Automata and $\mathsf{FO}[\mathbb{R}; +, <, 1]$-definable set of reals

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The main results

Theorem

It is decidable in linear time whether $R \subseteq \mathbb{R}^{\geq 0}$*, accepted by a minimal automaton in base* $b \geq 2$ *is* $\mathsf{FO}[\mathbb{R}; +, <, 1]$*-definable.*

Theorem

Let $\mathcal{A}$ *be a real automaton which accepts a* $\mathsf{FO}[\mathbb{R}; +, <, 1]$*-definable. There exists an existential* $\mathsf{FO}[\mathbb{R}; <, +\mathbb{Q}]$*-formula of length* $O\left(n^3 \log(n) b\right)$ *which defines* $\overline{\mathcal{A}}^{\mathbb{R}}$.

Outline:

Introduction

Methods for similar problems

Description of automata

Characterization of FO[$\mathbb{R}$; +, <, 1]-definable sets

Definition (Simple set)

A simple set $R \subseteq \mathbb{R}^{\geq 0}$ is a finite union of intervals with rational bounds.

Example:

$$\left[3, \frac{20}{3}\right) \cup \left(\frac{22}{3}, +\infty\right).$$

Theorem (Weispfenning 99)

A set is FO[$\mathbb{R}$; +, <, 1]*-definable iff it is simple.*

Theorem (Ferrante, Rackoff, 75)

FO[$\mathbb{R}$; +, <, 1] *admits quantifier elimination.*

Characterization of $\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$-definable sets

Definition (Periodically simple set)

A periodically simple set $R \subseteq \mathbb{R}^{\geq 0}$ is a set of the form $A \cup (m\mathbb{N} + B)$ with A and B simple and m rational.

Example:

$$\{3\} \cup \left(4\mathbb{N} + \left(2, \frac{5}{2}\right)\right)$$

Theorem (Weispfenning 99)

A set is $\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$*-definable iff it is periodically simple.*

Theorem (Weispfenning 99)

$\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1, \lfloor\rfloor, \mathsf{mod}]$ *admits quantifier elimination.*

Representation of reals.

	Base 2-expansion	Base 3-expansion
$\frac{2}{3}$	$\bullet(10)^\omega$	$\bullet 2(0)^\omega$
		$\bullet 1(2)^\omega$
$\frac{5}{4}$	$1 \bullet 01(0)^\omega$	$1 \bullet (02)^\omega$
	$1 \bullet 00(1)^\omega$	

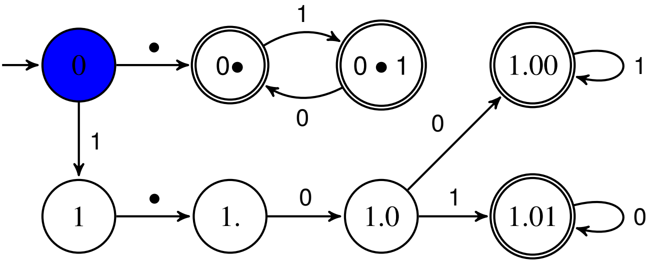

Figure: $\left\{\frac{2}{3}, \frac{5}{4}\right\}$ in base 2

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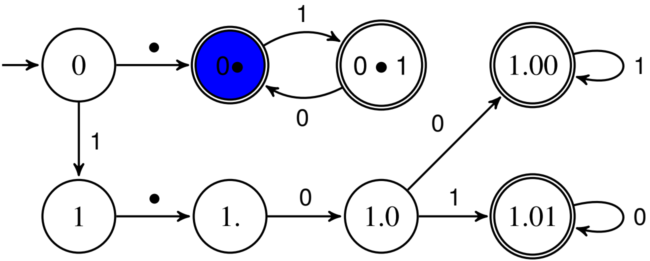

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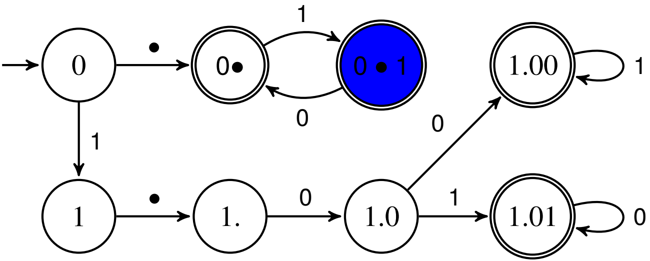

