Roadmap

- Last lecture(s):
  - Cole and Vishkin algorithm for 3-coloring the n-node cycle C_n
  - Generalization to $(\Delta + 1)$-coloring arbitrary graphs of maximum degree Δ in $O(\Delta^2 + \log^\star n)$ rounds
- Lecture today:
  - $(\Delta + 1)$-coloring algorithm in $O(\Delta + \log^\star n)$ rounds
  - Lower bound for 2-coloring C_n

$(\Delta + 1)$-Coloring in $O(\Delta + \log^\star n)$ rounds

Four phases:

1. 3^Δ-coloring in $O(\log^\star n)$ rounds (cf. previous lecture)
2. Reducing to $O(\Delta^3)$-coloring in 1 round
3. Reducing number of colors to $O(\Delta^2)$ in 1 round
4. Reducing number of colors to $\Delta + 1$ in $O(\Delta)$ rounds

Phase 2: From 3^Δ to $O(\Delta^3)$ colors in a single round

Lemma [Erdös, Frankl, Füredi, 1985]
For any $k > \Delta \geq 2$, there exists a family $\mathcal{F}$ of k subsets of $\{1,\ldots,5\lceil\Delta^2 \log k\rceil\}$ such that, for any $\Delta + 1$ (distinct) sets $F_0, \ldots, F_\Delta$ in $\mathcal{F}$, we have $F_0 \nsubseteq \cup_{i=1}^{\Delta} F_i$

Algorithm:

- Range of colors $[1,k]$ with $k = 3^\Delta$
- Node u with color $c(u) \in \{1,\ldots,k\}$ picks set $F_{c(u)} \in \mathcal{F}$
- By the lemma, $\exists x \notin \cup_{v\in N(u)} F_{c(v)}$
- Node u updates its color $c(u)$ to x i.e. $c(u) \leftarrow x$
- Reduction of #colors: $3^\Delta \rightarrow O\left(\Delta^2 \log(3^\Delta)\right) = O(\Delta^3)$

Polynomials on Finite Fields

- For a prime integer q, let $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ i.e., $\mathbb{F}_q$ is $\{0,\ldots,q-1\}$ with arithmetic modulo q
- $\mathbb{F}_q$ is a finite field
- A polynomial of degree d on $\mathbb{F}_q$ is of the form

$$a_0 + a_1X + \ldots + a_dX^d$$

Lemma A polynomial of degree d on $\mathbb{F}_q$ has at most d roots

Corollary Two polynomials of degree d on $\mathbb{F}_q$ may coincide on at most d values.

Phase 3: From $O(\Delta^3)$ to $O(\Delta^2)$ colors in a single round

- Say colors in $[1, \alpha\Delta^3]$ for some $\alpha > 0$
- Let $q = O(\Delta)$ prime with $3\Delta < q$ and $q^4 \geq \alpha\Delta^3$
- There are q^4 polynomials of degree 3 in $\mathbb{F}_q$
- Node u with color $c(u) = i \in [1, \alpha\Delta^3]$ picks set

$$S_{c(u)} = S_i = \{(x, p_i(x)) : x \in \mathbb{F}_q\} \subseteq \mathbb{F}_q \times \mathbb{F}_q$$

- For every $i \neq j$ we have $|S_i \cap S_j| \leq 3$
- Thus $|S_{c(u)} \setminus \cup_{v \in N(u)} S_{c(v)}| \geq |S_{c(u)}| - 3\Delta > 0$
- Node u updates its colors by picking one element in $S_{c(u)} \setminus \cup_{v \in N(u)} S_{c(v)}$

Phase 4: From $O(\Delta^2)$ to $\Delta + 1$ colors in $O(\Delta)$ rounds

- Say colors in $[1, \beta\Delta^2]$ for some $\beta > 0$
- Let $q = O(\Delta)$ prime with $6\Delta < q$ and $q^4 \geq \beta\Delta^2$
- Node u with color $c(u) = i \in [1, \beta\Delta^2]$ picks sequence

$$\sigma_{c(u)} = \sigma_i = \big(p_i(0), p_i(1), \ldots, p_i(q-1)\big)$$

ith polynomial of degree 3 in $\mathbb{F}_q$

For $x = 0$ to $q - 1$ do

- if uncolored then propose color $p_i(x)$
- if no conflicts, then adopt color $p_i(x)$ and terminate

