Lower Bounds for $(\Delta + 1)$-coloring

- Ramsey lower bounds
- Linial's $\log^{\star} n$ lower bound

Simple Lower Bounds

Theorem For all $t \geq 0$, every t-round algorithm fails to 3-color some cycle.

In other words, 3-coloring the cycles cannot be done in $O(1)$ rounds, i.e., requires $\omega(1)$ rounds.

The proof is based on Ramsey's theory.

Ramsey Theorem

Theorem [Ramsey, 1920s] For every positive integers r and s there exists $R = R(r, s)$ such that every edge-coloring of the complete graph on R vertices with two colors blue and red contains a blue clique on r vertices or a red clique on s vertices.

Example: $R(3,3) = 6$

Extension to Hypergraphs

For $k \geq 2$, a k-hypergraph is a hypergraph whose hyperedges are sets of k vertices

Theorem [Ramsey, 1920s] For any integers k and c, and any integers $n_1, \ldots, n_c$, there is an integer $R = R(n_1, \ldots, n_c; k)$ such that if the hyperedges of a complete k-hypergraph of R vertices are colored with c different colors, then there exists $i \in [c]$ such that the hypergraph contain a complete sub-k-hypergraph of order n_i whose hyperedges are all colored i.

Theorem For all $t \geq 0$, every t-round algorithm fails to 3-color some cycle.

Proof: Let $t \geq 0$ and let A be a t-round algorithm.

$A(x_{-t}, \ldots, x_0, \ldots, x_t) \in \{1,2,3\}$ with $x_i \in \{1, \ldots, n\}$

$k = 2t + 1,\ \ c = 3$, and $\ n_1 = n_2 = n_3 = 2t + 2$

Cycle C_n with $n = R(n_1, n_2, n_3; 2t + 1)$ nodes

Color hyperedge $\{x_{-t}, \ldots, x_0, \ldots, x_t\}$ where $x_{-t} < \ldots < x_0 < \ldots < x_t$ with color $A(x_{-t}, \ldots, x_0, \ldots, x_t)$

Ramsey $\implies \exists x_{-t} < \ldots < x_0 < \ldots < x_t < x_{t+1}$ such that $A(x_{-t}, \ldots, x_0, \ldots, x_t) = A(x_{-t+1}, \ldots, x_1, \ldots, x_{t+1})$ ❐

Δ-coloring is hard

Theorem 2-coloring the $2n$-node cycle requires at least n rounds.

Proof: Let A be a t-round algorithm, for $t \leq n - 1$

$A(x_{-t}, \ldots, x_0, \ldots, x_t) \in \{1,2\}$ with $x_i \in \{1,\ldots,2n\}$

$$
\begin{aligned}
A(x_1, x_2, \ldots, x_{2t+1}) &= 1 \\
A(x_2, x_3, \ldots, x_{2t+1}, y) &= 2 \\
A(x_3, x_4 \ldots, x_{2t+1}, y, z) &= 1 \\
A(x_4, \ldots, x_{2t+1}, y, z, x_1) &= 2 \\
&\vdots \\
A(y, z, x_1, \ldots, x_{2t-1}) &= 2 \\
A(z, x_1, \ldots, x_{2t}) &= 1
\end{aligned}
$$

❐

Lower Bound 3-Coloring C_n

- **Theorem** [Linial 1992] Any deterministic algorithm for computing a 3-coloring of the n-node cycle C_n with IDs in $[1,n]$ takes at least $1/2 \cdot \log^\star n - 1$ rounds.

configuration graph

- Linial's original proof:
  - C_n can be c-colored in t rounds $\Longrightarrow \chi(G_{n,t}) \leq c$
  - C_n can be c-colored in t rounds $\Longrightarrow C_n$ can be 2^{2^c}-colored in $t-1$ rounds

- We present a direct proof by Laurinharju & Suomela (2014)

Proof

Definition $\mathscr{A}$ is a k-ary c-coloring function if

- For all $1 \leq x_1 < x_2 < \ldots < x_k \leq n$, $\mathscr{A}(x_1, \ldots, x_k) \in \{1, \ldots, c\}$
- For all $1 \leq x_1 < x_2 < \ldots < x_k < x_{k+1} \leq n$, $\mathscr{A}(x_1, \ldots, x_k) \neq \mathscr{A}(x_2, \ldots, x_{k+1})$

Claim 1: t-tound algorithm $\mathscr{A}$ for 3-coloring C_n

➥ $\mathscr{A}$ is $(2t+1)$-ary 3-coloring function

Claim 2. If $\mathscr{A}$ is a 1-ary c-coloring function then $c \geq n$.

Claim 3. If $\mathscr{A}$ is a k-ary c-coloring function, then there is a $(k-1)$-ary 2^c-colouring function $\mathscr{B}$.

Proof: The following function is a 2^c-colouring function:

$$\mathscr{B}(x_1, \ldots, x_{k-1}) = \{\mathscr{A}(x_1, \ldots, x_{k-1}, x_k) : x_k > x_{k-1}\}$$

For contradiction, let $1 \leq x_1^* < \ldots < x_k^* \leq n$ with

$$\mathscr{B}(x_1^*, \ldots, x_{k-1}^*) = \mathscr{B}(x_2^*, \ldots, x_k^*)$$

Let $d = \mathscr{A}(x_1^*, \ldots, x_k^*)$.

➥ $d \in \mathscr{B}(x_1^*, \ldots, x_{k-1}^*) \Longrightarrow d \in \mathscr{B}(x_2^*, \ldots, x_k^*)$

➥ $\exists x_{k+1}^* > x_k^* : d = \mathscr{A}(x_2^*, \ldots, x_{k+1}^*)$

➥ $\mathscr{A}$ is not proper. ∎

Let $\mathcal{A}$ be a t-tound algorithm for 3-coloring C_n

$\Rightarrow \mathcal{A}$ is a $(2t+1)$-ary 3-coloring function *(by Claim 1)*

$\Rightarrow \exists$ a $(2t)$-ary 2^3-coloring function *(by Claim 3)*

$\Rightarrow \exists$ a $(2t-1)$-ary 2^{2^3}-coloring function *(by Claim 3)*

$\Rightarrow \exists$ a $(2t-2)$-ary $2^{2^{2^3}}$-coloring function *(by Claim 3)*

$\vdots$

$\Rightarrow \exists$ a 1-ary $\left.2^{2^{\cdot^{\cdot^{\cdot^{2^3}}}}}\right\}{\scriptstyle 2t-1}$ -coloring function *(by Claim 3)*

$\Rightarrow 2^{2^{\cdot^{\cdot^{\cdot^{2^3}}}}} \geq n$ *(by Claim 2)*

$\Rightarrow$ t ≥ ½ log*n - 1. ❑

End Lecture 3