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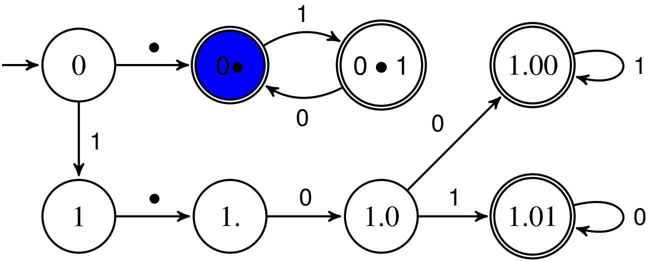

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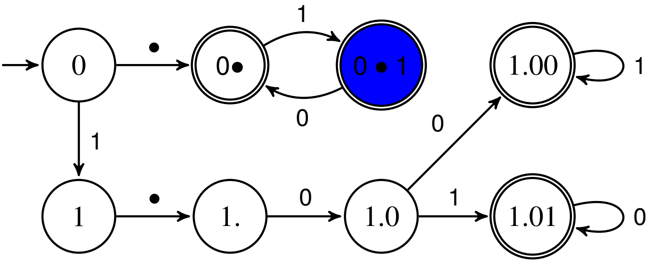

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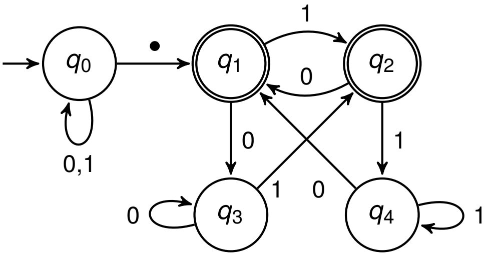
Expressivity of base b-automata

Theorem (Boigelot, Rassart, Wolper, 98)

A set $R \subseteq (\mathbb{R}^{\geq 0})^d$ is accepted by an automaton in base b iff it is $\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, X_b, 1]$-definable.

$X_b(x, u, k)$ holds iff $u = 2^z$ for $z \in \mathbb{Z}$ and the z-th bit of x is k.

Example: Digit with infinitely many 0 and 1



$$\forall r > 0.\exists u < n.X_2(x, u, 1) \wedge \exists u < n.X_2(x, u, 0)$$

Sets accepted in multiple basis.

Theorem (Boigelot, Brusten, Leroux, 09)

A set $R \subseteq (\mathbb{R}^{\geq 0})^d$ is accepted by automata in all basis iff it is $\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$*-definable.*

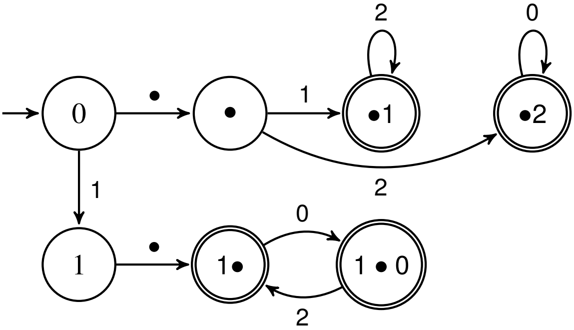


Figure: $\left\{\frac{2}{3}, \frac{5}{4}\right\}$ in base 3

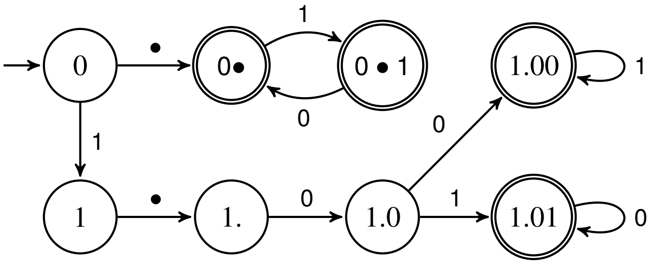
Weak Büchi Automata

Definition

An automaton is weak if all or none of the states of a strongly connected components are accepting.

Theorem (Löding - 2001)

Weak Büchi automata admits minimization in time $O(n\log(n))$.



Properties of $\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$

Theorem (Boigelot, Brusten,Bruyère, 08)

$\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$-*definable sets are accepted by weak-Büchi automata.*

Example

A weak Büchi automaton accepting a non-$\mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, <, 1]$-definable set.

