LCL Problems

- Randomized algorithm for $(\Delta + 1)$-coloring
- Four basic symmetry-breaking problems
- Luby’s randomized algorithm for MIS
- Locally Checkable Labelings
- Limit of randomization

Randomized (Δ+1)-coloring

- Assume each node picks colors in $\{1,\ldots,\Delta+1\}$ u.a.r.
- For every neighbor v of u we have $\Pr[c(u)=c(v)] = 1/(\Delta+1)$
- Thus $\Pr[\exists v \in N(u) : c(u)=c(v)] \leq \Delta/(\Delta+1)$
- If $\Delta = O(1)$ then each node terminates with constant probability, but not if $\Delta = \omega(1)$ (i.e., depends on n)
- There is however a simple trick resolving this issue

Randomized $(\Delta + 1)$-coloring in $O(\log n)$ rounds

Algorithm (Barenboim and Elkin, 2013) for node u

while uncolored **do**

$\mathscr{C}$ = {colors previously adopted by neighbors}

pick ℓ(u) at random in {0,1,…,Δ+1} - $\mathscr{C}$

- 0 is picked w/ probability ½
- ℓ(u) ∈ {1,…,Δ+1} - $\mathscr{C}$ is picket w/ proba 1/(2(Δ+1-|$\mathscr{C}$|))

if ℓ(u) ≠ 0 **and** ℓ(u) ∉ {colors picked by neighbors}

then adopt ℓ(u) as my color

else remain uncolored

inform neighbors of status

1 round

1 round

Theorem (Barenboim and Elkin, 2013) The $(\Delta+1)$-coloring algorithm takes, w.h.p., $O(\log n)$ rounds.

Claim For every node u, at any round, Pr[u terminates] ≥ ¼

$$\begin{aligned}
\Pr[u \text{ termine}] &= \Pr[\ell(u) \neq 0 \text{ et aucun } v \in N(u) \text{ satisfait } \ell(v) = \ell(u)] \\
&= \Pr[\forall v \in N(u), \ell(v) \neq \ell(u) \mid \ell(u) \neq 0] \cdot \Pr[\ell(u) \neq 0] \\
&= \frac{1}{2} \cdot \Pr[\forall v \in N(u), \ell(v) \neq \ell(u) \mid \ell(u) \neq 0]
\end{aligned}$$

$$\begin{aligned}
\Pr[\ell(v) = \ell(u) \mid \ell(u) \neq 0] &= \Pr[\ell(v) = \ell(u) \mid \ell(u) \neq 0 \wedge \ell(v) = 0]\ \Pr[\ell(v) = 0] \\
&\quad + \Pr[\ell(v) = \ell(u) \mid \ell(u) \neq 0 \wedge \ell(v) \neq 0]\ \Pr[\ell(v) \neq 0] \\
&= \Pr[\ell(v) = \ell(u) \mid \ell(u) \neq 0 \wedge \ell(v) \neq 0]\ \Pr[\ell(v) \neq 0] \\
&\leq \frac{1}{2}\ \Pr[\ell(v) = \ell(u) \mid \ell(u) \neq 0 \wedge \ell(v) \neq 0] \\
&= \frac{1}{2}\ \frac{1}{\Delta + 1 - |C(u)|}.
\end{aligned}$$

$$\Pr[\exists v \in N(u) : \ell(v) = \ell(u) \mid \ell(u) \neq 0] \leq (\Delta - |C(u)|) \frac{1}{2(\Delta + 1 - |C(u)|)} < \frac{1}{2} \quad \blacksquare$$

O(log n) rounds w.h.p.

Pr[u does not terminate in $k\ln(n)$ rounds]

$$\leq (3/4)^{k\ln(n)} = n^{-k\ln(4/3)}$$

Pr[$\exists u$ that does not terminate in $k\ln(n)$ rounds] $\leq n^{1-k\ln(4/3)}$

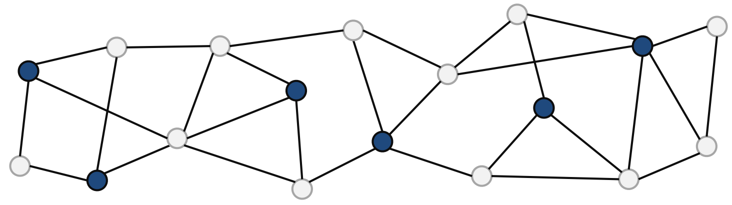
Let $c > 1$, by choosing $k = \dfrac{1+c}{\ln(4/3)}$, we get:

