Announcements

- No class on Oct 31 (DISC 2024)
- Class on Nov 7 (7th lecture)
- Nov 14 (8th lecture) will be dedicated to exercices
- No class on Nov 21
- Partial exam on Nov 28 or Dec 5 (to be specified ASAP)

Roadmap

$(\Delta + 1)$-coloring in polylog(n) rounds

- Network decomposition
- Derandomization

Network Decomposition

Network Decomposition

Definition A (c, d)-decomposition of an n-node graph $G = (V, E)$ is a partition of V into clusters such that each cluster has diameter at most d and the cluster graph is properly colored with colors in $\{1, \ldots, c\}$.

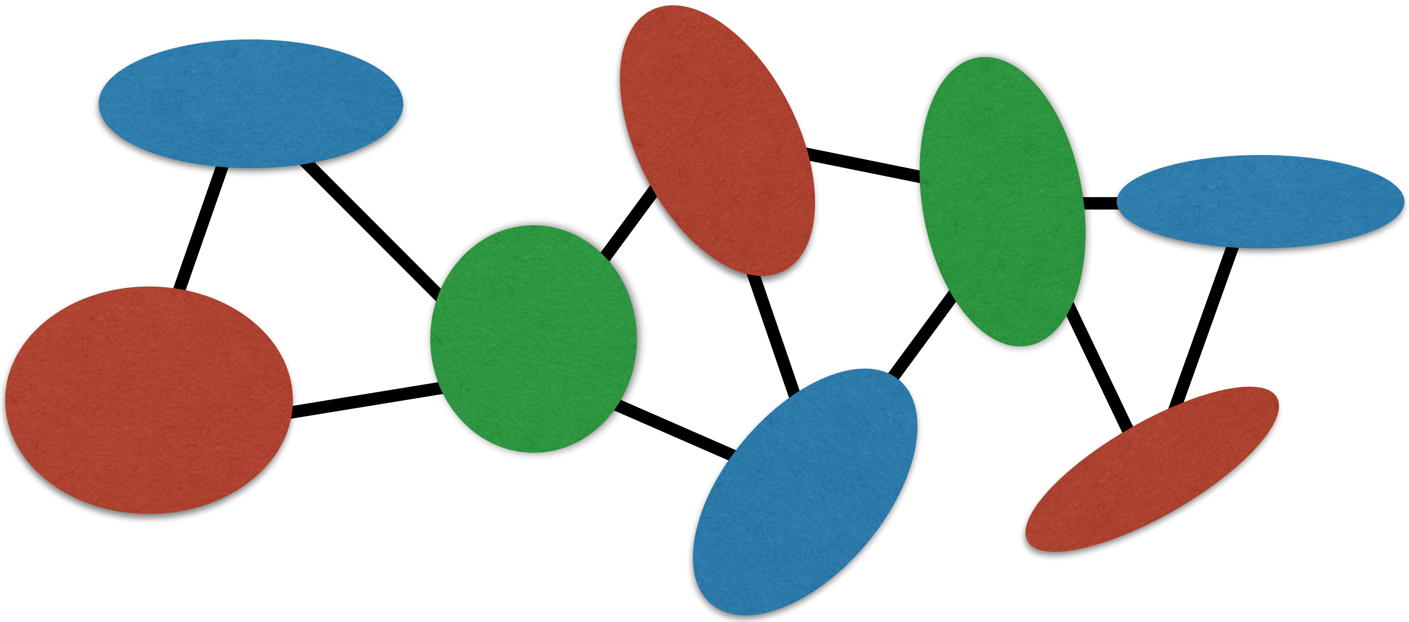
Theorem [Linial and Saks (1993)]
Every graph has a $(O(\log n), O(\log n))$ decomposition, and such a decomposition can be computed by a *randomized* algorithm in $O(\log^2 n)$ rounds in the LOCAL model.

Theorem [Panconesi and Srinivasan (1992)]
A $(2^{\sqrt{\log n}}, 2^{\sqrt{\log n}})$-decomposition can be computed *deterministically* in $2^{\sqrt{\log n}}$ rounds in the LOCAL model.

Impact on coloring and MIS

Lemma Given a (c, d)-decomposition, $(\Delta + 1)$-coloring and MIS can be solved in $O(cd)$ rounds in the LOCAL model.

Proof



Proceed in c phases, each of $O(d)$ rounds ❑

Fast Network Decomposition

Theorem [Rozhon and Ghaffari (2019)]
A $(O(\log n), O(\log n))$-decomposition can be computed deterministically in polylog(n) rounds in the LOCAL model.

Corollary $(\Delta + 1)$-coloring and MIS can be deterministically solved in polylog(n) rounds in the LOCAL model.

