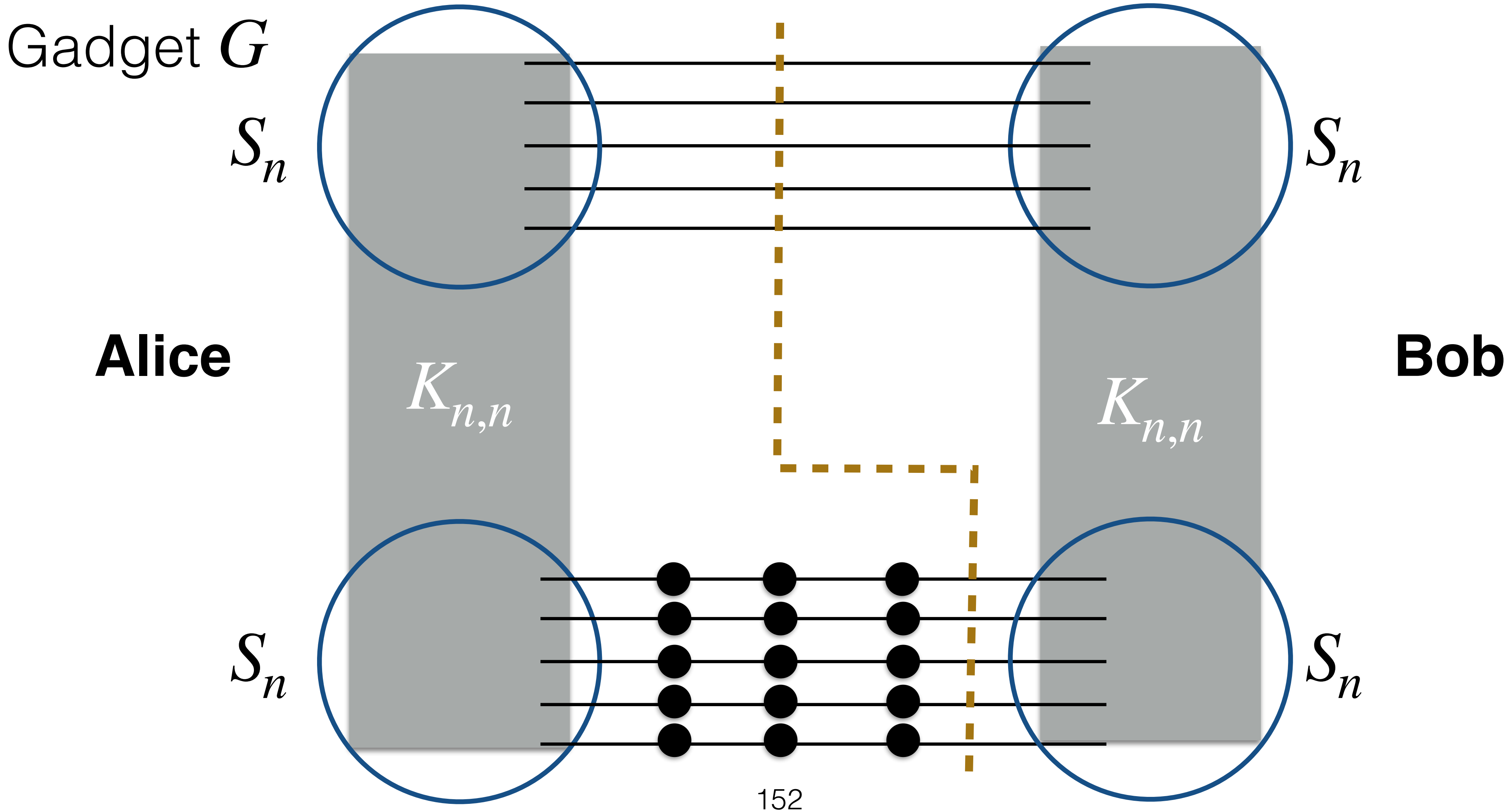
Correction

Show that, for every $k \geq 2$, deciding C_{2k+1}-freeness requires $\tilde{\Omega}(n)$ rounds in the CONGEST model



Proof

- We show that an R-round algorithm $\mathscr{A}$ for deciding C_{2k+1}-freeness in CONGEST can be used to solve DISJ(N).