Pr[all nodes terminates after $\dfrac{1+c}{\ln(4/3)}\ln(n)$ rounds] $\geq 1 - 1/n^c$

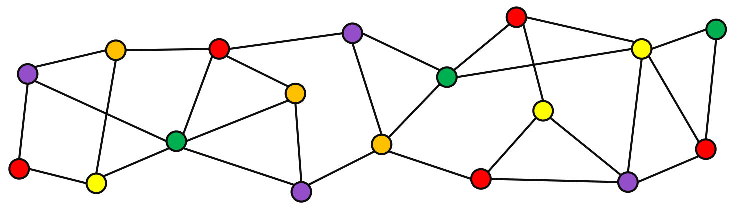
❏

Four classical problems

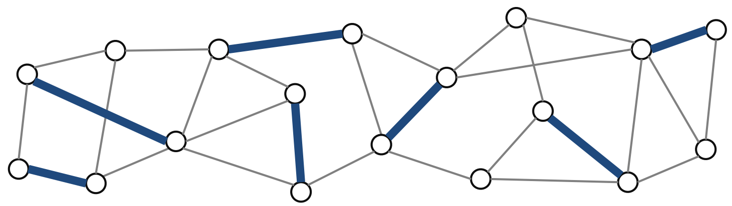
MIS



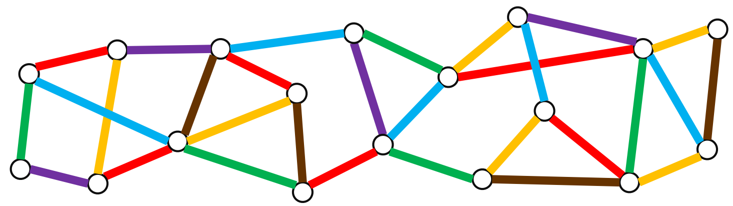
-Vertex Coloring



Maximal Matching



-Edge Coloring

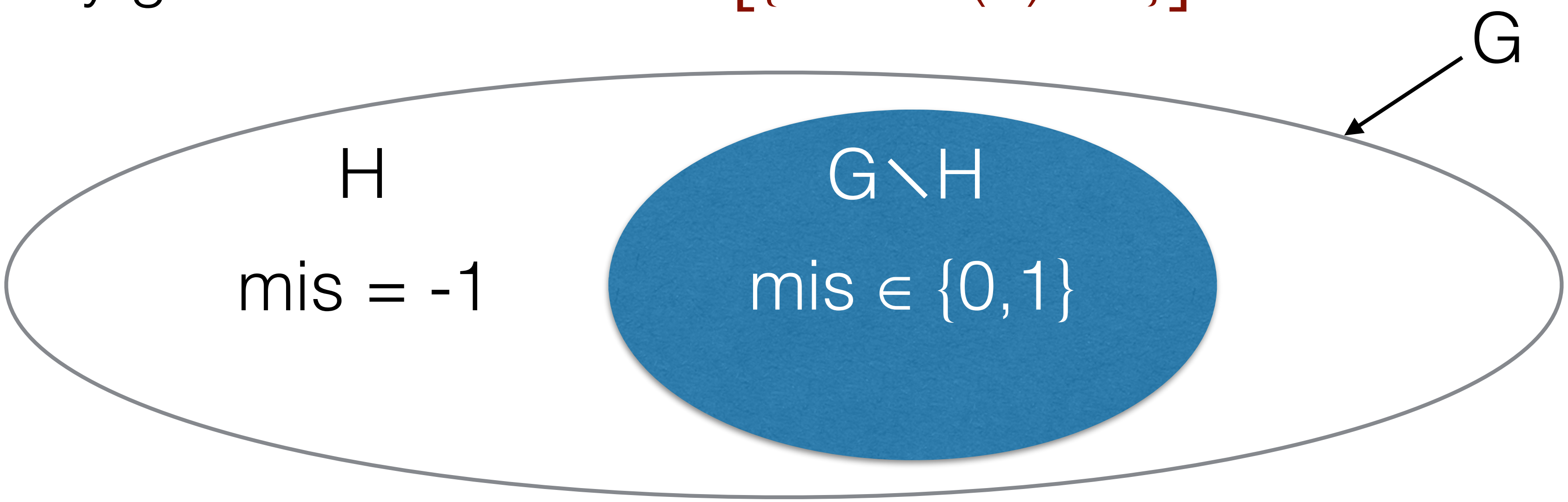


Randomized algorithm for MIS

Algorithm (Luby, 1985)

mis(u) $\in$ {-1,0,1} = {undecided, not in MIS, in MIS}

At any given round: $H = G\big[\{u : \text{mis}(u) = -1\}\big]$



<u>Trick:</u> enforcing an order between nodes:

