Satisfiability and Spectra of some logics.

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Example of Spectra

Example (Involution without fixpoint)

Let ψ be the formula $\forall y.\{f(f(y)) = y \wedge f(y) \neq y\}$ with f an unary function.

- Satisfiable over $\{1, \ldots, 2N\}$ and over $\mathbb{N}^*$ when f is such that:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases} .$$

- Unsatisfiable over $\{1, \ldots, 2N+1\}$.

The Spectrum of ψ is $2\mathbb{N}^*$.

Definition

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For ϕ a formula, let

$$\mathrm{SP}(\phi) = \{n \in \mathbb{N} \mid \exists U. |U| = n, U \models \phi\}.$$

Remark

The Spectrum of ϕ is not empty if and only if ϕ is finitely satisfiable.

Some results (f uninterpreted)

Theorem (Ehrenfeucht)

$FO[f(x)]$ *is decidable.*

Theorem (Durand, Fagin and Loescher)

$FO[f(x)]$*'s spectra are exactly the ultimately periodic sets.*

Theorem (Gurevich)

$FO[f_0(x), f_1(x)]$ *is undecidable.*

Theorem (Folklore)

$\Sigma_1\left[(f_{i,j}(x_0,\ldots,x_{j-1}))_{i,j\in\mathbb{N}}\right]$ *is decidable.*

Arithmetic

Theorem

Let $g : \mathbb{N} \to \mathbb{N}$ *be an increasing such that* $\exists^\infty n.g(n+1) > g(n) + 1$.

- $\exists\mathrm{MSO}[+1, g(x)]$ *is undecidable.*
- $\mathrm{FO}[+1, g(x)]$ *is undecidable over words.*

Theorem

An encoding of the set of halting times of a non-deterministic 2-counter automaton is a $\exists\mathrm{MSO}[+1, g(x)]$*-spectrum.*

Example (Possible values for g)

- $\times 2$,
- $n + \log(n)$.

b-Regular Sets

Definition (b-regular)

Let $b \geq 2$. $S \subseteq \mathbb{N}$ is b-regular if and only if S is accepted by an automaton reading numbers in base b.

Theorem (Büchi-Bruyère)

$S \subseteq \mathbb{N}$ is b-regular iff it is $\mathrm{FO}[+, V_b]$*-definable.*

Corollary (Büchi)

Büchi Arithmetic – $\mathrm{FO}[+, V_b]$ *– is decidable.*

Theorem

All b-regular sets are $\exists\mathrm{MSO}\,[+1, \times b]$*-spectra.*

Arithmetic with an uninterpreted function

Theorem (Shostak)

$\Sigma_1[+, <, (f_{i,j}(x_1, \ldots, x_j))_{i,j \in \mathbb{N}}]$ *is decidable.*

Theorem (Thomas)

$\mathrm{MSO}[+1, f(x)]$ *is undecidable.*

Arithmetic with an uninterpreted function

Theorem (Shostak)

$\Sigma_1[+,<,(f_{i,j}(x_1,\ldots,x_j))_{i,j\in\mathbb{N}}]$ *is decidable.*

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$\mathrm{FO}[+1, f(x)]$ *is undecidable.*

Theorem (Thomas)

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Theorem

$\mathrm{SP}(\exists\mathrm{MSO}[+1, \times b]) \subseteq \mathrm{SP}(\mathrm{FO}[+1, f(x)])$.

Open problem

- More precise characterization of the spectra.
- Considering universal (existential) formulas only.

Thank you.