# Stone spaces explained and exemplified

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**Abstract**

These are notes for a 1-hour introductory session on Stone spaces, which appear at the interface of computer science, logic, and general mathematics. We will give a gentle introduction to Stone spaces, distilling a few key examples that illustrate their peculiarities in an accessible way. We will also be hinting towards Stone duality and other related topics, such as applications in automata theory. No prerequisites.

Disclaimer: in this text, we will identify isomorphic objects, and transfer elements of one to elements of the other via the identification. This kind of notation finds a formal justification in Homotopy Type Theory.

## Topology and computer science

We first introduce topological spaces and related useful notions to deal with Stone spaces. We first do so in a mathematical manner.

A *topological space* is a pair $(X, \tau)$ where $X$ is a set whose elements are called points, and $\tau$ is a set of subsets of $X$, called *open sets*, such that:

1. $\tau$ is stable by all unions, i.e. for any (potentially infinite) set $I$ and family $(U_i)_{i \in I}$ of open sets,
$$\bigcup_{i \in I} U_i \quad \in \quad \tau \, , \tag{$\bigcup$}$$
2. $\tau$ is stable by finite intersections, i.e. for any finite family $U_1, \ldots, U_n$ of open sets,
$$U_1 \cap \cdots \cap U_n \quad \in \quad \tau \, . \tag{$\cap_f$}$$

A central topological space is the one denoted $\mathbb{R}$ whose points are real numbers and whose open subsets $U \subseteq \mathbb{R}$ are the ones such that

$$\text{for every } x \in U \text{, there exists } a, b \in \mathbb{R} \text{ such that} \quad x \in (a, b) \subseteq U \tag{$\tau_{\mathbb{R}}$}$$

where $(a, b)$ is the interval $\{y \in \mathbb{R} \mid a < y < b\}$.

A *closed set* $F$ is a subset $F \subseteq X$ whose complement $F^c$ is open. The notion of topology could very well be defined in terms of closed sets, which would then be stable under all intersections and finite unions. Nonetheless, open and closed sets correspond to very different intuitions:

- open sets represent subsets whose points are locally surrounded by all possible points: if one converges to the central point, then one has to pass through the surrounding points eventually.
- closed sets represent subsets that are stable under convergence and limits.

If $(X, \tau)$ is a topological space and $S$ is a subset of $X$, then we can define a topology on $S$, called the *subspace topology* whose open sets are the subsets $V \subseteq S$ such that

$$\text{there exists an open subset } U \subseteq X \text{ such that } V = S \cap U.$$

This construction can be applied to get new spaces out of old ones:

- The subset $[0, 1]$ of $\mathbb{R}$ gets the subspace topology where, for example, the two subsets
$$[0, 1/2) \quad \text{and} \quad (1/3, 1]$$
are both open sets of $[0, 1]$.
- the subset $\mathbb{Q}$ of $\mathbb{R}$, containing rational numbers, gets the subspace topology where, for example, the three subsets
$$(-\infty, 0) \cap \mathbb{Q} \quad \text{and} \quad (0, \sqrt{2}) \cap \mathbb{Q} \quad \text{and} \quad (\sqrt{2}, +\infty) \cap \mathbb{Q}$$
are all open sets of $\mathbb{Q}$.

A *clopen* is a subset that is both closed and open. Any space $X$ has at least the clopens $\varnothing$ and $X$ itself. A lot of spaces that one can meet in topology do not have any other: this is the case of the space $\mathbb{R}$ of real numbers, for instance.

However, an example of space with a lot of clopens is the space $\mathbb{Q}$ of rational numbers, which embeds in $\mathbb{R}$. Indeed, if $\alpha$ is an irrational number, i.e. a real number that is not rational, then by cutting the space in the two complement parts
$$(-\infty, \alpha) \cap \mathbb{Q} \quad \text{and} \quad (\alpha, +\infty) \cap \mathbb{Q}$$
which are both of them open – so clopens too! By contrast, in $\mathbb{R}$ we would have to choose where to include $\alpha$, which could result for example in the two sets

$$(-\infty, \alpha) \quad \text{and} \quad [\alpha, +\infty)$$

where only the first one is open and only the second one is closed, so neither are clopens. This is not surprising given that, as said above, the only clopens of the space $\mathbb{R}$ of real numbers are $\varnothing$ and $\mathbb{R}$ which are the complement one of the other.