$v \succ u \iff \deg_H(v) > \deg_H(u)$
$\quad$ or $(\deg_H(v) = \deg_H(u)$ and $ID(v) > ID(u))$

Luby's algorithm

One phase of the algorithm for node u with mis(u) = -1

if $deg_H(u) = 0$ **then** mis(u) ← 1
else join(u) ← true with proba $1/(2\, deg_H(u))$, false otherwise
    exchange join with every $v \in N(u)$
    free(u) ← $\nexists\, v \in N(u)$ such that $v \succ u$ and join(v)=true
    if (join(u) = true **and** free(u) = true) **then** mis(u) ← 1
    exchange mis with every $v \in N(u)$
    if (mis(u) = -1 **and** $\exists v \in N(u)$ mis(v)=1) **then** mis(u) ← 0
    exchange mis with every $v \in N(u)$

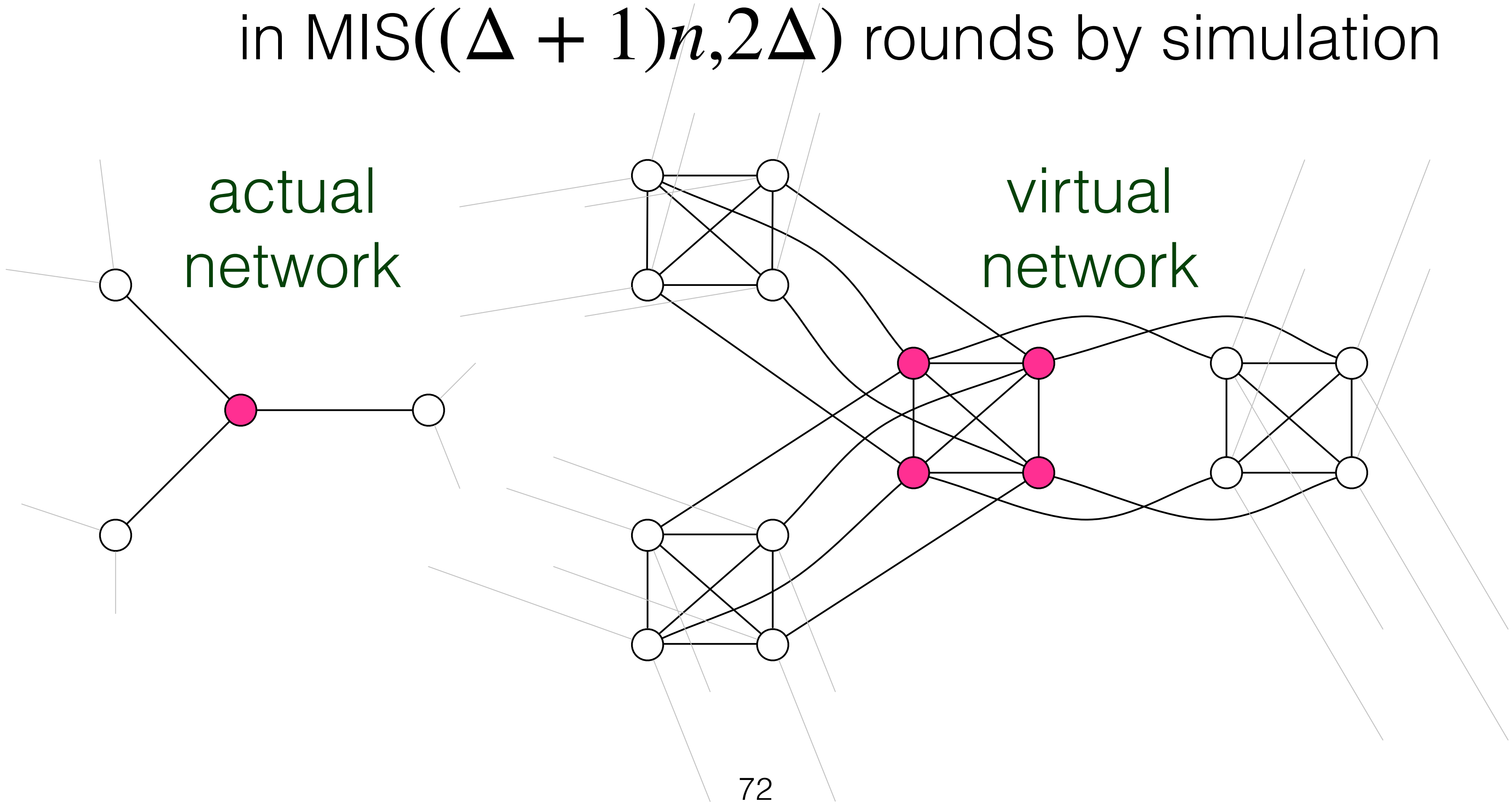
Round-Complexity of Luby's Algorithm

Remark A very similar algorithm was independently discovered by Alon, Babai, and Itai (1986).

Theorem Luby's algorithm terminates in $O(\log n)$ rounds, w.h.p.

