Hybrid Intersection Types for PCF

Pablo Barenbaum[1] Delia Kesner[2] Mariana Milicich[2*]

[1]UNQ (CONICET) and ICC, UBA, Argentine

[2]Université Paris-Cité, CNRS, IRIF, France

Logic for Programming, Artificial Intelligence and Reasoning
May 2024

* *This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 945332*

Evaluation strategies

Let f x = f x and g y = 42

Call-by-Name	Call-by-Value
> g (f id)	> g (f id)
> 42	> Infinite loop!

Evaluation strategies govern the behavior of programs.

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Evaluation strategies govern the behavior of programs.

May/Must termination

P **may terminate** if there is a computation from P to NF.

P **(must) terminate** if there is no infinite computation from P.

Intersection types

Syntax of types

$$\tau, \sigma ::= \alpha \mid \tau \to \sigma \mid \tau \cap \sigma$$

Intersection types

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Theorem (Intersection types characterize termination)

Type systems $\quad \vdash P : \tau \iff P \text{ } \textit{terminates}$ $\quad$ **Programming languages**

- *CBN* *(CoppoDezani80, Gardner94)*
- *CBV* *(Ehrhard12)*

We generalize this result to an **explicitly hybrid** evaluation strategy, characterizing termination in **PCFH** through the **type system** $\mathcal{H}$.

PCF$_{\text{H}}$

Syntax

Terms: $t, s, u ::= x \mid \lambda x.\, t \mid t\, s \mid \mathbf{0} \mid \mathbf{S}(t) \mid \mathrm{if}(t, s, x.\, u) \mid \mathtt{fix}\ x.\, t$
Values: $v ::= \lambda x.\, t \mid k \qquad k ::= \mathbf{0} \mid \mathbf{S}(k)$

Semantics

$$
\begin{array}{rcll}
(\lambda x.\, t)\, v & \to_{\mathtt{B}} & t\{x := \boldsymbol{v}\} & (\text{CBV}) \\
\mathrm{if}(\mathbf{0}, t, x.\, s) & \to_{\mathtt{IO}} & t & \\
\mathrm{if}(\mathbf{S}(k), t, x.\, s) & \to_{\mathtt{IS}} & s\{x := \boldsymbol{k}\} & (\text{CBV}) \\
\mathtt{fix}\ x.\, t & \to_{\mathtt{F}} & t\{x := \mathtt{fix}\ \boldsymbol{x.\, t}\} & (\text{CBN})
\end{array}
$$

B, IO, IS, and F are **names**.

▸ Further details

PCF$_H$

Example

Let $\texttt{double} := \texttt{fix}\ \mathsf{rec}.\, \lambda n.\, \mathsf{if}\big(n, \mathbf{0}, m.\, \mathbf{S}(\mathbf{S}(\mathsf{rec}\ m))\big)$

$$\texttt{double}\, \mathbf{S}(\mathbf{0})$$

PCF$_\text{H}$

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Example

Let $\texttt{double} := \texttt{fix}\ \mathsf{rec}.\,\lambda n.\,\mathsf{if}(n, \mathbf{0}, m.\,\mathbf{S}(\mathbf{S}(\mathsf{rec}\, m)))$

$$(\mathit{CBN}) \quad \texttt{double}\,\mathbf{S}(\mathbf{0}) \qquad \to_{\texttt{F}}$$

$$(\lambda n.\,\mathsf{if}(n, \mathbf{0}, m.\,\mathbf{S}(\mathbf{S}(\texttt{double}\, m))))\,\mathbf{S}(\mathbf{0})$$

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PCF$_{\mathrm{H}}$

Example

Let $\texttt{double} := \texttt{fix}\ \mathrm{rec}.\ \lambda n.\ \mathrm{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\mathrm{rec}\ m)))$

$$(CBN) \quad \texttt{double}\ \mathbf{S}(\mathbf{0}) \qquad \to_{\texttt{F}}$$

$$(CBV) \quad (\lambda n.\ \mathrm{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m))))\ \mathbf{S}(\mathbf{0}) \qquad \to_{\texttt{B}}$$

$$\mathrm{if}(\mathbf{S}(\mathbf{0}), \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m)))$$

PCF$_{\text{H}}$

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$\mathrm{PCF_H}$

Example

Let $\texttt{double} := \texttt{fix}\ \mathrm{rec}.\ \lambda n.\ \mathrm{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\mathrm{rec}\ m)))$

(*CBN*)	$\texttt{double}\ \mathbf{S}(\mathbf{0})$	$\to_{\texttt{F}}$
(*CBV*)	$(\lambda n.\ \mathrm{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m))))\ \mathbf{S}(\mathbf{0})$	$\to_{\texttt{B}}$
(*CBV*)	$\mathrm{if}(\mathbf{S}(\mathbf{0}), \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m)))$	$\to_{\texttt{IS}}$
	$\mathbf{S}(\mathbf{S}(\texttt{double}\ \mathbf{0}))$	