We say that a topological space $X$ is *totally separated* if, for any two distinct points $x$ and $y$ of $X$, there exists a clopen set $U$ such that

$$x \in U \quad \text{and} \quad y \in U^c \ .$$

Notice the symmetry of the condition: as $U$ is a clopen, its complement $U^c$ is also one. The space $\mathbb{Q}$ of rational numbers yields an example of totally separated space: indeed, if $x < y$ are two rational numbers, let $\alpha$ be any irrational number such that $x < \alpha < y$, for example, $\alpha = x + \sqrt{2}/n$, for $n$ sufficiently big, is suitable. Then, we have that

$$x \ \in \ (-\infty, \alpha) \cap \mathbb{Q} \quad \text{and} \quad y \ \in \ (\alpha, +\infty) \cap \mathbb{Q}$$

which are two clopens of $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is totally separated. On the other hand, neither $\mathbb{R}$ nor $[0,1]$ have non-trivial clopens, so these spaces are not totally separated. In general, if a space $X$ has two points $x$ and $y$ which are related by a (continuous) path $\gamma$, i.e.

$$\gamma \quad : \quad [0,1] \ \longrightarrow \ X \qquad \text{s.t.} \qquad \gamma(0) = x \quad \text{and} \quad \gamma(1) = y$$

then $x$ and $y$ cannot be separated by clopens. Therefore, any space that has at least one non-constant path is not totally separated.

The topological condition of *compactness* is a bit more technical. Instead of spilling its technicalities, we choose to explain it informally.

Consider a finite space and a sequence of points of it. By the pigeonhole principle, the sequence must visit infinitely many times at least one point of the space. In a way, we must loop, and get back to where we have already been. Compactness is a generalization of this property of finite spaces, to general topological spaces. In a compact space with a notion of distance, to any sequence, there exists some point to which the sequence must come closer, from time to time. In the case of finite spaces, closer means equal so we get back that some point is visited an infinite number of times.

The discussion has been informal until now, but in the case of the space $\mathbb{R}$ of real numbers, we have the following mathematical criterion: the compact subspaces of $\mathbb{R}$ are exactly the closed and *bounded* subsets, where bounded means that it can be included in a closed interval $[a, b]$. For instance, the space $[0,1]$ is compact, as it is closed and bounded. On the contrary, $\mathbb{R}$ itself and $\mathbb{Q}$ are not bounded so they are not compact, and $[0,1] \cap \mathbb{Q}$ is not closed so it is not compact either.

A *Stone space* is a space that verifies these two conditions. Therefore:

$$\text{Stone} \quad = \quad \text{compact} \ + \ \text{totally separated}.$$

## Open sets and algorithms

The definition of topological spaces is very simple, yet it was still a bit mysterious to me about what it really meant. Following [Smy93] and [Sim02], we now explore an analogy between topology and computability.

Consider the space $\mathbb{R}$ of real numbers, and an open set of that space, say the open interval $(-1, 1)$. Cauchy sequences, i.e. sequences whose elements are closer and closer to each other, provide a way to represent real numbers. If $(x_n) \in \mathbb{Q}^{\mathbb{N}}$ is a sequence of real numbers converging to some limit $l \in \mathbb{R}$, then the fact that

$$-1 \ < \ l \ < \ 1$$

i.e. the fact that the limit of the sequence $(x_n)$ belongs to the open interval $(-1, 1)$, is equivalent to the fact that

$$\text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N, \quad -1 \ < \ x_n \ < \ 1$$

i.e. the fact that the sequence $(x_n)$ eventually stays in the open interval $(-1, 1)$.

