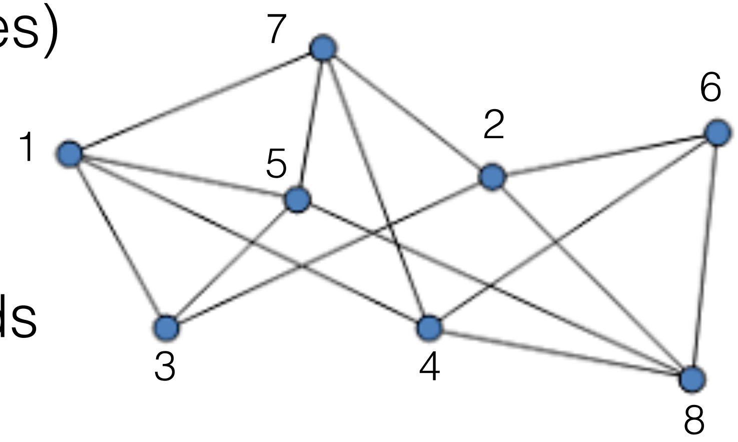
Local Computing

- The LOCAL model
- Deterministic $(\Delta + 1)$-coloring arbitrary graphs with maximum degree Δ

LOCAL Model

- Each process is located at a node of a network modeled as an n-node graph (n = #processes)

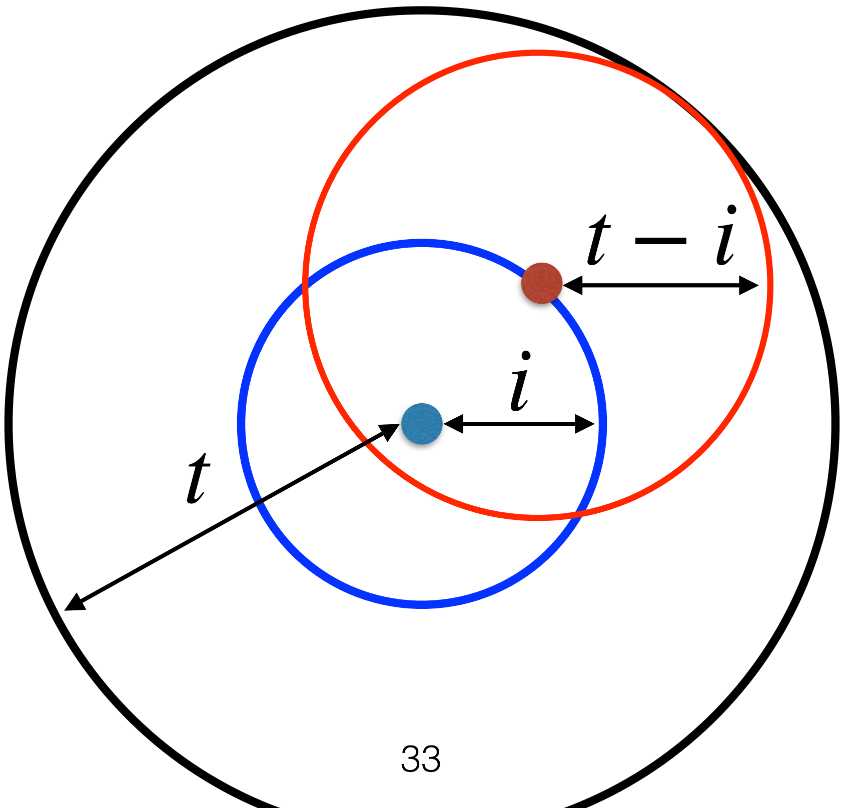


- Each process has a unique ID in $\{1, \ldots, n\}$
- Computation proceeds in synchronous rounds during which every process:

    1. Sends a message to each neighbor
    2. Receives a message from each neighbor
    3. Performs individual computation (same algorithm for all nodes)

Complexity = #rounds

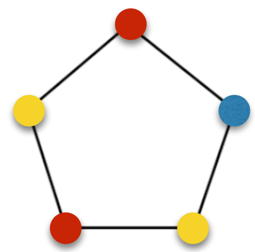
Lemma If a problem P can be solved in t rounds in the LOCAL model by an algorithm A, then there is a t-round algorithm B solving P in which every node proceeds in two phases: (1) Gather all data in the t-ball around it; (2) Simulate and compute the solution.