- Alice and Bob agree beforehand on the gadget G, and on a ordering $e_1, \ldots, e_N$ of the $N = n^2$ edges of the two copies $K^A_{n,n}$ and $K^B_{n,n}$ of $K_{n,n}$

- Alice receives input $x = (x_1, \ldots, x_N) \in \{0,1\}^N$. For every $i \in [N]$, Alice keeps e_i in $K^A_{n,n}$ if and only if $x_i = 1$. This results in graph K^A.

- Bob receives input $y = (y_1, \ldots, y_N) \in \{0,1\}^N$. For every $i \in [N]$, Bob keeps e_i in $K^B_{n,n}$ if and only if $y_i = 1$. This results in graph K^B.

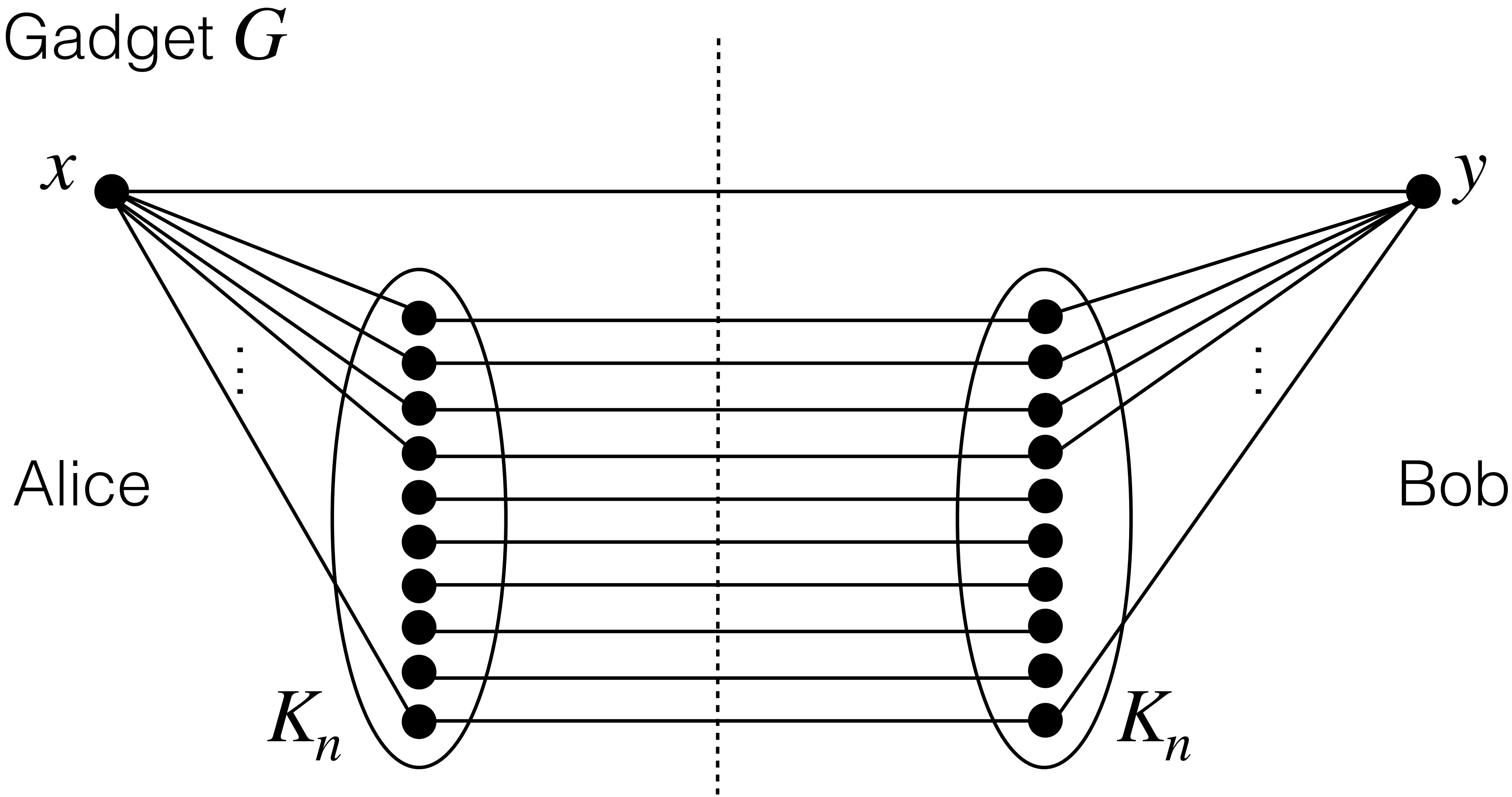
Fact. DISJ$(x, y) \iff (K^A, K^B)$ is C_{2k+1}-free