PCF$_\mathsf{H}$

Example

Let $\texttt{double} := \texttt{fix}\ \mathsf{rec}.\, \lambda n.\, \mathsf{if}(n, \mathbf{0}, m.\, \mathbf{S}(\mathbf{S}(\mathsf{rec}\ m)))$

$$
\begin{array}{lll}
(\mathit{CBN}) & \texttt{double}\ \mathbf{S}(\mathbf{0}) & \to_{\texttt{F}} \\
(\mathit{CBV}) & (\lambda n.\, \mathsf{if}(n, \mathbf{0}, m.\, \mathbf{S}(\mathbf{S}(\texttt{double}\ m))))\ \mathbf{S}(\mathbf{0}) & \to_{\texttt{B}} \\
(\mathit{CBV}) & \mathsf{if}(\mathbf{S}(\mathbf{0}), \mathbf{0}, m.\, \mathbf{S}(\mathbf{S}(\texttt{double}\ m))) & \to_{\texttt{IS}} \\
(\mathit{CBN}) & \mathbf{S}(\mathbf{S}(\texttt{double}\ \mathbf{0})) & \to_{\texttt{F}} \\
 & \ldots & \to^{*}
\end{array}
$$

PCF$_{\text{H}}$

Example

Let $\texttt{double} := \texttt{fix}\ \text{rec}.\ \lambda n.\ \text{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\text{rec}\ m)))$

$$
\begin{array}{lll}
(\mathit{CBN}) & \texttt{double}\ \mathbf{S}(\mathbf{0}) & \to_{\texttt{F}} \\
(\mathit{CBV}) & (\lambda n.\ \text{if}(n, \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m))))\ \mathbf{S}(\mathbf{0}) & \to_{\texttt{B}} \\
(\mathit{CBV}) & \text{if}(\mathbf{S}(\mathbf{0}), \mathbf{0}, m.\ \mathbf{S}(\mathbf{S}(\texttt{double}\ m))) & \to_{\texttt{IS}} \\
(\mathit{CBN}) & \mathbf{S}(\mathbf{S}(\texttt{double}\ \mathbf{0})) & \to_{\texttt{F}} \\
 & \dots & \to^{*} \\
 & \mathbf{S}(\mathbf{S}(\mathbf{0})) &
\end{array}
$$

$\mathbf{S}(\mathbf{S}(\mathbf{0}))$ is reached after **6 steps**.

Type system $\mathcal{H}$

Definitions

Typing judgments: $\Phi; \Gamma \vdash^{\mathfrak{m}} t : \mathcal{S}$

- Φ: variables bound by fixed-point operators (CBN)
- Γ: variables bound by abstractions and conditionals (CBV)
- $\mathfrak{m}$: multiset of names (*aka* multi-counter)

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Key details

- One typing context (Φ/Γ) for each (CBN/CBV) behavior.
- Axioms for typing variables:

$$\frac{}{x : \{\!\{\mathcal{T}\}\!\}; \varnothing \vdash^{[]} x : \mathcal{T}} \qquad \frac{}{\varnothing; x : \mathcal{T} \vdash^{[]} x : \mathcal{T}}$$

$$\boxed{\vdash P : \tau \iff P \text{ terminates}}$$

Theorem (Upper bounds)

There exists $\Phi, \Gamma, \mathcal{S}$ such that $\Phi; \Gamma \vdash^{\mathfrak{m}} t : \mathcal{S}$

$$\Updownarrow$$

There exists a reduction sequence $t = t_0 \to t_1 \ldots \to t_n$ where t_n is in normal form and $n \leq \#(\mathfrak{m})$

Type system $\mathcal{H}$

Results

$$\vdash P : \tau \iff P \text{ terminates}$$

Theorem (Upper bounds)

There exists $\Phi, \Gamma, \mathcal{S}$ *such that* $\Phi; \Gamma \vdash^{\mathfrak{m}} t : \mathcal{S}$

$$\Updownarrow$$

There exists a reduction sequence $t = t_0 \to t_1 \ldots \to t_n$ *where* t_n *is in normal form and* $n \leq \#(\mathfrak{m})$

Corollary (Exact measures)

$$\varnothing; \varnothing \vdash^{\mathfrak{m}} t : [\,]$$

$$\Updownarrow$$

There exists a reduction sequence $t = t_0 \to_{\rho_1} t_1 \ldots \to_{\rho_n} t_n$ *where* t_n *is in normal form and* $[\rho_1, \ldots, \rho_n] = \mathfrak{m}$

Type system $\mathcal{H}$

Results

Standard techniques suffice the results:

- Subject reduction
- Subject expansion
- **Two** substitution lemmas, one for CBN and another for CBV.