(Δ+1)-coloring

Δ = maximum node degree of the graph

$(\Delta+1)$-coloring = assign colors to nodes such that every pair of adjacent nodes are assigned different colors.



Lemma Every graph is $(\Delta+1)$-colorable

Theorem (Brooks, 1941)
Every graph G is Δ-colorable, unless G is a complete graph, or an odd cycle.

Lemma $(\Delta+1)$-coloring can be sequentially computed by a simple greedy algorithm treating each node individually.

Remark

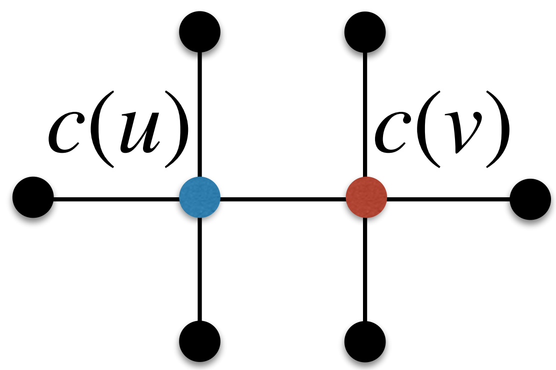
Let $k \geq \Delta + 1$

If there exists a t-round k-coloring algorithm then there exists a $(\Delta + 1)$-coloring algorithm running in $t + \big(k - (\Delta + 1)\big)$ rounds.

Coloring graphs of max degree Δ with $\Delta^{O(\Delta)}$ colors in $O(\log^\star n)$ rounds

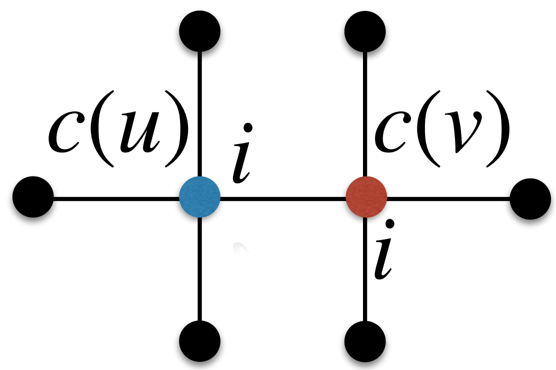
Every node u maintains an array $c(u) = (c_1(u), \ldots, c_\Delta(u))$ of colors, ordered according to the IDs of its neighbors.

- Initially $c(u) = (\text{ID}(u), \ldots, \text{ID}(u))$
- Repeat
  - performs C&V with each neighbor independently, in parallel.



Correctness

Claim: Proper coloring is preserved after each iteration of C&V, transforming color $c(u)$ of u into $c'(u)$