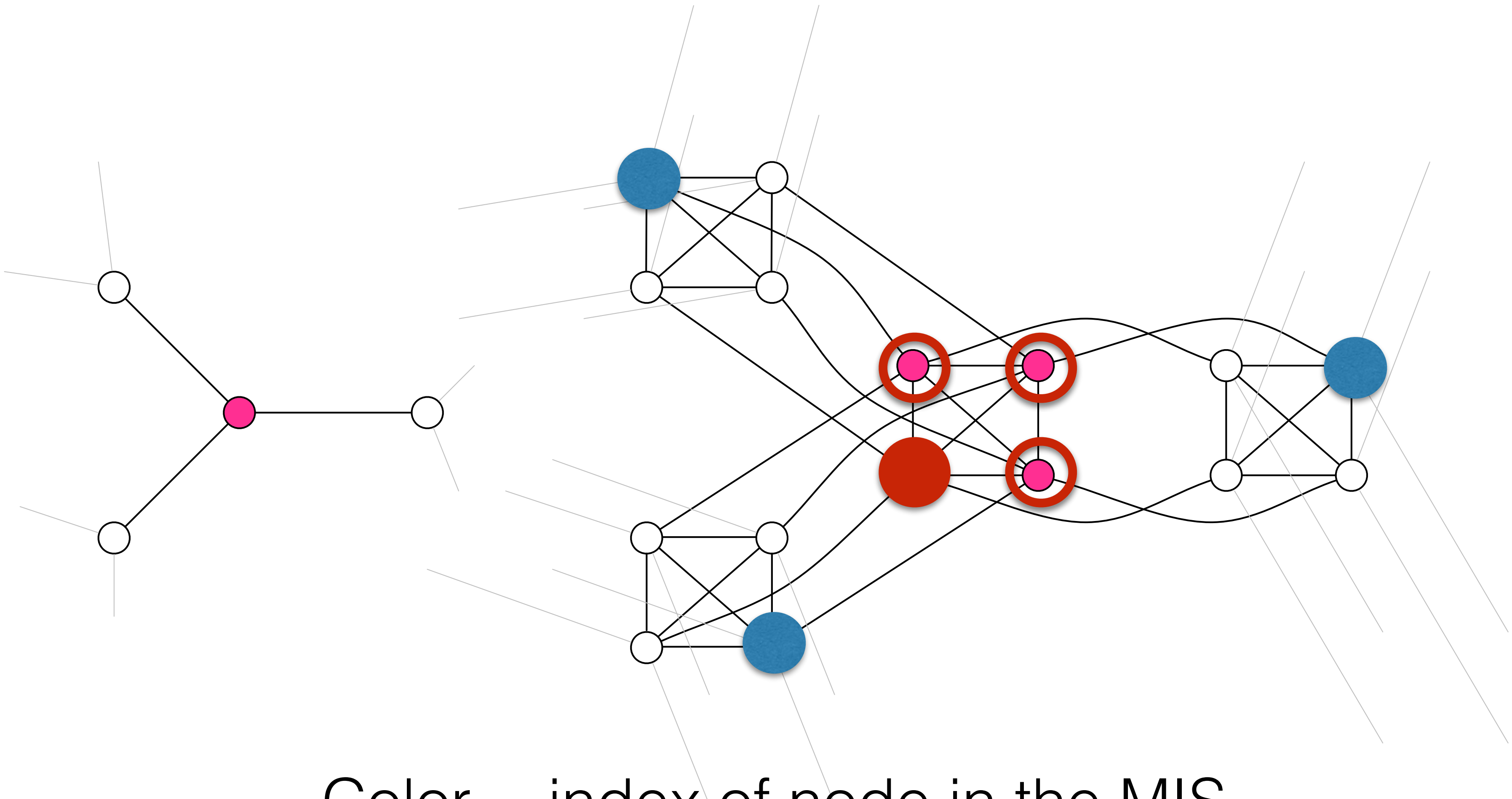
Reduction

- (Δ+1)-coloring ➜ MIS
  in Δ rounds by maximizing {1}
- MIS ➜ (Δ+1)-coloring
  in MIS$((\Delta + 1)n, 2\Delta)$ rounds by simulation



Claim 1. At most one node of each clique in the MIS