This kind of topological facts can be understood from a computational point of view. Indeed, suppose that $(x_n)$ is not any converging sequence, but verifies some kind of explicit Cauchy condition[1], like

$$\text{for all } m, n \in \mathbb{N}, \quad |x_m - x_n| \ < \ 2^{-\max(m,n)} \ . \tag{B}$$

A natural example of such sequence is given by the sequence of binary expansions of some real number $x \in [0, 1]$, which is of the form

$$x_0 \ = \ 0 \ , \quad x_1 \ = \ 0.c_1 \ , \quad x_2 \ = \ 0.c_1c_2 \ , \quad \ldots \quad x_n \ = \ 0.c_1 \ldots c_n \quad \ldots$$

Then, there is an algorithm taking any sequence $(x_n)$ verifying B and terminating if and only if the limit of $(x_n)$ lies in the open interval $(-1, 1)$. The algorithm is as follows: The idea behind this algorithm is that, knowing $x_n$ for some natural

**Algorithm 1** Semidecision of limits in the open interval $(-1, 1)$

```
1: n ← 0
2: while x_n − 2^{−n} ≤ −1 or 1 ≤ x_n + 2^{−n} do
3:     n ← n + 1
4: end while
5: return ⊤
```

number $n$, by the condition B the next point $x_{n+1}$ of the sequence belongs to the open interval $(x_n - 2^{-n}, x_n + 2^{-n})$. Therefore, if we have

$$(x_n - 2^{-n} \ , \ x_n + 2^{-n}) \quad \subseteq \quad (-1, 1) \tag{1}$$

we know that all the following points of the sequence belong to $(-1, 1)$, so

$$l \quad \in \quad (-1, 1)$$

and Algorithm 1 rightly stops.

On the other hand, if Equation (1) does not hold, one cannot be sure that the limit point won't belong to $(-1, 1)$. For example, if we have $x_n = 1$ for some $n$, then the limit might be less or more than 1. In the case where the limit does not belong to $(-1, 1)$, Algorithm 1 will never terminate.

[1] which amounts to a suitable modulus of continuity

## The semidecidability analogy

In computer science, a subset $U \subseteq X$ is *semidecidable* if there is an algorithm taking points of $X$ as inputs, that terminates exactly on points belonging to $U$ and continues running infinitely on others. A typical example of a semidecidable set is a subset defined by some nice enough existential property, in which case it is then possible to use an exhaustive search algorithm: for example, the set of real numbers that have a 1 in their canonical decimal expansion. This is related to recursively enumerable sets.

We talk of *decidability* in the case where there is an algorithm that always terminates and outputs a Boolean representing membership to the subset.

We note $\mathbb{S}$ the Sierpiński space, whose set of points is $\{\bot, \top\}$ and whose open sets are

$$\varnothing \quad , \quad \{\top\} \quad \text{and} \quad \{\bot, \top\} \ .$$

This space has a very special role in topology: indeed, for any topological space, the data of an open set $U \subseteq X$ is the same thing as a continuous map

$$\chi_U \quad : \quad X \longrightarrow \mathbb{S} \ .$$

The continuous map $\chi_U$ is the characteristic function of the subset $U$, which is continuous if and only if $U$ is open. This permits us to think of processes like Algorithm 1 as continuous functions

$$X \longrightarrow \mathbb{S}$$

where $\top$ represents termination $\bot$ represents non-termination. Notice that the fact that $\{\bot\}$ is not an open set of $\mathbb{S}$ corresponds exactly to the fact that non-termination is not observable.

In the light of Algorithm 1 and the Sierpiński space $\mathbb{S}$, we consider the *analogy*

$$\boxed{\text{open} \quad = \quad \text{semidecidable}}$$

which says that an open set $U \subseteq X$ can be thought of as an algorithm taking points of $X$ as inputs that terminate only when the input point belongs to $U$.

With this analogy in mind, explained in considerable details in [Vic89], we now revisit the definition of a topological space and its two conditions:

**Equation ($\bigcup$)** If $(U_i)_{i \in I}$ is a family of open sets, each semidecided by an algorithm that we think of as the continuous characteristic function

$$\chi_i \quad : \quad X \longrightarrow \mathbb{S} \ ,$$

then to semidecide the union

$$\bigcup_{i \in I} U_i$$

run all the algorithms $\chi_i$ in parallel and terminate as soon as one terminates.

**Equation** $(\cap_f)$ If $U_1, \ldots, U_n$ is a finite family of open sets, each semidecided by an algorithm $\chi_i$, then to semidecide the intersection

$$U_1 \cap \cdots \cap U_n$$

run all the algorithms $\chi_1, \ldots, \chi_n$ in parallel and terminate as soon as all have terminated.

