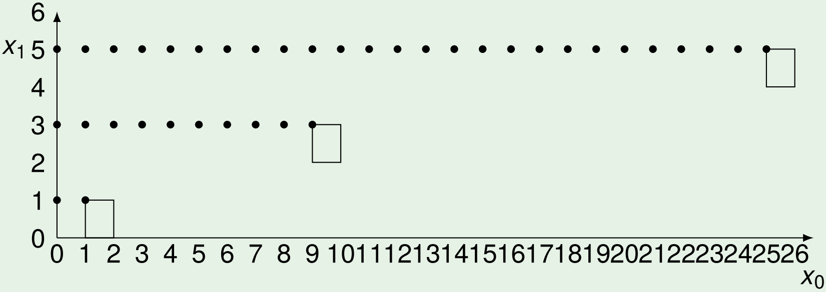
1) $R \in \mathrm{FO}[<, \mathrm{mod}\, m]$*, if and only if*
2) - *Each section and diagonal of R are* FO$[<, \mathrm{mod}\, m]$*-definable and,*
   - *$R \triangle (R + (m, \ldots, m))$, is included in a finite number of section.*

Example - FO[+]

Example ($R = \left\{(x_0^2, x_1) \mid x_0, x_1 \in \mathbb{N}\right\}$)

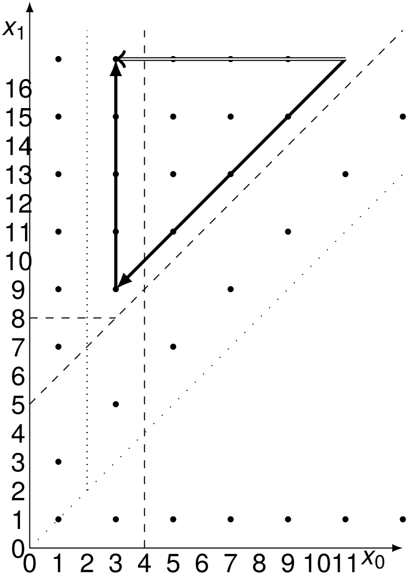
The section $x_1 = 0$ is not FO[+]-definable.

Example ($R = \left\{(x_0, x_1) \in \mathbb{N}^2 \mid x_1 \equiv 1 \mod 2, x_0^2 \leq x_1\right\}$)



The periodicities near the square increase.

Sketch of the proof for FO[$<, \bmod m$]



Reducing the dimension. $R \subseteq \mathbb{N}^d$

Theorem (Michaux-Villemaire 95)

Assume that R is not FO$[+]$*-definable. There exists a* FO$[+, R]$*-definable set of integer which is not* FO$[+]$*-definable.*

The definition can be chosen independently of R.

Theorem

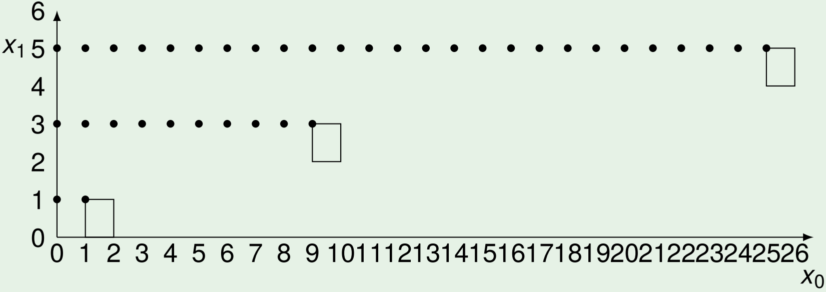
Assume that R is not FO$[<, \mathrm{mod}\, m]$*-definable. There exists a* FO$[<, R]$*-definable set of integer which is not* FO$[<, \mathrm{mod}\, m]$*-definable.*

Theorem

Assume R is FO$[+]$*-definable and not* FO$[<, \mathrm{mod}]$*-definable. There exists* FO$[<, R]$*-definable unary function not* FO$[<, \mathrm{mod}]$*-definable.*

The function is of the form $n \mapsto rn + g(n)$ with $r > 1$ and g bounded.

Example ($R = \left\{(x_0,x_1) \in \mathbb{N}^2 \mid x_1 \equiv 1 \mod 2, x_0^2 \leq x_1\right\}$)



The norm of the positions of the square is not FO[+]-definable.

Open question

Problem

Give a more precise bound on the least logical fragment whose satisfiability is undecidable.

How to consider sets which are not FO[+]*-definable.*

Theorem (Elgot - Rabin (consequence))

Satisfiability of $\mathrm{FO}[\mathbb{N}, <, \{n! \mid n \in \mathbb{N}\}, (P_a)_{a\in A}]$ *and* $\mathrm{FO}\left[\mathbb{N}, <, \{n^2 \mid n \in \mathbb{N}\}, (P_a)_{a\in A}\right]$ *are decidable.*

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Thank you.