Linear logic and canonical extensions

A topic explored with Sam van Gool and Paul-André Melliès.

Vincent Moreau

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A quick presentation


Now a PhD student at IRIF! On higher-order automata and recursion schemes.

You can meet me in 4033. More on my webpage: irif.fr/~moreau.
Canonical extensions: \( L \leftrightarrow L^\delta \) which is a very special complete lattice.
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Historically developed for modal logic, [JT52]:

$\square, \diamond : L \to L$ lift to maps $L^\delta \to L^\delta$ with good properties.
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\[ \square, \Diamond : L \to L \text{ lift to maps } L^\delta \to L^\delta \text{ with good properties.} \]

Our contribution: \( L^\delta \) can be seen as the fixpoints of an adjunction

\[
\begin{align*}
\begin{array}{cccc}
L^\delta & \downarrow \Box & \downarrow \Diamond & \downarrow L^\delta \\
F_\cup & F_\vee & F_\wedge & F_\cap \end{array}
\end{align*}
\]

\[ \lambda, \lambda' \]
Why is it useful?

Provides a link between algebras and relational semantics:

\[ \text{algebra } \delta \rightarrow \text{perfect algebra } \simeq \text{relational frame} \]
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Today: an application from the article

*Relational semantics for full linear logic* [CGv14]

published by Dion Coumans, Mai Gehrke and Lorijn van Rooijen in 2013.
Linear logic sees formulas as resources.

Definition of linear logic
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Linear logic sees formulas as resources. Linear formulas with atoms in $P$:

$$\phi ::= p \in P$$

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<thead>
<tr>
<th>$\phi \otimes \phi$</th>
<th>$1$</th>
<th>$\phi &amp; \phi$</th>
<th>$\bot$</th>
<th>$\phi \multimap \phi$</th>
<th>$\phi \circ \phi$</th>
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<td>multiplicative</td>
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<th>$\phi &amp; \phi$</th>
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<th>$\phi \oplus \phi$</th>
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Definition of linear logic

Linear logic sees formulas as resources. Linear formulas with atoms in $P$:

$$
\varphi ::= p \in P \\
\mid \varphi \otimes \varphi \mid 1 \mid \varphi \& \varphi \mid \perp \mid \varphi \multimap \varphi \\
\text{multiplicative} \\
\mid \varphi \text{ & } \varphi \mid \top \mid \varphi \oplus \varphi \mid 0 \\
\text{additive} \\
\mid \!\varphi \mid ?\varphi \\
\text{exponentials}
$$

Examples:

$$(A \otimes B) \multimap A \quad (A \& B) \multimap A \quad !A \multimap ?A \quad A \multimap (B \otimes C) \quad !A \multimap !!A$$
Examples

Lafont’s restaurant, from *An Introduction to Linear Logic: Expressiveness and Phase Semantics* [Oka98]:

\[
(Vegetable\ Soup \oplus \ Consommé\ Soup) \& \ Salad \\
\otimes (Fish \& \ Meat) \\
\otimes !\ coffee \\
\otimes (2€ \rightarrow\ Cake \& \ Ice\ cream)
\]
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\[\otimes (\text{Fish} \& \text{Meat}) \]
\[\otimes !\text{coffee} \]
\[\otimes (2€ \rightarrow \text{Cake} \& \text{Ice cream}) \]

The formula

\[\forall x \exists y, x < y\]

does not imply infinity as it may only be used once!
Just like boolean algebras or Heyting algebras, but for MALL.
CL-algebras

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A lattice $A$ with min $0$ and max $\top$ together with

- a commutative monoid operation $\otimes : A \times A \to A$ with unit $1 \in A$
- a map $\rightarrow : A^{\text{op}} \times A \to A$ such that
  \[ a \otimes b \leq c \iff a \leq b \rightarrow c \]
- an element $\bot \in A$ such that $(\cdot)^\bot : a \mapsto a \rightarrow \bot$ is an involution.
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We obtain the $\otimes$ as the De Morgan dual of $\otimes$

\[ a \otimes b = (a^\perp \otimes b^\perp)^\perp \]
Given a valuation $u : P \rightarrow A$, one may define by induction $[\varphi]_u \in A$. 