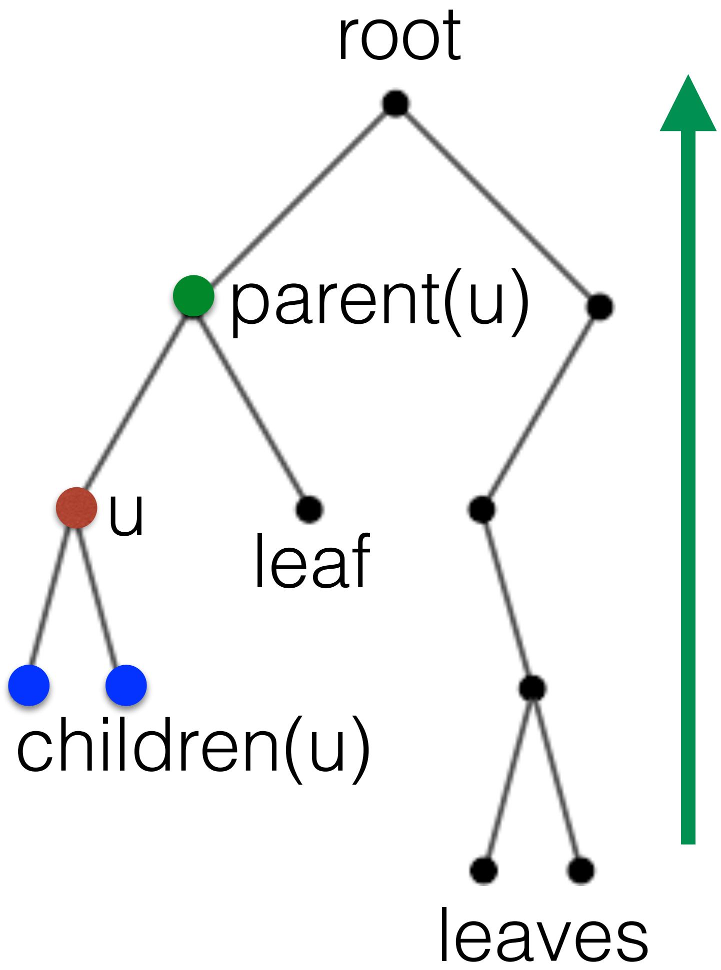
- Let $c_i'(u) = (p, b)$ and $c_i'(v) = (p', b')$
- If $p \neq p'$ then $c'(u) \neq c'(v)$
- If $p = p'$ then, as p is the first bit-position at which $c_i(u)$ and $c_i(v)$ differ, we have $b \neq b'$, and thus $c'(u) \neq c'(v)$

Complexity

- Colors are initially on $\Delta \cdot \lceil \log_2 n \rceil$ bits
- Assuming colors on k bits
- After one iteration: colors on $f(k) = \Delta(\lceil \log_2 k \rceil + 1)$ bits
- For $k = \alpha \, \Delta \log \Delta$ with α sufficiently large, we have $f(k) < k$
- Thus, after $O(\log^\star n)$ iterations, colors on $O(\Delta \log \Delta)$ bits
- That is, $2^{O(\Delta \log \Delta)} = \Delta^{O(\Delta)}$ colors.

3-Coloring Rooted Trees

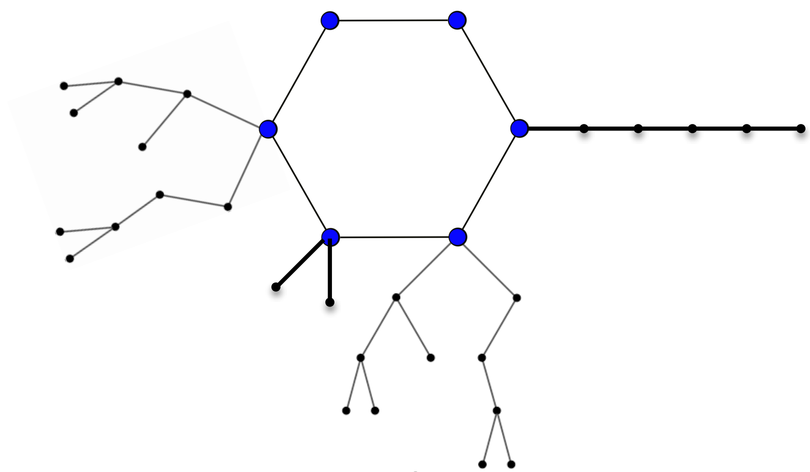
- Apply C&V with parent for $O(\log^{\star} n)$ rounds, to 6-color the tree
- For $i = 6$ down to 4 do
  - adopt color of parent
  - recolor nodes colored i with a color in $\{1,2,3\}$