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System $\mathcal{H}$ characterizes **must termination** of PCFH:

1. Our previous Thm. and Cor. shows *may* termination.
2. PCFH evaluation satisfies the **diamond property**,
   so **all** normalization sequences have the same length.

Tightness

The notion of emptiness is related to **tightness** (AccattoliGraham-LengrandKesner18).

- **Exact** information from *minimal* typing derivations.
- $\emptyset; \emptyset \vdash^{\mathfrak{m}} t : [\,]$ is tight
  $\emptyset; \emptyset \vdash^{\mathfrak{n}} t : [Nat \to Nat]$ is not

Type system $\mathcal{H}$

Example

Recall `double` := `fix` rec. $\lambda n.\,\text{if}(n, \mathbf{0}, m.\,\mathbf{S}(\mathbf{S}(\text{rec}\,m)))$
For typing `double` $\mathbf{S}(\mathbf{0})$,
consider π_1 deriving $\varnothing; \varnothing \vdash^{[\,]} \mathbf{S}(\mathbf{0}) : \mathcal{N}_1$ and

$$\pi_{\text{tree}_{\text{base}}} \left\{ \quad \dfrac{\begin{matrix}\vdots \\ \varnothing; \varnothing \vdash^{[\ldots]} \lambda n.\,\text{if}(n, \mathbf{0}, m.\,\mathbf{S}(\mathbf{S}(\text{rec}\,m))) : [\mathcal{M}_0 \to [\,]]\end{matrix}}{\varnothing; \varnothing \vdash^{[\ldots]} \texttt{double} : [\mathcal{M}_0 \to [\,]]} \right.$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \text{rec} : \{\!\{[\mathcal{M}_0 \to [\,]]\}\!\}; \varnothing \vdash^{[\ldots]} \lambda n.\,\text{if}(n, \mathbf{0}, m.\,\mathbf{S}(\mathbf{S}(\text{rec}\,m))) : [\mathcal{N}_1 \to [\,]]\end{matrix} \quad \pi_{\text{tree}_{\text{base}}}}{\varnothing; \varnothing \vdash^{[\ldots]} \texttt{double} : [\mathcal{N}_1 \to [\,]]} \qquad \pi_1}{\varnothing; \varnothing \vdash^{[\mathtt{B},\mathtt{F},\mathtt{IS},\mathtt{B},\mathtt{F},\mathtt{I0}]} \texttt{double}\,\mathbf{S}(\mathbf{0}) : [\,]}$$

The multi-counter has cardinality **6**.

Conclusions:

- System $\mathcal{H}$ provides a **quantitative interpretation** for PCFH:
  - **Upper** bounds for the number of steps required to reach normal form (Thm. 2).
  - **Exact** measures by restricting typing derivations (Cor. 3).
- We generalize standard proof techniques.

Future work:

- Extend our results to other hybrid evaluation strategies.
- Solve the *inhabitation problem* in a hybrid-type setting.
- Embed PCFH into the *Bang Calculus* (unifying framework).


Thank you! Questions?

Programming Computable Functions (PCF)

G. Plotkin (1977)

Syntax:

$$t, s, u ::= \ldots \mid \underline{n} \mid \mathbf{S}(t) \mid \text{if}(t, s, x.\, u) \mid \texttt{fix}\ x.\, t$$

where $n \in \mathbb{N}$.

Semantics:

$$\frac{}{(\lambda x.\, t)\, s \to t\{x := s\}} \qquad \frac{}{\texttt{fix}\ x.\, t \to t\{x := \texttt{fix}\ x.\, t\}} \qquad \frac{}{\mathbf{S}(\underline{n}) \to \underline{n+1}}$$

$$\frac{}{\text{if}(\underline{\mathbf{0}}, t, x.\, s) \to t} \qquad \frac{}{\text{if}(\underline{n+1}, t, x.\, s) \to s\{x := \underline{n}\}}$$

$$\frac{t \to t'}{t\, s \to t'\, s} \qquad \frac{t \to t'}{\mathbf{S}(t) \to \mathbf{S}(t')} \qquad \frac{t \to t'}{\text{if}(t, s, x.\, u) \to \text{if}(t', s, x.\, u)}$$

Back to PCFH

Evaluation rules for PCF$_\mathrm{H}$

Let $\rho \in \{\mathtt{B}, \mathtt{IO}, \mathtt{IS}, \mathtt{F}\}$ be a **name**.

