Announcements

- Nov 14 (8th lecture) will be dedicated to exercices
- No class on Nov 21
- Partial exam on Nov 28

Roadmap

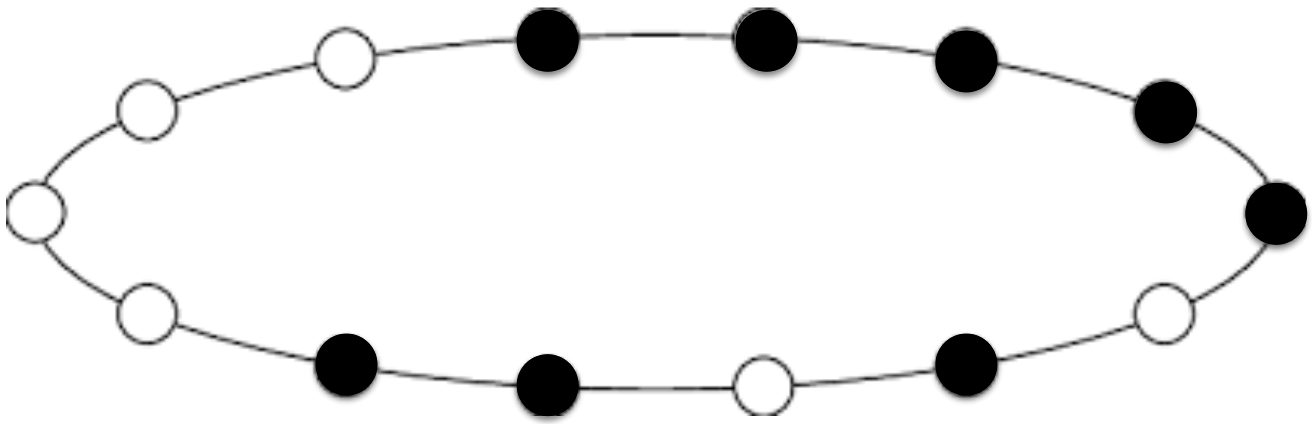
Randomized $(\Delta + 1)$-coloring in $poly(\log \log n)$ rounds

- Graph Shattering
- A Monte-Carlo algorithm
- Concentration bounds

Graph Shattering

Randomized Algorithms using Shattering

Pick ● or ○ u.a.r.



W.h.p., max length monochromatic interval $\leq O(\log n)$

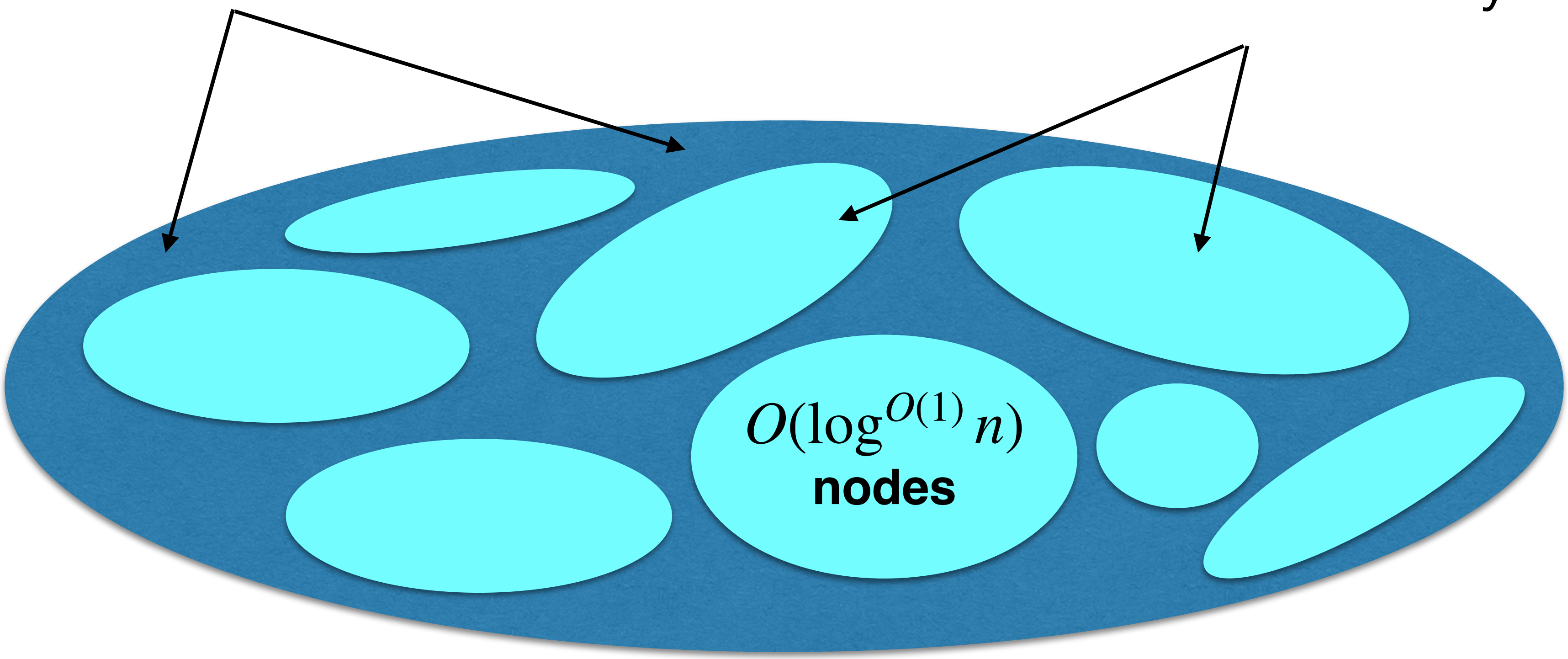
$\Longrightarrow$ 3-coloring or MIS in $O(\log^\star \log n)$ rounds