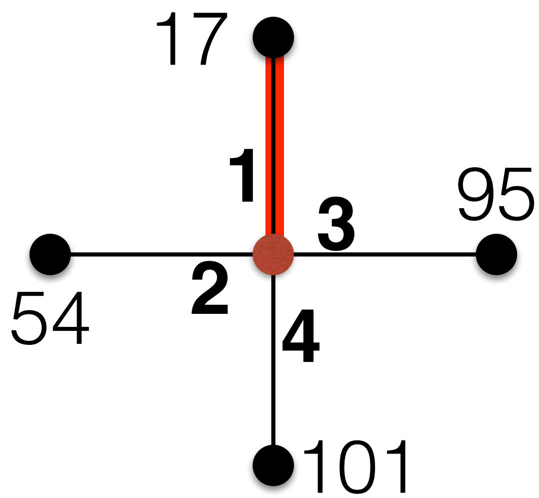
1-Factors

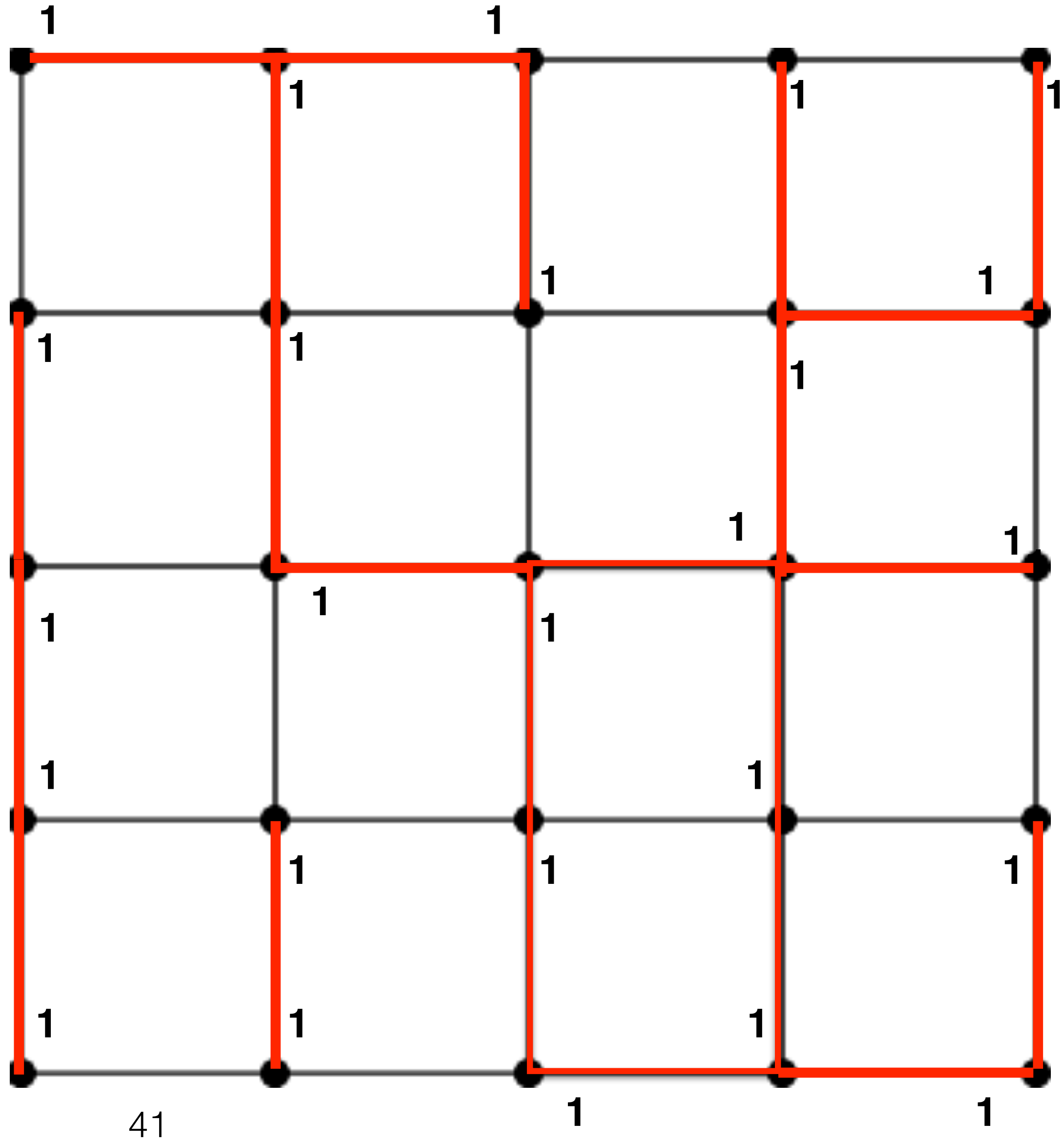
- Let $G = (V, E)$ be a graph
- Assume each node $v \in V$ selects one of its incident edges
- Let $F \subseteq E$ be the set of selected edges

