Linear logic and canonical extensions

A topic explored with Sam van Gool and Paul-André Melliès.

ASV day

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A quick presentation

MPRI student 2020–2021. Internship with Paul-André Melliès and Sam van Gool.

Now a PhD student at IRIF! On higher-order automata and recursion schemes.

You can meet me in 4033. More on my webpage: `irif.fr/~moreau`.

About my internship

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Our contribution: L^δ can be seen as the fixpoints of an adjunction

$$\begin{tikzcd} \mathbf{F}_\sqcup \mathbf{F}_\vee \mathbf{F}_\sqcap \arrow[rr, bend left, "\lambda"] \arrow[rr, phantom, "\perp"] & & \mathbf{F}_\sqcap \mathbf{F}_\wedge \mathbf{F}_\sqcup \arrow[ll, bend left, "\lambda'"] \end{tikzcd}$$

Why is it useful?

Provides a link between algebras and relational semantics:

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Today: an application from the article

Relational semantics for full linear logic [CGv14]

published by Dion Coumans, Mai Gehrke and Lorijn van Rooijen in 2013.

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\begin{aligned}
\varphi ::= \quad & p \in P \\
& \mid \underbrace{\varphi \otimes \varphi \mid \mathbf{1} \mid \varphi \parr \varphi \mid \bot \mid \varphi \multimap \varphi}_{\text{multiplicative}} \\
& \mid \underbrace{\varphi \,\&\, \varphi \mid \top \mid \varphi \oplus \varphi \mid \mathbf{0}}_{\text{additive}} \\
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Examples:

$$(A \otimes B) \multimap A \qquad (A \,\&\, B) \multimap A \qquad !A \parr ?A \qquad A \parr (B \otimes C) \qquad !A \multimap !!A$$

Examples

Lafont's restaurant, from *An Introduction to Linear Logic: Expressiveness and Phase Semantics* [Oka98]:

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\begin{aligned}
&(\text{Vegetable Soup} \oplus \text{Consommé Soup}) \,\&\, \text{Salad} \\
\otimes\ &(\text{Fish} \,\&\, \text{Meat}) \\
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The formula

$$\forall x \exists y, x < y$$

does not imply infinity as it may only be used once!

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A lattice A with min 0 and max $\top$ together with

- a commutative monoid operation $\otimes : A \times A \to A$ with unit $\mathbf{1} \in A$
- a map $\multimap : A^{\mathrm{op}} \times A \to A$ such that

$$a \otimes b \leq c \iff a \leq b \multimap c$$

- an element $\perp \in A$ such that $(\cdot)^{\perp} : a \mapsto a \multimap \perp$ is an involution.

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We obtain the $\parr$ as the De Morgan dual of $\otimes$

$$a \parr b = (a^{\perp} \otimes b^{\perp})^{\perp}$$

Interpretation of formulas

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The axioms on CL-algebras ensure that any formula that has a proof in the sequent calculus of linear logic is true with respect to all valuations.

- We apply this to show that the formula $A \otimes B \multimap A \parr B$ has no proof.

A counterexample for a specific formula

Let φ be $A \otimes B \multimap A \parr B$. We consider the CL-algebra:

$$A = \{0, 1, 2, 3, 4, 5\}$$

which is a bounded lattice where we consider that:

- the $\otimes$ is the bounded addition: $1 \otimes 2 = 3$ but $3 \otimes 4 = 5$.
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Let us take the valuation u such that $u(A) = 1$ and $u(B) = 2$. Then:

- $[\![A \otimes B]\!]_u = 3$
- $[\![A \parr B]\!]_u = \left(u(A)^{\perp} \otimes u(B)^{\perp}\right)^{\perp} = (4 \otimes 3)^{\perp} = 5^{\perp} = 0$

Therefore, we have $[\![A \otimes B]\!]_u \not\leq [\![A \parr B]\!]_u$ hence the counterexample.

Main theorem

Gehrke et al. proved the following general theorem:

Theorem

Let $\mathcal{E}$ be a set of inequalities of formulas. If we have a class of frames $\mathcal{K}$ such that $\mathrm{Alg}_{\mathcal{E}}^{\delta} \subseteq \mathcal{K}^{+} \subseteq \mathrm{Alg}_{\mathcal{E}}$, then the semantics given by $\mathcal{K}$ are sound and complete with respect to the one given by $\mathrm{Alg}_{\mathcal{E}}$.

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Notice that, because of this theorem, we could have worked in the relational frame associated to the previous algebra.

Conclusion

Canonical extensions are useful to link algebras and relational semantics.

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Thank you for your attention!

Any question?

Bibliography

[CGv14] Dion Coumans, Mai Gehrke, and Lorijn van Rooijen. "Relational semantics for full linear logic." In: *Journal of Applied Logic* 12.1 (2014). Logic Categories Semantics, pp. 50–66. ISSN: 1570-8683. DOI: https://doi.org/10.1016/j.jal.2013.07.005. URL: https://www.sciencedirect.com/science/article/pii/S1570868313000505.

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