Claim 2. At least one node of each clique in the MIS

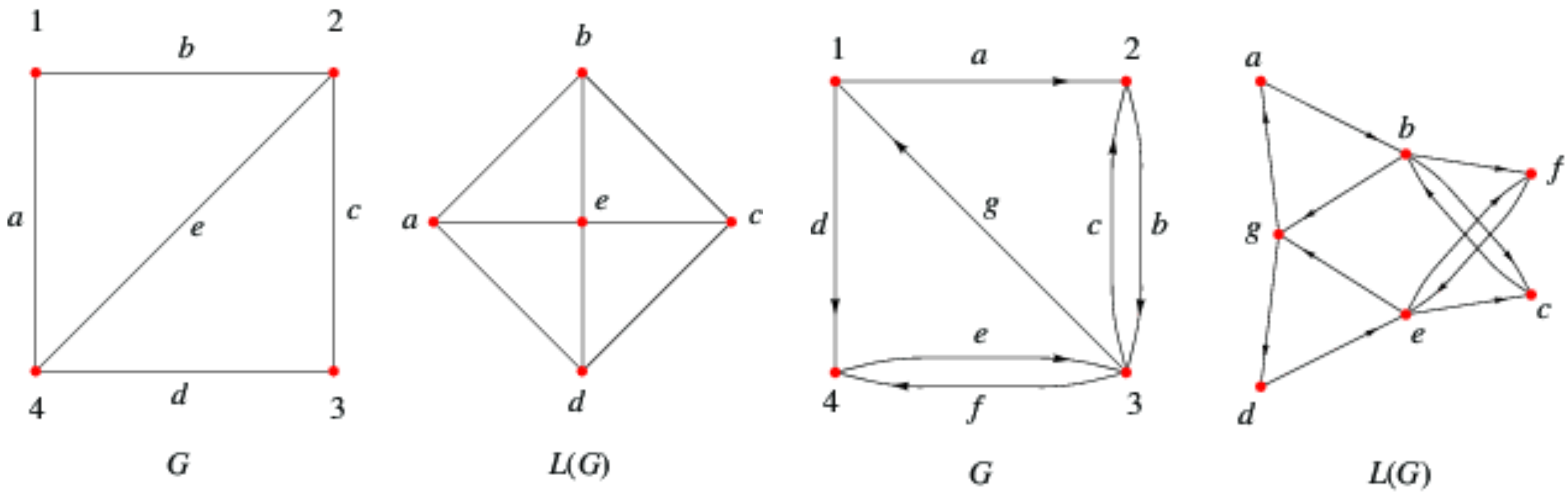


Color = index of node in the MIS

Line Graphs

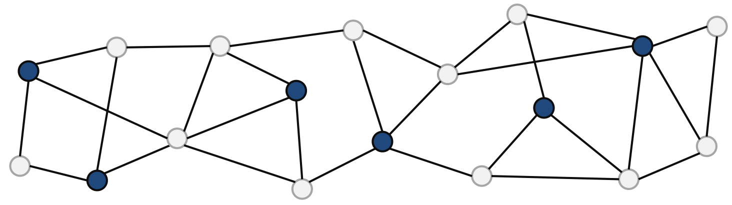
Definition The line graph of a graph G is the graph L(G) such that