In the case of intersections, notice that the fact that we only have $n$ algorithms in parallel, i.e. a finite number, is the key point: indeed, the termination status at infinity is then realized at some point in time.

## Stone spaces revisited

We now revisit the two properties defining Stone spaces, i.e. compactness and total separation, using the semidecidability analogy and the Sierpiński space.

Let $U \subseteq X$ be a clopen set. As it is open, we think of the predicate

$$x \;\in\; U \quad \text{where } x \in X$$

as being semidecidable, and as its complement $U^c$ is also open, we think of the complement predicate

$$x \;\notin\; U \quad \text{where } x \in X$$

as being semidecidable too. Therefore, for $U$ clopen, there is an algorithm that takes any point $x \in X$ as input and decides whether or not this point belongs to $U$. Therefore, we get a corollary analogy

$$\boxed{\text{clopen} \quad = \quad \text{decidable}}$$

It should then be clear that formulas $\psi$ of classical logic, which verify the syntactic, logical laws

$$\psi \wedge \neg\psi \;=\; \bot \qquad \text{and} \qquad \psi \vee \neg\psi \;=\; \top$$

can be soundly interpreted as clopens $U$ of any topological space $X$, as we have

$$U \cap U^c \;=\; \varnothing \qquad \text{and} \qquad U \cup U^c \;=\; X$$

where $U^c$ is still a clopen set of $X$.

With that perspective in mind, we can interpret total separation as the following: for any distinct points $x$ and $y$ of a topological space $X$, having a clopen set $U \subseteq X$ such that

$$x \in U \quad \text{and} \quad y \in U^c$$

correspond to a decidable property on points of $X$, which holds at $x$ and does not hold at $y$.

Compactness is slightly more technical. Its computational interpretation has been described in [Esc04], and appeals to higher-order functions. Indeed, for every space $X$, we have a set-theoretic function

$$\forall \quad : \quad (X \to \mathbb{S}) \longrightarrow \mathbb{S}$$

which takes as input an open set $U$ of $X$ and outputs $\top$ if $U = X$, and $\bot$ otherwise.

To even state the continuity of the $\forall$ function, one needs to equip the set of continuous maps $X \to \mathbb{S}$, which we think of as open sets of $X$, with the natural topology [Esc04, §8.1]. Then, it turns out that the $\forall$ function is continuous exactly when $X$ is compact. Again, the link with finite spaces is clear, where the $\forall$ function is always continuous as it consists of running its input (the open set) at all points. Continuity, in the general case, amounts to an algorithmic exhaustive search.

# Stone spaces

We now give examples of Stone spaces, and how they may appear in natural mathematical practice.

## Finite spaces

The easiest examples of Stone spaces are discrete finite spaces. For any natural number $n$, we equip the set $\{1, \ldots, n\}$ with the discrete topology, i.e. the one where every subset $U \subseteq \{1, \ldots, n\}$ is open. We can think of it as $n$ distinct points far from each other:

$$1 \qquad 2 \qquad \ldots \qquad n-1 \qquad n$$

One can consider the set $\{1, \ldots, n\}$ as a subset of the space $\mathbb{R}$ of real numbers. As such, the subspace topology that we get is exactly the discrete one.

## One-point compactification of natural numbers

We consider the topological space $\overline{\mathbb{N}}$, called the one-point compactification of $\mathbb{N}$, whose set of points is $\mathbb{N} \cup \{\infty\}$, i.e. the natural numbers together with a new point $\infty$. They can be represented as the set $\mathbb{N}$ to which we add a new point $\infty$:

$$0 \qquad 1 \qquad 2 \qquad 3 \qquad \ldots \qquad \infty$$

A subset $U \subseteq \overline{\mathbb{N}}$ is open if

$$\infty \notin U \qquad \text{or} \qquad \infty \in U \text{ and } U^c \text{ is finite.}$$

In the present case, it might be enlightening to think in terms of closed sets. By dualizing the definition, we get that a subset $F \subseteq \overline{\mathbb{N}}$ is closed if

$$\infty \in F \qquad \text{or} \qquad \infty \notin F \text{ and } F \text{ is finite.}$$