Derandomization

Derandomization

Assume every node v of $G = (V, E)$ maintains

$$D(v) = (p_1, \ldots, p_{\Delta+1})$$

$\Pr[c(v) = x] = p_x$

Initially $D(v) = (\dfrac{1}{\Delta + 1}, \ldots, \dfrac{1}{\Delta + 1})$

Objective: $D(v) = (0,\ldots,0,1,0,\ldots,0)$

Framework

- Assume given a proper $O(\Delta^2)$-coloring of G
- Assume (for simplicity) that $\Delta + 1 = 2^k$
- Derandomization proceeds in a series of k phases
- At phase $i = 1, \ldots, k$: $D(v) = (p_1, \ldots, p_{\Delta+1})$ with
  - 2^{k-i+1} entries equal to $1/2^{k-i+1}$
  - all other entries equal to 0

Initially

$X =$ #monochromatic edges $= \sum_{e \in E} X_e$

$$\mathbb{E}X = \sum_{e \in E} \mathbb{E}X_e \leq \frac{1}{\Delta + 1} \cdot \frac{n\Delta}{2} \leq \frac{n}{2}$$

Let $X_v =$ #monochromatic edges incident to $v \in E$

Ordering the probability

$D(v) = (p_1, \ldots, p_{\Delta+1})$

For $x \in \{1, \ldots, \Delta + 1\}$, let $E_x = \mathbb{E}[X_v \mid c(v) = x]$

$C_{small} = \{2^{k-i}$ colors x with smallest non null $E_x\}$

$C_{large} = \{2^{k-i}$ colors x with largest non null $E_x\}$

Action: double p_x for $x \in C_{small}$ (and others p_x set to 0)

Impact on #monochromatic edges

Treat each color class in $\{1,\ldots,O(\Delta^2)\}$ separately

$$\mathbb{E}X_v^{new} = \sum_{x\in[\Delta+1]} \mathbb{E}[X_v \mid c(v)=x]\cdot \Pr[c(v)=x]$$

$$= \sum_{x\in C_{small}} \Big(\sum_{u\in N(v)} \Pr[c(u)=x]\Big)\cdot \Pr[c(v)=x]$$

$$\leq \sum_{x\in C_{small}\cup C_{large}} \Big(\sum_{u\in N(v)} \Pr[c(u)=x]\Big)\cdot \Big(\frac{1}{2^{k-i+1}}+\frac{1}{2^{k-i+1}}\Big)$$

$$\leq \mathbb{E}X_v^{old}$$

After $k = \log(\Delta + 1)$ phases

- For every v the distribution $D(v)$ is integral, i.e., D uses $(\Delta + 1)$ colors, but it is not necessarily proper.
- #monochromatic edges $\leq n/2$
- Let $G' = (V, E')$ where $E' =$ {monochromatic edges}
- Number of nodes with degree > 4 is at most $n/4$
- Compute MIS in subgraph G'' of nodes with degree ≤ 4, in $O(\log^\star \Delta)$ rounds
- Nodes in the MIS adopt their colors, and terminate
- Any MIS in G'' is of size $\dfrac{3n/4}{5} \implies \dfrac{3n}{20}$ nodes terminate

In Total…

- $O(\log n)$ iterations
- each iteration takes $O(\log \Delta)$ phases
- But… each phase takes $O(\Delta^2)$ rounds!

Defective Coloring

A coloring $\gamma : V \to \{1,\ldots,k\}$ of $G = (V, E)$ is d-defective if every node has at most d neighbors with the same color.

Let $w : E \to \mathbb{R}^+$ and $\epsilon > 0$. A coloring γ is an ϵ-average defective coloring if

$$\sum_{e=\{u,v\} \mid \gamma(u)=\gamma(v)} w(e) \leq \epsilon \sum_{e \in E} w(e)$$

Lemma There exists an $O(\log \Delta)$-round deterministic algorithm that computes an $(1/\log \Delta)$-average defective coloring using $O(\log \Delta)$ colors.

Application

- For weight w defined as $\forall e \in E, w(e) = \mathbb{E}X_e$ compute an $(1/\log \Delta)$-average defective $O(\log \Delta)$-coloring γ at the beginning of each phase, in $O(\log \Delta)$ rounds

- Ignore monochromatic edges in γ, and treats each color classes separately.

- $X =$ #monochromatic edges

- $\mathbb{E}X_{new} \leq (1 + 4/\log \Delta) \cdot \mathbb{E}X_{old}$

- $\mathbb{E}X_{final} \leq (1 + 4/\log \Delta)^{\log \Delta} \cdot \mathbb{E}X_{initial} \leq O(n)$

Wrap Up

Theorem [Ghaffari and Kuhn, 2021]
There exists an $O(\log n \cdot \log^2 \Delta)$-round deterministic algorithm solving $(\Delta + 1)$-coloring in any n-node graph with maximum degree Δ.

For large Δ (e.g., $\Delta = \Omega(n^{\epsilon})$) there is a faster algorithm (using network decomposition) running in $O(\log^2 n)$ rounds.

For small degrees, $O(\sqrt{\Delta} + \log^{\star} n)$ rounds

End Lecture 5