- $V(L(G)) = E(G)$
- $\{e,e'\} \in E(L(G)) \iff e$ and e' are incident in G



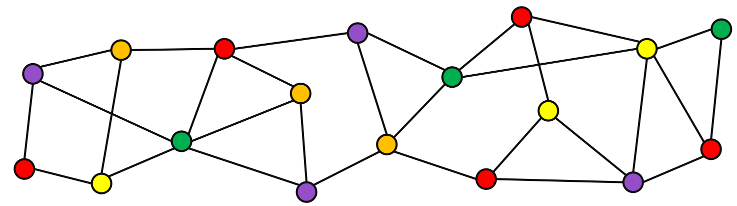
Reductions

MIS



clique reduction

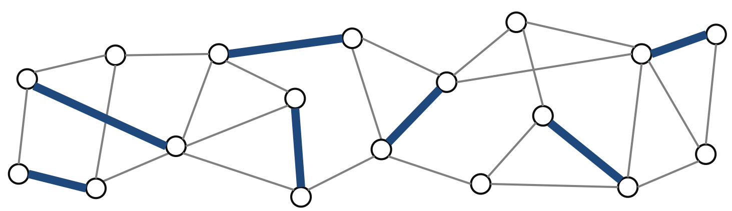
-Vertex Coloring



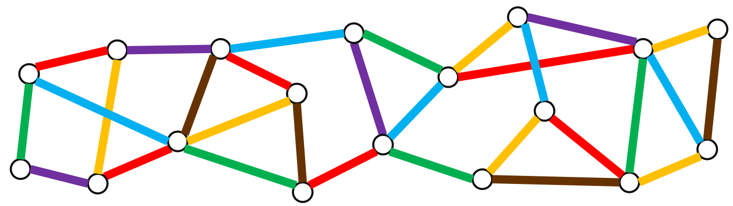
MIS on line graph

coloring on line graph

Maximal Matching



-Edge Coloring



State of the Art and Open Problems

Each of the four problems $(\Delta + 1)$-coloring, MIS, Maximal Matching, and $(2\Delta - 1)$-edge-coloring has

- a deterministic algorithm running in $O(\log^c n)$ rounds, $c \geq 1$
- a randomized algorithm running in $O(\log^c \log n)$ rounds, $c \geq 1$

➡ Improving the exponent of the logs and loglogs?

➡ Lower bounds?

Recent breakthrough: $\Omega(\log n / \log \log n + \Delta)$-round lower bound for deterministic MIS (this will be taught later in the course)

For $(\Delta + 1)$-coloring, no better deterministic lower bound that $\Omega(\log^\star n)$ rounds

Locally Checkable Labeling

Let $\mathcal{F}_\Delta$ be the set of all (connected) graphs with maximum degree Δ.

Definition (Naor and Stockmeyer, 1995) An LCL in $\mathcal{F}_\Delta$ is specified by a <u>finite</u> set of labels, and a <u>finite</u> set of labeled balls with maximum degree Δ, called good balls.

Examples:

- k-coloring, k-edge-coloring
- maximal independent set (MIS)
- maximal matching
- Etc.

Focus is on LCL tasks solvable sequentially by a greedy algorithm selecting nodes in arbitrary order, like, e.g., k-coloring for $k \geq \Delta+1$.

Solving an LCL Problem

Let $\Delta \geq 2$, and let L be an LCL in $\mathscr{F}_\Delta$

The LCL problem associated to L consists, for all nodes of any graph $G \in \mathscr{F}_\Delta$, to compute a label such that the collection of labels results in good balls centered at every node of G.

The definition can be generalized with inputs given to the nodes, in addition to their IDs.