Graph Shattering

1. Shatter the graph using randomization
2. Complete each piece deterministically

parts that are fixed after 1.

parts that remain to be fixed by 2.



$$\text{Rand}(n) \approx \text{Det}(O(\log^{O(1)} n))$$

A Monte-Carlo Algorithm

Algorithm *OneShotColoring*

[L. Barenboim, M. Elkin, S. Pettie, J. Schneider (2015)]

For $G = (V, E)$, let $U = \{v \in V \mid c(v) = \perp\}$

$$\Psi(v) = \{1, \ldots, \deg(v) + 1\} \setminus \cup_{u \in N_G(v)} \{c(u)\}$$

For all $v \in U$ in parallel:

- pick $c_{tmp}(v) \in \Psi(v)$ u.a.r.
- if v satisfies $\mathrm{ID}(v) > \mathrm{ID}(u)$ for all $u \in N_U(v)$ such that $c_{tmp}(u) = c_{tmp}(v)$, then $c(v) \leftarrow c_{tmp}(v)$

Notations

Let $N_U^+(v) = \{u \in N_U(v) \mid \mathrm{ID}(u) > \mathrm{ID}(v)\}$

For every $q \in \Psi(v)$, let

$$\Psi^{-1}(q) = \{u \in N_U^+(v) \mid q \in \Psi(u)\}$$

Let $w(q) = \displaystyle\sum_{u \in \Psi^{-1}(q)} 1/|\Psi(u)|$

Remark: $1/|\Psi(u)| \leq 1/(\deg_U(u) + 1) \leq 1/2$

In particular: $w(q) \leq 1$

Analysis (1)

$$\Pr[q \notin c_{tmp}(N_U^+(v))] = \Pi_{u \in \Psi^{-1}(q)}(1 - 1/|\Psi(u)|)$$

$\forall x \in [0,1/2],\ 1 - x \geq (1/4)^x$

$$\geq \Pi_{u \in \Psi^{-1}(q)}(1/4)^{1/|\Psi(u)|} = (1/4)^{w(q)}$$

Lemma 1. $\Pr[v \text{ is colored}] \geq 1/4$

Let $X_q \in \{0,1\}$ indicator of whether color q can be adopted by v

For $X = \sum_{q \in \Psi(v)} X_q$ we have $\mathbb{E}X \geq \sum_{q \in \Psi(v)} (1/4)^{w(q)}$

$$\implies \mathbb{E}X \geq |\Psi(v)| \cdot (1/4)^{\sum_q w(q)/|\Psi(v)|}$$

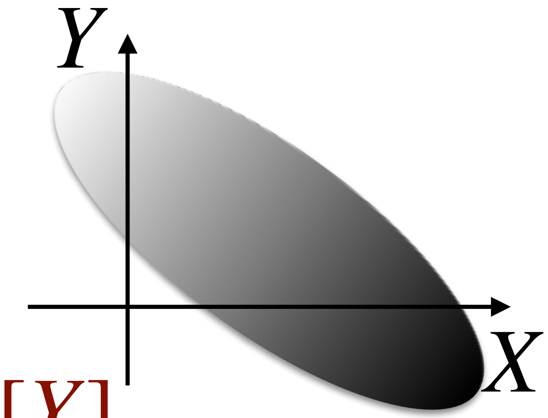
$$\geq |\Psi(v)| \cdot (1/4)^{\deg_U(v)/|\Psi(v)|} > |\Psi(v)|/4$$