$$\frac{}{(\lambda x.\, t)\, v \to_{\mathtt{B}} t\{x := v\}} \qquad \frac{}{\mathrm{if}(\mathbf{0}, t, x.\, s) \to_{\mathtt{IO}} t}$$

$$\frac{}{\mathrm{if}(\mathbf{S}(k), t, x.\, s) \to_{\mathtt{IS}} s\{x := k\}} \qquad \frac{}{\mathtt{fix}\ x.\, t \to_{\mathtt{F}} t\{x := \mathtt{fix}\ x.\, t\}}$$

$$\frac{t \to_\rho t'}{t\, s \to_\rho t'\, s} \qquad \frac{s \to_\rho s'}{t\, s \to_\rho t\, s'}$$

$$\frac{t \to_\rho t'}{\mathbf{S}(t) \to_\rho \mathbf{S}(t')} \qquad \frac{t \to_\rho t'}{\mathrm{if}(t, s, x.\, u) \to_\rho \mathrm{if}(t', s, x.\, u)}$$

System $\mathcal{H}$

Typing rules (1)

$$\frac{}{\varnothing; x : \mathcal{T} \vdash^{[]} x : \mathcal{T}} \qquad \frac{}{x : \{\!\{\mathcal{T}\}\!\}; \varnothing \vdash^{[]} x : \mathcal{T}}$$

$$\frac{(\Phi_i; \Gamma_i, x : \mathcal{T}_i^? \vdash^{\mathfrak{m}_i} t : \mathcal{S}_i)_{i \in I}}{+_{i \in I} \Phi_i; +_{i \in I} \Gamma_i \vdash^{+_{i \in I} \mathfrak{m}_i} \lambda x. t : [\mathcal{T}_i^? \to \mathcal{S}_i]^{\mathsf{abs}}_{i \in I}}$$

$$\frac{\Phi_1; \Gamma_1 \vdash^{\mathfrak{m}_1} t : [\mathcal{T}^? \to \mathcal{S}]^{\mathsf{abs}} \qquad \mathcal{T}^? \lhd \mathcal{T} \qquad \Phi_2; \Gamma_2 \vdash^{\mathfrak{m}_2} s : \mathcal{T}}{\Phi_1 + \Phi_2; \Gamma_1 + \Gamma_2 \vdash^{[\mathrm{B}] + \mathfrak{m}_1 + \mathfrak{m}_2} t\, s : \mathcal{S}}$$

$$\frac{\Phi, x : \{\!\{\mathcal{T}_i\}\!\}_{i \in I}; \Gamma \vdash^{\mathfrak{m}} t : \mathcal{S} \qquad (\Phi_i; \Gamma_i \vdash^{\mathfrak{m}_i} \mathtt{fix}\ x. t : \mathcal{T}_i)_{i \in I}}{\Phi +_{i \in I} \Phi_i; \Gamma +_{i \in I} \Gamma_i \vdash^{[\mathrm{F}] + \mathfrak{m} +_{i \in I} \mathfrak{m}_i} \mathtt{fix}\ x. t : \mathcal{S}}$$

System $\mathcal{H}$

Typing rules (2)

$$\frac{}{\varnothing; \varnothing \vdash^{[]} \mathbf{0} : [0]^{\mathsf{nat}}_{i \in I}} \qquad \frac{\Phi; \Gamma \vdash^{\mathfrak{m}} t : \mathcal{N} \quad \mathcal{N} = +_{i \in I} \mathcal{N}_i}{\Phi; \Gamma \vdash^{\mathfrak{m}} \mathbf{S}(t) : [\mathbb{S}(\mathcal{N}_i)]^{\mathsf{nat}}_{i \in I}}$$

$$\frac{\Phi_1; \Gamma_1 \vdash^{\mathfrak{m}_1} t : [0]^{\mathsf{nat}} \quad \Phi_2; \Gamma_2 \vdash^{\mathfrak{m}_2} s : \mathcal{T}}{\Phi_1 + \Phi_2; \Gamma_1 + \Gamma_2 \vdash^{[\mathtt{I0}] + \mathfrak{m}_1 + \mathfrak{m}_2} \mathsf{if}(t, s, x.\, u) : \mathcal{T}}$$

$$\frac{\Phi_1; \Gamma_1 \vdash^{\mathfrak{m}_1} t : [\mathbb{S}(\mathcal{N})]^{\mathsf{nat}} \quad \mathcal{N}^? \lhd \mathcal{N} \quad \Phi_2; \Gamma_2, x : \mathcal{N}^? \vdash^{\mathfrak{m}_2} u : \mathcal{T}}{\Phi_1 + \Phi_2; \Gamma_1 + \Gamma_2 \vdash^{[\mathtt{IS}] + \mathfrak{m}_1 + \mathfrak{m}_2} \mathsf{if}(t, s, x.\, u) : \mathcal{T}}$$