Claim F is a collection of « pseudo-trees » of the form

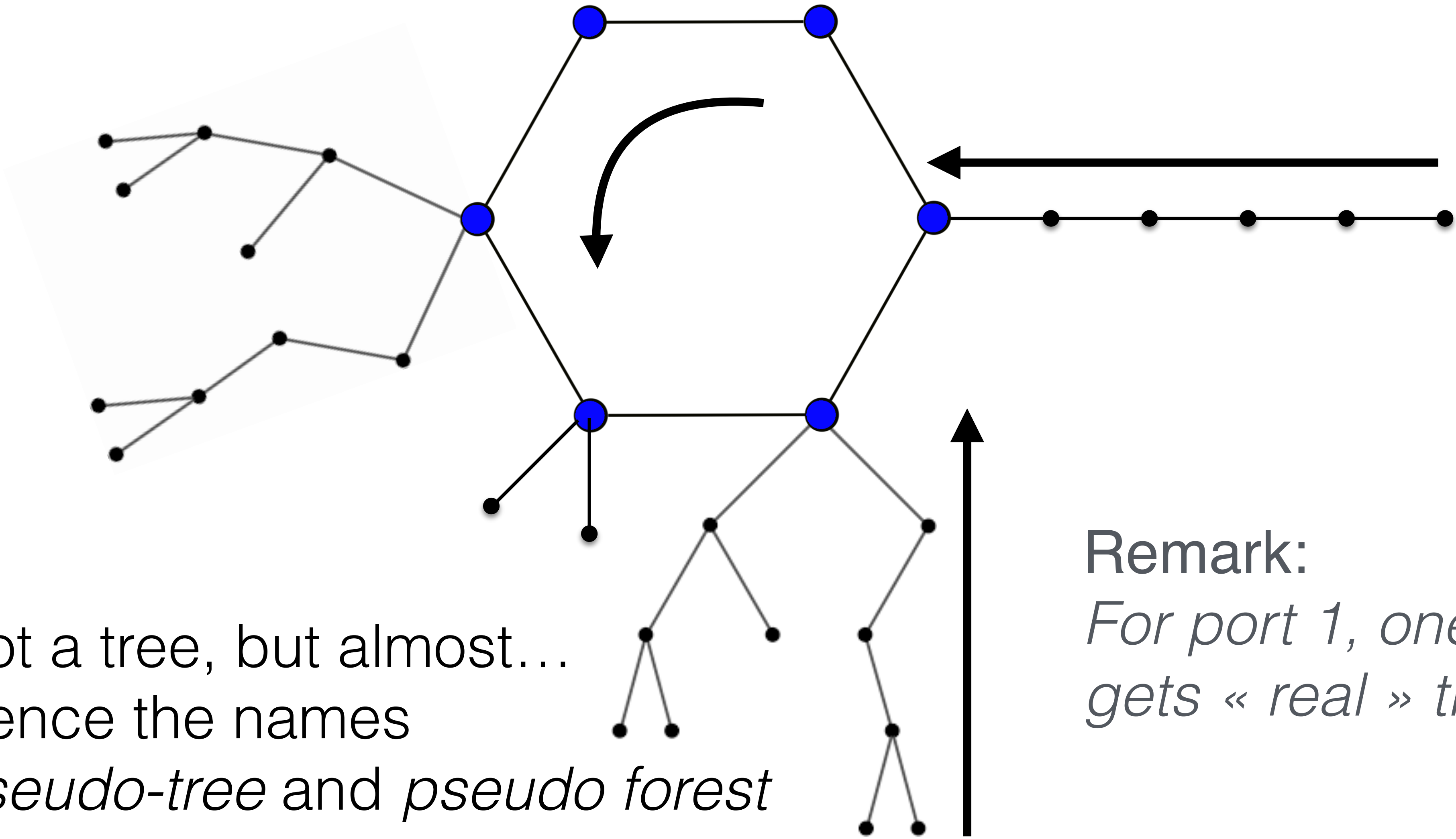


Pseudo-Forest Decomposition





A Connected Component



Remark:
For port 1, one gets « real » trees

Not a tree, but almost…
Hence the names
pseudo-tree and *pseudo forest*

Coloring with 3^Δ colors in $O(\log^\star n)$ rounds

- Every node u orders its incident links from 1 to $\deg(u)$ according to the IDs of its neighbors

- This results in Δ pseudo-forests $F_1, \ldots, F_\Delta$

- Color each pseudo tree in each pseudo forest in parallel, in $O(\log^\star n)$ rounds

- Each node gets a color $c(u) = (c_1(u), \ldots, c_\Delta(u))$ where $c_i(u) \in \{1,2,3\}$, hence 3^Δ colors.

$(\Delta + 1)$-Coloring in $O(\Delta^2 + \log^\star n)$ rounds

- Nodes first compute a 3^Δ-coloring in $O(\log^\star n)$ rounds
- For each u, $c(u) = (c_1(u), \ldots, c_\Delta(u))$ with $c_i(u) \in \{1,2,3\}$
- Iteratively compute a $(\Delta + 1)$-coloring c_i' of $\cup_{j=1}^{i} F_j$ for $i = 1, \ldots, \Delta$
  - $c_1' = c_1$ is a 3-coloring of F_1