The Limited Power of Randomized Algorithms

$\Pr[\text{success}] \geq 1 - 1/n$

Theorem [Y.-J. Chang, T. Kopelowitz, S. Pettie (2016)]

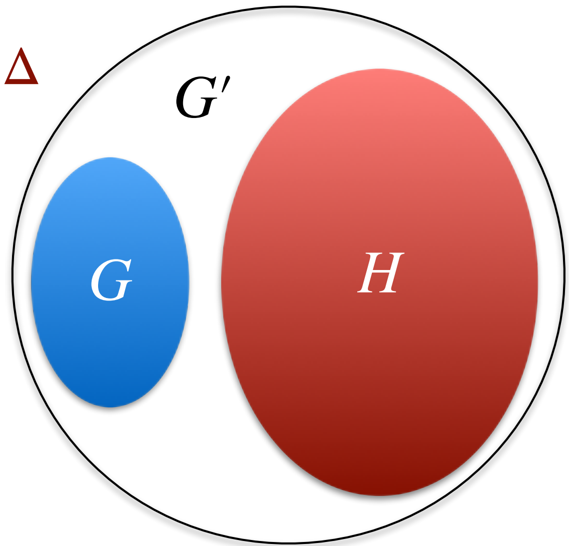
For any LCL problem, its randomized (Monte Carlo) complexity on instances of size n is **at least** its deterministic complexity on instances of size $\sqrt{\log n}$

One needs to design better deterministic algorithms for improving the performances of randomized algorithms!

Proof

- Let Π be an LCL problem
- Let $\mathsf{Det}_{\Pi}(n, \Delta)$ and $\mathsf{Rand}_{\Pi}(n, \Delta)$ be the LOCAL deterministic and randomized complexity of Π for instance of size n and maximum degree Δ.
- We show that $\mathsf{Det}_{\Pi}(n, \Delta) \leq \mathsf{Rand}_{\Pi}(2^{n^2}, \Delta)$
- We assume that, initially, each node v knows its ID, as well as n and Δ, with $\mathsf{ID}(v) \in \{0,1\}^{c \log n}$, for some $c \geq 1$.
- Let $\mathcal{G}_{n,\Delta}$ be the family of n-node graphs with maximum degree Δ, and nodes with IDs on $c \log n$ bits.

- Let $\mathscr{R}$ be an optimal randomized algorithm for Π
- Our aim is to construct a deterministic algorithm $\mathscr{D}$ for Π
- Assume that $\mathscr{R}$
  - performs in $t(n, \Delta)$ rounds, and
  - uses $r(n, \Delta)$ random bits at each node.
- Note that $|\mathscr{G}_{n,\Delta}| \leq 2^{\binom{n}{2}+cn\log n} \ll 2^{n^2}$
- We consider graphs $G \in \mathscr{G}_{n,\Delta}$ and let $N = 2^{n^2}$
- We also consider $G' = G \cup H$ with $H \in \mathscr{G}_{N-n,\Delta}$
- In G', nodes are given (N, Δ) as input
- We have: $\Pr[\mathscr{R} \text{ fails in } G'] \leq 1/N$



- Let us consider any function ϕ of the form

$$\phi : \{0,1\}^{c \log n} \to \{0,1\}^{r(N,\Delta)}$$

- Let $\mathscr{R}[\phi]$ be the deterministic algorithm equal to $\mathscr{R}$ with the fixed random strings determined by ϕ, i.e., node v uses $\mathscr{R}$ with bit-string $\phi(\mathrm{ID}(v))$.

- $\mathscr{R}[\phi]$ runs in $t(N, \Delta)$ rounds in G'

- We said that ϕ is *bad* if $\mathscr{R}[\phi]$ fails on some $G \in \mathscr{G}_{n,\Delta}$

$$\Pr_{\phi \sim unif}[\phi \text{ bad}] \leq \sum_{G \in \mathcal{G}_{n,\Delta}} \Pr_{\phi \sim unif}[\mathcal{R}[\phi] \text{ errs on } G]$$

$$= \sum_{G \in \mathcal{G}_{n,\Delta}} \Pr[\mathcal{R} \text{ errs on } G \text{ with input } (N, \Delta)]$$

$$\leq |\mathcal{G}_{n,\Delta}|/N < 1$$

It follows that there exists $\phi^\star : \{0,1\}^{c \log n} \to \{0,1\}^{r(N,\Delta)}$ good

Deterministic algorithm $\mathcal{D}$: on input (n, Δ), every node v

- computes $N = 2^{n^2}$, $t(N, \Delta)$ and $r(N, \Delta)$
- performs $\mathcal{R}[\phi^\star]$ for $t(N, \Delta)$ rounds

By construction $\mathcal{D}$ never errs in $\mathcal{G}_{n,\Delta}$ ❑

End Lecture 4