- Simulation of $\mathscr{A}$ on (K^A, K^B) can be done by exchanging $O(R\,n\,\log n)$ bits.

- Since DISJ(N) has communication complexity $\Omega(N)$ bits, i.e., $\Omega(n^2)$ bits, we get that $R = \Omega(n/\log n)$

Correction

Show that deciding between $D = 2$ and $D = 3$ requires $\tilde{\Theta}(n)$ rounds in the CONGEST model.



Proof

- We show that an R-round algorithm $\mathscr{A}$ for deciding DIAM 2 vs. 3 in CONGEST can be used to solve DISJ(N).

- Alice and Bob agree beforehand on the gadget G, and on a ordering $e_1, \ldots, e_N$ of the $N = n(n-1)/2$ edges of the two copies K_n^A and K_n^B of K_n

- Alice receives input $x = (x_1, \ldots, x_N) \in \{0,1\}^N$. For every $i \in [N]$, Alice keeps e_i in K_n^A if and only if $x_i = 0$. This results in graph K^A.

- Bob receives input $y = (y_1, \ldots, y_N) \in \{0,1\}^N$. For every $i \in [N]$, Bob keeps e_i in K_n^B if and only if $y_i = 0$. This results in graph K^B.

Fact. DISJ$(x, y) \iff (K^A, K^B)$ has diameter 2

- Simulation of $\mathscr{A}$ on $K^A, K^B)$ can be done by exchanging $O(R\, n \, \log n)$ bits.

- Since DISJ(N) has communication complexity $\Omega(N)$ bits, i.e., $\Omega(n^2)$ bits, we get that $R = \Omega(n/\log n)$