Analysis (2)

Let $\mathscr{E}_v$ be the event « $X \geq |\Psi(v)|/8$ », i.e., 1/8 of v's colors are « good »

The random variables X_q are not independent, but they are negatively correlated.

Negative Correlation



$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X) \cdot (Y - \mathbb{E}Y)]$$

Lemma $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Definition X and Y are negatively correlated if

$$\mathbb{E}[XY] \leq \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Definition A collection $X_1, \ldots, X_k$ of random variables are negatively correlated if, for every $S \subseteq [k]$,

$$\mathbb{E}[\Pi_{i \in S} X_i] \leq \Pi_{i \in S} \mathbb{E} X_i$$

Concentration

Theorem Let $X = X_1 + \ldots + X_k$ be the sum of k independent or negatively correlated random variables X_i with values in $[a_i, b_i]$ for all $i \in [k]$. For every $\lambda > 0$,

$$\Pr[X \leq \mathbb{E}X - \lambda] \leq \exp\left(-\frac{2\lambda^2}{\sum_{i=1}^{k}(b_i - a_i)^2}\right)$$

The same bound applies to $\Pr[X \geq \mathbb{E}X + \lambda]$.

Analysis (2)

Let $\mathscr{E}_v$ be the event « $X \geq |\Psi(v)|/8$ », i.e., 1/8 of v's colors are « good »

The random variables X_q are not independent, but they are negatively correlated. Therefore, with $\lambda = \mathbb{E}X/2$, we get:

$$\Pr[\neg \mathscr{E}_v] < \exp\left(-\frac{2 \cdot (|\Psi(v)|/8)^2}{|\Psi(v)|}\right) = \exp\left(-\frac{|\Psi(v)|}{32}\right)$$

Lemma 2. Let U be the uncolored nodes, and let $v \in U$

$$\Pr[\mathscr{E}_v] \geq 1 - \exp\left(-\frac{\deg_U(v) + 1}{32}\right)$$

Shattering Property

Let Δ_U be the max degree of the subgraph $G[U]$ induced by U.

Let $\alpha \geq 1$.

Property 1. With probability at least $1 - 1/n^\alpha$, after $5 \log_{4/3} \Delta_U$ iterations of *OneShotColoring*, all uncolored components have at most $\alpha \cdot \Delta_U^2 \cdot \log_{\Delta_U} n$ nodes.

Corollary There exists a randomized Monte-Carlo $(\Delta + 1)$-coloring algorithm running in $O(\log^2 \Delta + \log^2 \log n)$ rounds, succeeding w.h.p.

Proof. After $O(\log \Delta)$ rounds, each ν-node component can be colored deterministically in $O(\log^2 \nu)$ rounds, where $\nu \leq \alpha \cdot \Delta^2 \cdot \log_\Delta n$. $\square$

Basic Facts

Claim 1. The number of rooted unlabeled t-node trees is less than 4^t.

Proof. The Euler tour of such a tree can be encoded as a bit-vector with length $2t$. $\square$

Claim 2. The number of ways to embed a t-node tree in an n-node graph of maximum degree Δ is less than $n\Delta^{(t-1)}$.

Proof. There are n choices for the root, and less than Δ choices for each subsequent node. $\square$

Proof of Property 1

If $\text{dist}(v, v') \geq 3$ then $\mathcal{E}_v$ and $\mathcal{E}_{v'}$ are independent.

Let $T \subseteq U$ such that

1. $|T| = \alpha \log_{\Delta_U} n$
2. $\forall v, v' \in T, \text{dist}(v, v') \geq 3$
3. T is a tree in $G^3[U]$

Remark. If the property does not hold, then a partially uncolored set T as above exists.

There are at most $4^{\alpha \log_{\Delta_U} n} \cdot \Delta_U^{3(\alpha \log_{\Delta_U} n - 1)} \cdot n < n^{4\alpha}$ such sets T.