We now show how to realize the space $\overline{\mathbb{N}}$: we consider the following injection:

$$\begin{array}{cccccc} & & \overline{\mathbb{N}} & \longrightarrow & \mathbb{R} \\ \varphi & : & n & \longmapsto & \frac{1}{1+n} \\ & & \infty & \longmapsto & 0 \end{array}$$

We can then see the set of points of $\overline{\mathbb{N}}$ as a subset of $\mathbb{R}$, i.e. the subset

$$\overline{\mathbb{N}} \quad = \quad \{0\} \cup \left\{ \frac{1}{1+n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R} .$$

The subspace topology that we get is exactly the one described above in terms of finite sets. As this subset of $\mathbb{R}$ is closed and bounded, we thus get that $\overline{\mathbb{N}}$ is compact. Moreover, it is totally separated: seen as a subset of $\mathbb{R}$, if $x$ and $y$ are different points of $\overline{\mathbb{N}}$, i.e. are each either 0 or of the form $1/(1+n)$, then the two clopens

$$\left(-\infty, \frac{x+y}{2}\right) \cap \overline{\mathbb{N}} \quad \text{and} \quad \left(\frac{x+y}{2}, \infty\right) \cap \overline{\mathbb{N}}$$

seperate $x$ and $y$.

The space $\overline{\mathbb{N}}$ is very special as it has the following property. Indeed, for any topological space $X$, any sequence of points $(x_n)$ of $X$ can be seen as a function

$$\mathbb{N} \xrightarrow{x} X$$

Now, a sequence $(x_n)$ which has a limit $l$ can be thought of as a continuous function

$$\overline{\mathbb{N}} \xrightarrow{x \to l} X$$

from the Stone space $\overline{\mathbb{N}}$ to $X$. The discrete space $\mathbb{N}$ embeds into the Stone space $\overline{\mathbb{N}}$, and a limit of the sequence $x$ is exactly the same thing as a continuous lifting of $x$ to $\overline{\mathbb{N}}$, as described in the diagram below:

$$\begin{tikzcd} \overline{\mathbb{N}} \arrow[dr, dashed] & \\ \mathbb{N} \arrow[u, hook] \arrow[r, "x"] & X \end{tikzcd}$$

A lot of spaces of interest in topology are sequential, in the sense that their topology is fully determined by the way sequences of points converge, i.e. continuous maps from $\overline{\mathbb{N}}$. This idea has been used successfully by Peter Johnstone to construct a convenient category of spaces, the topological topos [Joh79], whose objects can be thought of as generalizations of sequential spaces.

## Cantor space

We now define the *Cantor space $K$*, whose set of points is the set $2^{\mathbb{N}}$ of binary sequences, where $2 = \{0, 1\}$. To define the topology of the Cantor space, we first consider, for any binary word $w$, i.e. **finite** sequence $w = [b_0, \ldots, b_n]$ of elements of 2, the set

$$V_w \quad = \quad \{x \in 2^{\mathbb{N}} \mid \text{for all } 0 \leq i \leq n, x_i = b_i\}$$

which is the set of sequences whose prefix of length $n+1$ is $w$. We say that a subset $U \subseteq K$ is open if for every $x \in U$,

$$\text{there exists a binary word } w \text{ such that} \quad x \in V_w \subseteq U \ . \tag{$\tau_K$}$$

Notice the analogy between Equation ($\tau_K$) and Equation ($\tau_{\mathbb{R}}$). However, in the case of the Cantor space, each $V_w$ is a clopen set: indeed, we have

$$(V_w)^c \quad = \quad \bigcup_{\substack{w' \in 2^n \\ w' \neq w}} V_{w'}$$

where the union is finite. This shows that the Cantor space $K$ is as follows: looking only at the first Boolean of sequences, it is partitioned into 2 clopens, associated with 0 and 1. Then, looking at the first two Booleans of sequences, each of the two clopens is itself partitioned into 2 clopens namely 00 and 01 for 0, and 10 and 11 for 1. For any natural number $n$, by looking at the prefix of length $n$ of the sequences, we partition the Cantor space into $2^n$ clopens.