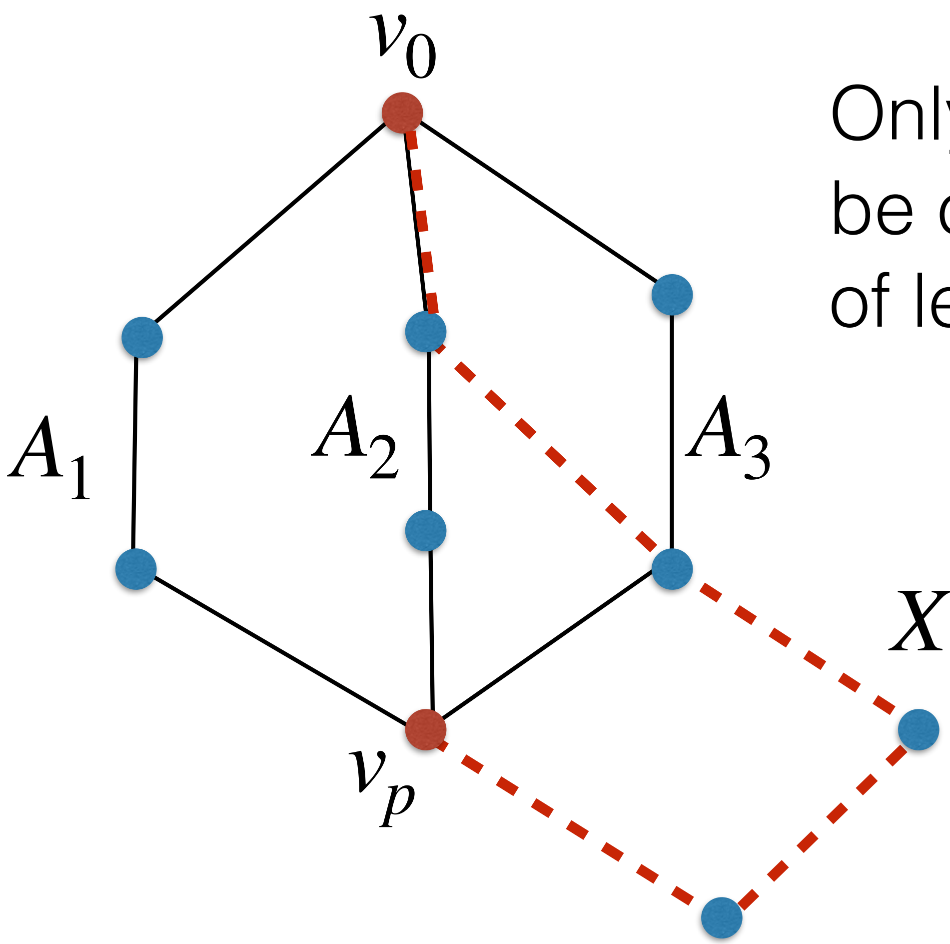
$O(n)$ rounds Algorithm for Deciding C_k-freeness, $k \geq 3$

General idea:

- $k - 1$ phases
- At first phase, every node sends its ID to all its neighbors (i.e., this phase performs 1 round).
- At each phase $i > 1$, every node forwards $O(1)$ simple paths $(v_0, v_1, \ldots, v_{i-1})$ for each source v_0. Hence each phase takes $O(n)$ rounds.
- After $k - 1$ phases, if a node v has received a simple path $(v_0, v_1, \ldots, v_{k-2})$ at phase $k - 1$ such that $(v_0, v_1, \ldots, v_{k-2}, v)$ is a k -cycle, then v rejects, otherwise v accepts.

Question: How to select $O(1)$ simple paths $(v_0, v_1, \ldots, v_{i-1})$ for each source v_0 at each phase without « missing » a path?

Selecting a path



Only A_1 (of length p) can be continued by a path X of length $q = k - p$

Filtering

Definition For every integer $n \geq 1$, every family $\mathscr{A}$ of subsets of $[n]$, and every $q \in [n]$, a family of sets $\mathscr{B} \subseteq \mathscr{A}$ is a q-representative of $\mathscr{A}$ if, for every $X \subseteq [n]$ of size $|X| \leq q$,

$$\exists A \in \mathscr{A} \mid A \cap X = \emptyset \iff \exists B \in \mathscr{B} \mid B \cap X = \emptyset$$

Lemma [Monien (1985)]