By Lemma 1, $\Pr[T \text{ uncolored}] \leq \left(\frac{3}{4}\right)^{(\alpha \log_{\Delta_U} n) \cdot (5 \log_{4/3} \Delta_U)} = n^{-5\alpha}$

By union bound, $\Pr[\exists T, T \text{ partially uncolored}] \leq n^{4\alpha - 5\alpha} = 1/n^{\alpha}$. $\square$

Decreasing Degrees

Let $U^{+} = \{v \in U \mid \deg_U(v) > c \ln n\}$

Let $\mathscr{E} = \bigwedge_{v \in U^{+}} \mathscr{E}_v$

Property 2. $\Pr[\neg \mathscr{E}] < n^{-c/32+1}$, and if U_0 and U_1 denotes the set of uncolored nodes before and after one iteration of *OneShotColoring,* then, for every node $v \in U_0^{+}$

$$\Pr[\deg_{U_1^{+}}(v) \leq \frac{15}{16} \deg_{U_0^{+}}(v)] > 1 - n^{-c/512} - n^{-c/32+1}$$

Fast Randomized $(\Delta + 1)$-Coloring

Corollary There exists a randomized Monte-Carlo $(\Delta + 1)$-coloring algorithm running in $O(\log \Delta + \log^2 \log n)$ rounds, succeeding w.h.p.

Proof. After $\log_{16/15} \Delta$ iterations:

- for every $v \in U^+$, $\deg_{U^+}(v) \leq c \ln n$ by Property 2
- for every $v \in U \setminus U^+$, $\deg_{U \setminus U^+}(v) \leq c \ln n$ by definition.

By Property 1, after $O(\log(c \ln n)) = O(\log \log n)$ iterations, uncolored components of size at most $O((c \ln n)^3)$, and each can be colored (deterministically) in $O(\log^2 \log n)$ rounds. $\square$

Proof of Property 2

By Lemma 2, $\Pr[\neg \mathscr{E}_v] \leq \exp\left(-\frac{c \ln n + 1}{32}\right)$.

By Union bound,

$$\Pr[\neg \mathscr{E}] \leq |U^+| \cdot \exp\left(-\frac{c \ln n + 1}{32}\right) \leq n^{-c/32+1}$$

Yet Another Concentration Bound

Let $X_1, \ldots, X_k$ be k random variables with values in $[a_i, b_i]$ for all $i \in [k]$

Let $X = \sum_{i=1}^{k} X_i$

Let $Y_0, Y_1, \ldots, Y_k$ be $k + 1$ random variables, and assume that X_i is determined by $Y_0, \ldots, Y_i$ only.

Let $\mu_i = \mathbb{E}[X_i \mid Y_0, \ldots, Y_i]$ and $\mu = \sum_{i=1}^{k} \mu_i$

Lemma 3. For every $\lambda > 0$, $\Pr[X \leq \mathbb{E}X - \lambda] \leq \exp\left(-\frac{\lambda^2}{2\sum_{i=1}^{k}(b_i - a_i)^2}\right)$

The same bound applies to $\Pr[X \geq \mathbb{E}X + \lambda]$.

Proof of Property 2

(Continued)

Let $v \in U^+$ and let $u_1, \ldots, u_k$ be the k neighbors of v in $G[U^+]$ ordered in decreasing order of IDs.

At step 0, reveal $c_{tmp}(u)$ for all $u \notin N_{U^+}(v)$, resulting in Y_0

At step $i \geq 1$, reveal $c_{tmp}(u_i)$, resulting in Y_i

Let X_i indicator variable of whether u_i is colored

Let $X = \sum_{i=1}^{k} X_i$

Assuming $\mathscr{E}$ we have $\Pr[X_i = 1 \mid Y_{i-1}] \geq 1/8$.

Proof of Property 2

(Continued 2)

By Lemma 3, we get that

$$\begin{aligned}\Pr[X < \deg_{U_0^+}(v)/16 \mid \mathscr{E}] &\leq \exp\left(\frac{(\deg_{U_0^+}(v)/16)^2}{2\deg_{U_0^+}(v)}\right) \\ &= \exp\left(-\frac{1}{512}\deg_{U_0^+}(v)\right) \\ &\leq n^{-c/512}\end{aligned}$$

Complete the proof by applying union bound. $\square$

End Lecture 6