A formula $\varphi$ is true in $A$ together with the valuation $u$ iff $1 \leq J\varphi_K u$. If $\varphi$ is of the form $\varphi \rightarrow \varphi'$, then this is equivalent to $J\varphi_K u \leq J\varphi'_K u$.

The axioms on CL-algebras ensure that any formula that has a proof in the sequent calculus of linear logic is true with respect to all valuations.

We apply this to show that the formula $A \otimes B \nrightarrow A`B$ has no proof.
Interpretation of formulas

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The axioms on CL-algebras ensure that any formula that has a proof in the sequent calculus of linear logic is true with respect to all valuations.

▶ We apply this to show that the formula \( A \otimes B \vdash A \multimap B \) has no proof.
A counterexample for a specific formula

Let $\varphi$ be $A \otimes B \rightarrow A \bowtie B$. We consider the CL-algebra:

$$A = \{0, 1, 2, 3, 4, 5\}$$

which is a bounded lattice where we consider that:

- the $\otimes$ is the bounded addition: $1 \otimes 2 = 3$ but $3 \otimes 4 = 5$.
- the involution $(\cdot)^\perp$ is the symmetry $a \mapsto 5 - a$. 

Therefore, we have $J_{A \otimes B}^K u \nleq J_{A \bowtie B}^K u$, hence the counterexample.
A counterexample for a specific formula

Let $\varphi$ be $A \otimes B \rightarrow A \nbigvee B$. We consider the CL-algebra:

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- the $\otimes$ is the bounded addition: $1 \otimes 2 = 3$ but $3 \otimes 4 = 5$.
- the involution $(\cdot)^\perp$ is the symmetry $a \mapsto 5 - a$.

Let us take the valuation $u$ such that $u(A) = 1$ and $u(B) = 2$. Then:

- $J_{A \otimes B}^K u = 3$
- $J_{A \nbigvee B}^K u = (u(A)^\perp \otimes u(B)^\perp)^\perp = (4 \otimes 3)^\perp = 5^\perp = 0$

Therefore, we have $J_{A \otimes B}^K u \ngtr J_{A \nbigvee B}^K u$ hence the counterexample.
A counterexample for a specific formula

Let \( \varphi \) be \( A \otimes B \to A \supset B \). We consider the CL-algebra:

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which is a bounded lattice where we consider that:

- the \( \otimes \) is the bounded addition: \( 1 \otimes 2 = 3 \) but \( 3 \otimes 4 = 5 \).
- the involution \( (\cdot)^\perp \) is the symmetry \( a \mapsto 5 - a \).

Let us take the valuation \( u \) such that \( u(A) = 1 \) and \( u(B) = 2 \). Then:

\[
\begin{align*}
[A \otimes B]_u &= 3 \\
[A \supset B]_u &= (u(A)^\perp \otimes u(B)^\perp)^\perp = (4 \otimes 3)^\perp = 5^\perp = 0
\end{align*}
\]

Therefore, we have \( [A \otimes B]_u \not\leq [A \supset B]_u \) hence the counterexample.
Main theorem

Gehrke et al. proved the following general theorem:

Theorem
Let $\mathcal{E}$ be a set of inequalities of formulas. If we have a class of frames $\mathcal{K}$ such that $\text{Alg}_\delta^\mathcal{E} \subseteq \mathcal{K}^+ \subseteq \text{Alg}_\mathcal{E}$, then the semantics given by $\mathcal{K}$ are sound and complete with respect to the one given by $\text{Alg}_\mathcal{E}$. Notice that, because of this theorem, we could have worked in the relational frame associated to the previous algebra.
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Conclusion

Canonical extensions are useful to link algebras and relational semantics.

Modal algebras $\leftrightarrow$ Kripke frames

CL-algebras $\leftrightarrow$ CL-frames
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Future work: extend this construction to categories to use it for first-order logic
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Thank you for your attention!

Any question?