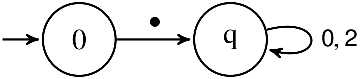


Figure: Reals without 1 in its base-3 representation

Similar problem for FO[+] and integers

Theorem (Recall: main result)

It is decidable in linear time whether $R \subseteq \mathbb{R}^{\geq 0}$*, accepted by a minimal automaton in base* $b \geq 2$ *is* $\mathsf{FO}[\mathbb{R}; +, <, 1]$*-definable.*

Theorem

Whether $R \subseteq \mathbb{N}^d$ *accepted by a minimal automaton is* $\mathsf{FO}[\mathbb{N}; +]$*-definable is decidable.*

dimension d	*time complexity*	
1		*(Honkala 86)*
any	*3EXP*	*(Muchnik 91)*
any	*polynomial*	*(Leroux 05)*
1	*quasi-linear*	*(Marsault-Sakarovitch 13)*

Honkala's algorithm

Theorem (Honkala 86)

Whether $R \subseteq \mathbb{N}$ *accepted by an automaton is* $\mathsf{FO}[\mathbb{N}; +]$*-definable is decidable.*

Lemma

The threshold t *and the period* p *is at most* b^n *where* n *is the number of states of the automaton* $\mathcal{A}$*.*

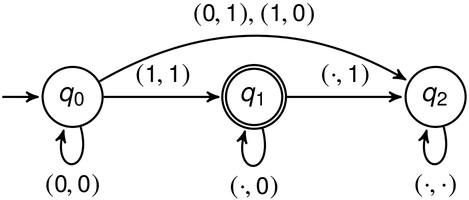
Algorithm

(1) Runs over all sets $R \subseteq \mathbb{N}$ with threshold and period less than b^n.

(2) Generates the minimal automaton $\mathcal{A}_R$ in base b accepting R.

(3) Accepts if $\mathcal{A}_R = \mathcal{A}$.

Example of application of Honkala's method



If this automaton accepts an ultimately periodic set, the threshold and period are at most $2^3 = 8$.

Proposition

This methods allows to recognize any class C of languages such that:

–there exists a function s from language to integers,

–for all language $L \in C$, the minimal automaton accepting L has at least $s(L)$ states,

–For each $c \in \mathbb{N}$, the set $\{L \mid s(L) = c\}$ is finite and computable.

Application

For C the periodically simple sets, take $s(R)$ as the length of the greatest rational.

Muchnik's algorithm

Theorem (Muchnik 91)

There exists $\phi_d \in \mathsf{FO}[\mathbb{N}; +, R]$ *which states "*$R \subseteq \mathbb{N}^d$ *is* $\mathsf{FO}[\mathbb{N}; +]$ *– definable".*

Corollary (Muchnik 91)

Whether $R \subseteq \mathbb{N}^d$ *accepted by an automaton* $\mathcal{A}$ *is* $\mathsf{FO}[\mathbb{N}; +]$*-definable is decidable in* $\mathsf{4EXP}$*-time.*

Algorithm

(1) Transform ϕ_d into an automaton $\mathcal{A}'$ where R is encoded by $\mathcal{A}$.

(2) Accepts if $\mathcal{A}'$ accepts a non-empty language.

Example of application of Muchnik's example.

This method works for any class C of set of tuple of numbers such that there exists a $\phi \in \mathsf{FO}[\mathbb{R}, \mathbb{Z}; +, X_b, R]$ which holds if and only if $R \in C$.

Example

Whether an automaton accepts a subsemigroup of $(\mathbb{R}^d, +)$ is decidable.

$$\forall \begin{matrix} x_1, \ldots, x_d \\ y_1, \ldots, y_d \end{matrix} . \left\{ \begin{matrix} (x_1, \ldots, x_d) \in R \wedge \\ (y_1, \ldots, y_d) \in R \end{matrix} \right\} \Longrightarrow (x_1 + y_1, \ldots, x_d + y_d) \in R$$

asserts that R is a subsemigroup of $(\mathbb{R}^d, +)$.

Marsault-Sakarovitch's algorithm

Theorem (Marsault-Sakarovitch 13)

Whether $R \subseteq \mathbb{N}$ accepted by a minimal automaton $\mathcal{A}$ is $\mathsf{FO}[\mathbb{N}; +]$-definable is decidable in linear time.