  - Given c_i', let us view (c_i', c_{i+1}) as a $3(\Delta + 1)$-coloring of $\cup_{j=1}^{i+1} F_j$
  - The coloring (c_i', c_{i+1}) can be transformed into a $(\Delta + 1)$-coloring c_{i+1}' of $\cup_{j=1}^{i+1} F_j$ in $2(\Delta + 1)$ rounds
- The coloring c_Δ' is a $(\Delta + 1)$-coloring of $\cup_{j=1}^{\Delta} F_j = G$, obtained in $(\Delta - 1)\big(2(\Delta + 1)\big) = O(\Delta^2)$ rounds.

$(\Delta + 1)$-Coloring in $O(\Delta + \log^\star n)$ rounds

Four phases:

1. 3^Δ-coloring in $O(\log^\star n)$ rounds (cf. previous slides)
2. Reducing to $O(\Delta^3)$-coloring in 1 round
3. Reducing number of colors to $O(\Delta^2)$ in 1 round
4. Reducing number of colors to $\Delta + 1$ in $O(\Delta)$ rounds

Phase 2: From 3^Δ to $O(\Delta^3)$ colors in a single round

Lemma [Erdös, Frankl, Füredi, 1985]
For any $k > \Delta \geq 2$, there exists a family $\mathscr{F}$ of k subsets of $\{1,\ldots,5\lceil\Delta^2 \log k\rceil\}$ such that, for any $\Delta + 1$ sets $F_0, \ldots, F_\Delta$ in $\mathscr{F}$, we have $F_0 \nsubseteq \cup_{i=1}^{\Delta} F_i$