- At most 3 conflicting iterations for each non-terminated neighbor *and* at most 3 conflicting iterations for each terminated neighbor
- Reduce #colors from q to $\Delta + 1$ in $q - (\Delta + 1) = O(\Delta)$ rounds

State of the Art and Open Problems

Best known algorithm performs $(\Delta + 1)$-coloring in

- $O(\log^\star n + \sqrt{\Delta \log \Delta})$ rounds
- $O(\log n \cdot \log^2 \Delta) \leq O(\log^3 n)$ rounds
- $O(\log^2 n)$ rounds

Can we improve these complexities?

Is there a distributed algorithm running in $O(\log^\star n)$ rounds in LOCAL that properly colors every graph of maximum degree Δ with $o(\Delta^2)$ colors?

Lower Bounds for $(\Delta + 1)$-coloring

- Ramsey lower bounds
- 2-coloring C_{2n} is hard

Simple Lower Bounds

Theorem For all $t \geq 0$, every t-round algorithm fails to 3-color some cycle.

In other words, 3-coloring the cycles cannot be done in $O(1)$ rounds, i.e., it requires $\omega(1)$ rounds.

The proof is based on Ramsey's theory.

Ramsey Theorem

Theorem [Ramsey, 1920s] For every positive integers r and s there exists $R = R(r, s)$ such that every edge-coloring of the complete graph on R vertices with two colors blue and red contains a blue clique on r vertices or a red clique on s vertices.

Example: $R(3,3) = 6$

Extension to Hypergraphs

For $k \geq 2$, a k-hypergraph is a hypergraph whose hyperedges are sets of k vertices

Theorem [Ramsey, 1920s] For any integers k and c, and any integers $n_1, \ldots, n_c$, there is an integer $R = R(n_1, \ldots, n_c; k)$ such that:

- if the hyperedges of a complete k-hypergraph of R vertices are colored with c different colors,
- then there exists $i \in [c]$ such that the hypergraph contain a complete sub-k-hypergraph of order n_i whose hyperedges are all colored i.

Theorem For all $t \geq 0$, every t-round algorithm fails to 3-color some cycle.

Proof: Let $t \geq 0$ and let A be a t-round algorithm.

$A(x_{-t}, \ldots, x_0, \ldots, x_t) \in \{1,2,3\}$ with $x_i \in \{1,\ldots,n\}$

$k = 2t + 1,\ c = 3$, and $n_1 = n_2 = n_3 = 2t + 2$

Cycles C_n with $n = R(n_1, n_2, n_3; 2t + 1)$ nodes

Color hyperedge $\{x_{-t}, \ldots, x_0, \ldots, x_t\}$ where $x_{-t} < \ldots < x_0 < \ldots < x_t$ with color $A(x_{-t}, \ldots, x_0, \ldots, x_t)$

Ramsey $\implies \exists x_{-t} < \ldots < x_0 < \ldots < x_t < x_{t+1}$ such that $A(x_{-t}, \ldots, x_0, \ldots, x_t) = A(x_{-t+1}, \ldots, x_1, \ldots, x_{t+1})$ ❐

Δ-coloring is hard

Theorem 2-coloring the $2n$-node cycle requires at least n rounds.

Proof: Let A be a t-round algorithm, for $t < n - 1$

$A(x_{-t}, \ldots, x_0, \ldots, x_t) \in \{1,2\}$ with $x_i \in \{1,\ldots,2n\}$

$$A(x_1, x_2, \ldots, x_{2t+1}) = 1$$
$$A(x_2, x_3, \ldots, x_{2t+1}, y) = 2$$
$$A(x_3, x_4 \ldots, x_{2t+1}, y, z) = 1$$
$$A(x_4, \ldots, x_{2t+1}, y, z, x_1) = 2$$
$$\vdots$$
$$A(y, z, x_1, \ldots, x_{2t-1}) = 2$$
$$A(z, x_1, \ldots, x_{2t}) = 1$$

❒

End Lecture 2