Proposition

Let $\mathbb{L}$ be a class of language and $\mathbb{A}$ a class of automata such that:

(1) each language of $\mathbb{L}$ is accepted by an automaton of $\mathbb{A}$,

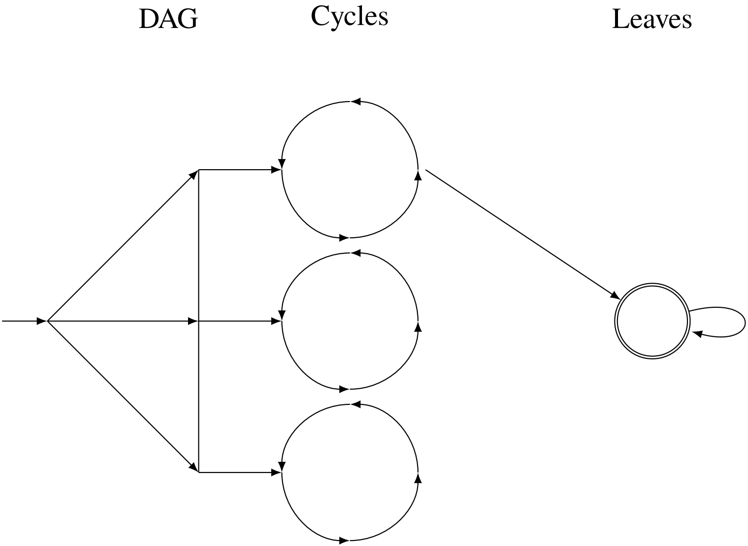
(2) all automata of $\mathbb{A}$ accepts a language belonging to $\mathbb{L}$,

(3) $\mathbb{A}$ is closed under quotient and

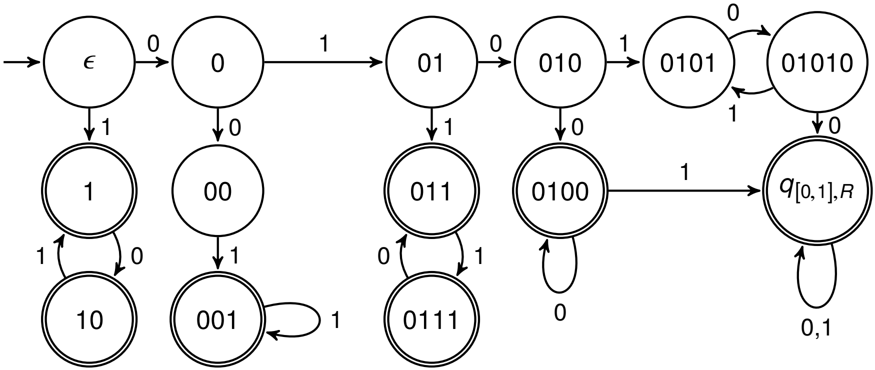
(4) it is decidable in time $t(n)$ whether an automaton belongs to $\mathbb{A}$.

It is decidable in time $t(n)$ whether a minimal automaton accepts a language of $\mathbb{L}$.

Form of an automaton accepting a FO[$\mathbb{R}$; +, <, 1]-definable subset of [0, 1]



Example $\left[\frac{1}{4}, \frac{1}{3}\right) \cup \left\{\frac{11}{24}, \frac{2}{3}\right\}$



Lemma

Cyclic states accepts a set which is a finite union of elements of $\{[0, r), \{r\}, (r, 1]\}$ *for some rational* r.

Describing $\mathbb{A}$.

Using Marsault-Sakarovitch's method, $\mathbb{A}$ is the set of automata satisfying:

(1) At most two leaves, accepting $\emptyset$ and $[0, 1]$,

(2) Let C be a strongly connected component which is not a leaf. Let $q \in C$, $a \in [b-1]$. If $\delta(q, a)$ does not belong to C then it belongs to a leaf.

(3) C is a cycle.

(4) ...

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(2) Let C be a strongly connected component which is not a leaf. Let $q \in C$, $a \in [b-1]$. If $\delta(q, a)$ does not belong to C then it belongs to a leaf.

(3) For each q in C, there exists a single digit $d_q \in [b-1]$ such that $\delta(q, d_q) \in C$.

(4) for each $q, q' \in C$, for each $a < d_q$, $a' < d_{q'}$, $\delta(q, a) = \delta(q', a')$.

Future research

Solving the similar problem for:

$R \subseteq [0, 1]^d$,

$R \subseteq \mathbb{N}^d$ and the logic $\mathsf{FO}[<, \mathsf{mod}]$.