Algorithm:

- Range of colors $[1,k]$ with $k = 3^\Delta$
- Node u with color $c(u) \in \{1,\ldots,k\}$ picks set $F_{c(u)} \in \mathscr{F}$
- By the lemma, $\exists x \notin \cup_{v \in N(u)} F_{c(v)}$
- Node u updates its color $c(u)$ to x i.e. $c(u) \leftarrow x$
- Reduction of #colors: $3^\Delta \rightarrow O\left(\Delta^2 \log(3^\Delta)\right) = O(\Delta^3)$

Polynomials on Finite Fields

- For a prime integer q, let $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ i.e., $\mathbb{F}_q$ is $\{0,\ldots,q-1\}$ with arithmetic modulo q
- $\mathbb{F}_q$ is a finite field
- A polynomial of degree d on $\mathbb{F}_q$ is of the form

$$a_0 + a_1X + \ldots + a_dX^d$$

Lemma A polynomial of degree d on $\mathbb{F}_q$ has at most d roots

Corollary Two polynomials of degree d on $\mathbb{F}_q$ may coincide on at most d values.

Phase 3: From $O(\Delta^3)$ to $O(\Delta^2)$ colors in a single round

- Say colors in $[1, \alpha\Delta^3]$ for some $\alpha > 0$
- Let $q = O(\Delta)$ prime with $3\Delta < q$ and $q^4 \geq \alpha\Delta^3$
- There are q^4 polynomials of degree 3 in $\mathbb{F}_q$
- Node u with color $c(u) = i \in [1, \alpha\Delta^3]$ picks set

$$S_{c(u)} = S_i = \{(x, p_i(x)) : x \in \mathbb{F}_q\} \subseteq \mathbb{F}_q \times \mathbb{F}_q$$

- For every $i \neq j$ we have $|S_i \cap S_j| \leq 3$
- Thus $|S_{c(u)} \backslash \cup_{v \in N(u)} S_{c(v)}| \geq |S_{c(u)}| - 3\Delta > 0$
- Node u updates its colors by picking one element in $S_{c(u)} \backslash \cup_{v \in N(u)} S_{c(v)}$

Phase 4: From $O(\Delta^2)$ to $\Delta + 1$ colors in $O(\Delta)$ rounds

- Say colors in $[1, \beta\Delta^2]$ for some $\beta > 0$
- Let $q = O(\Delta)$ prime with $6\Delta < q$ and $q^4 \geq \beta\Delta^2$
- Node u with color $c(u) = i \in [1, \beta\Delta^2]$ picks sequence

$$\sigma_{c(u)} = \sigma_i = \big(p_i(0), p_i(1), \ldots, p_i(q-1)\big)$$

ith polynomial of degree 3 in $\mathbb{F}_q$

For $x = 0$ to $q - 1$ do

- if uncolored then propose color $p_i(x)$
- if no conflicts, then adopt color $p_i(x)$ and terminate

- At most 3 conflicting iterations for each non-terminated neighbor *and* at most 3 conflicting iterations for each terminated neighbor
- Reduce #colors from q to $\Delta + 1$ in $q - (\Delta + 1) = O(\Delta)$ rounds

State of the Art and Open Problems

Best known algorithm performs $(\Delta + 1)$-coloring in

- $O(\log^\star n + \sqrt{\Delta \log \Delta})$ rounds
- $O(\log n \cdot \log^2 \Delta) \leq O(\log^3 n)$ rounds

Can we improve this complexity?

Is there a distributed algorithm running in $O(\log^\star n)$ rounds in LOCAL that properly colors every graph of maximum degree Δ with $o(\Delta^2)$ colors?

End Lecture 2