The Cantor space contains the space $\overline{\mathbb{N}}$, as we have the continuous injection

$$\iota \quad : \quad \begin{array}{ccc} \overline{\mathbb{N}} & \longrightarrow & K \\ n & \longmapsto & 0^n 1^\omega \\ \infty & \longmapsto & 0^\omega \end{array} \ .$$

We now describe the Cantor space $K$ as a subset of $\mathbb{R}$. It is built iteratively, like a fractal. Indeed, we set, for every subset $S \subseteq [0, 1]$:

$$T(S) \quad = \quad \left\{\frac{s}{3} \ : \ s \in S\right\} \cup \left\{\frac{2+s}{3} \ : \ s \in S\right\}$$

The Cantor space is then defined, in that fashion, as the infinite intersection

$$K \quad = \quad \bigcap_{n=0}^{\infty} T^n([0, 1])$$

The first seven steps of the construction are represented in the figure below. We have a continuous function

$$\begin{array}{ccc} 2^{\mathbb{N}} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sum_{i=1}^{\infty} \frac{2 \times x_i}{3^i} \end{array}$$

Figure 1: Graphical representation of the Cantor space

whose image is the fractal description of $K$. With that identification in mind, the embedding $\iota : \overline{\mathbb{N}} \to K$ now becomes

$$n \mapsto \sum_{i=n}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} \quad \text{and} \quad \infty \mapsto 0$$

The Cantor space appears naturally in game theory, especially in Gale-Stewart games. In that game (with Boolean moves), a move is a choice of a Boolean, and points $x \in 2^{\mathbb{N}}$ of the Cantor space represent a play of the game: each move $x_{2n}$ has been chosen by Player I and each move $x_{2n+1}$ has been chosen by Player II. Given a payoff set $P$, i.e. subset

$$P \quad \subseteq \quad 2^{\mathbb{N}} \ ,$$

we say that a play $x \in 2^{\mathbb{N}}$ is winning for Player I if $x$ belongs to $P$.

A strategy is a choice of a move to choose for each of the possible prior moves of the play. A strategy is said to be winning if, whatever move the opponent chooses, the player always wins the game. We say that the game is determined if Player I or Player II has a winning strategy.

As defined, the Gale-Stewart game does not use the topology of the Cantor space, only its set of points. A deep result due to [GS53] states that, if $P$ is an open or a closed subset of $K$, then the game is determined. For example:

- if $P$ is the clopen $V_w$ of all plays starting with the finite prefix $w$, for $|w| \geq 2$, then Player II has a winning strategy, which consists of playing the opposite of $w_2$,
- we think of 0 as an open parenthesis `(` and of 1 as a closing parenthesis `)`, then if $P$ is the closed set of sequences whose finite prefixes all have as much or more open than closing parentheses, can be written as

$$P \quad = \quad \bigcap_{n \in \mathbb{N}} \bigcup_{\substack{w \in 2^n \\ |w|_0 \geq |w|_1}} V_w$$

which is a closed set as an intersection of clopens: and indeed, Player I has a strategy to win it, which is to always output 0, i.e. always open parentheses.

This result has been extended by Martin [Mar75] to subsets $P$ which are Borel sets, and is hence called the Borel determinacy theorem. This kind of considerations has lead mathematicians to consider the axiom of determinacy in the case where the set of moves is no longer 2 but $\mathbb{N}$, which is incompatible with the usual axiom of choice.

## A few directions

The connection with classical propositional logic can be made more precise than the mere analogy that we have described here and is then called Stone duality. Roughly, it says that each Boolean algebra, which models classical logic, can be obtained as the algebra of clopen sets of a Stone space, and conversely, every Stone space can be constructed as the space of ultrafilters of a Boolean algebra, a generalization of the notion of atom to the possibly infinite case. The ideas of Stone duality have been extended to a lot of different situations, see [GG22][2], and have also been applied to programming language theory [Abr91].

Stone spaces can be thought of as some kind of generalization of finite sets, and are also called profinite sets or profinite spaces. As such, they are related to fields like finite model theory and automata theory [Pin09]. For example, the set of finite words

$$\Sigma^*$$

can be seen as a discrete subspace of a Stone space

$$\widehat{\Sigma^*}$$

whose points are called profinite words. This Stone space makes it possible to think of the limiting behavior of finite automata, by using topological tools to understand it, and can provide a new and elegant point of view on already known phenomena.

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