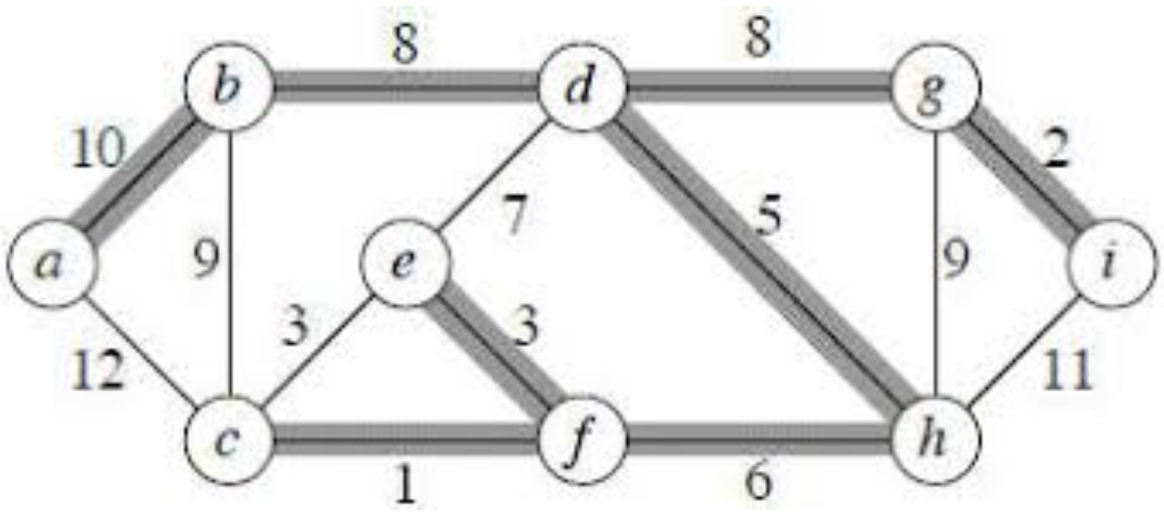
Let $n \geq 1$ be an integer, let $(p, q) \in [n] \times [n]$, and let $\mathscr{A} \subseteq 2^{[n]}$. If $|A| \leq p$ for all $A \in \mathscr{A}$, then there exists a q-representative family $\mathscr{B} \subseteq \mathscr{A}$ of $\mathscr{A}$ such that

$$|\mathscr{B}| \leq \binom{p+q}{p}.$$

CONGEST Model (cont.)

- Global problems
- Minimum-weight Spanning Tree (MST)
    - Borůvska’s algorithm
    - Matroïd algorithm
    - Sublinear algorithm
- Lower bound for MST

Minimum Spanning Tree (MST)



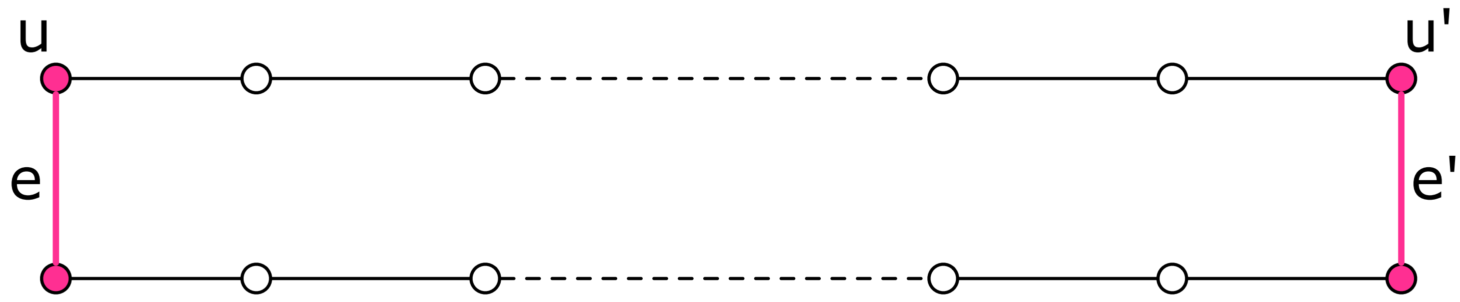
Input of node u : ID(u), w(e) for every e ∈ E(u)
Output of node u : list of edges e ∈ E(u) belonging to MST

Facts about MST

Let $G = (V,E)$ be a connected weighted graph

- Without loss of generality, all weights can be assumed distinct ➠ for every $e = \{u,v\}$ with $ID(u) > ID(v)$, replace $w(e)$ by $(w(e),ID(u),ID(v))$.
- For every **cut** $(S,V\setminus S)$ in G, the edge of **minimum** weight in the cut belongs to the MST.
- For every **cycle** C in G, the edge of **maximum** weight in C does not belong to the MST

MST is a non-local problem



$$I_1 = (1,3) \quad I_2 = (3,2) \quad I_3 = (1,2)$$

Remark MST requires at least D rounds in the cycle.

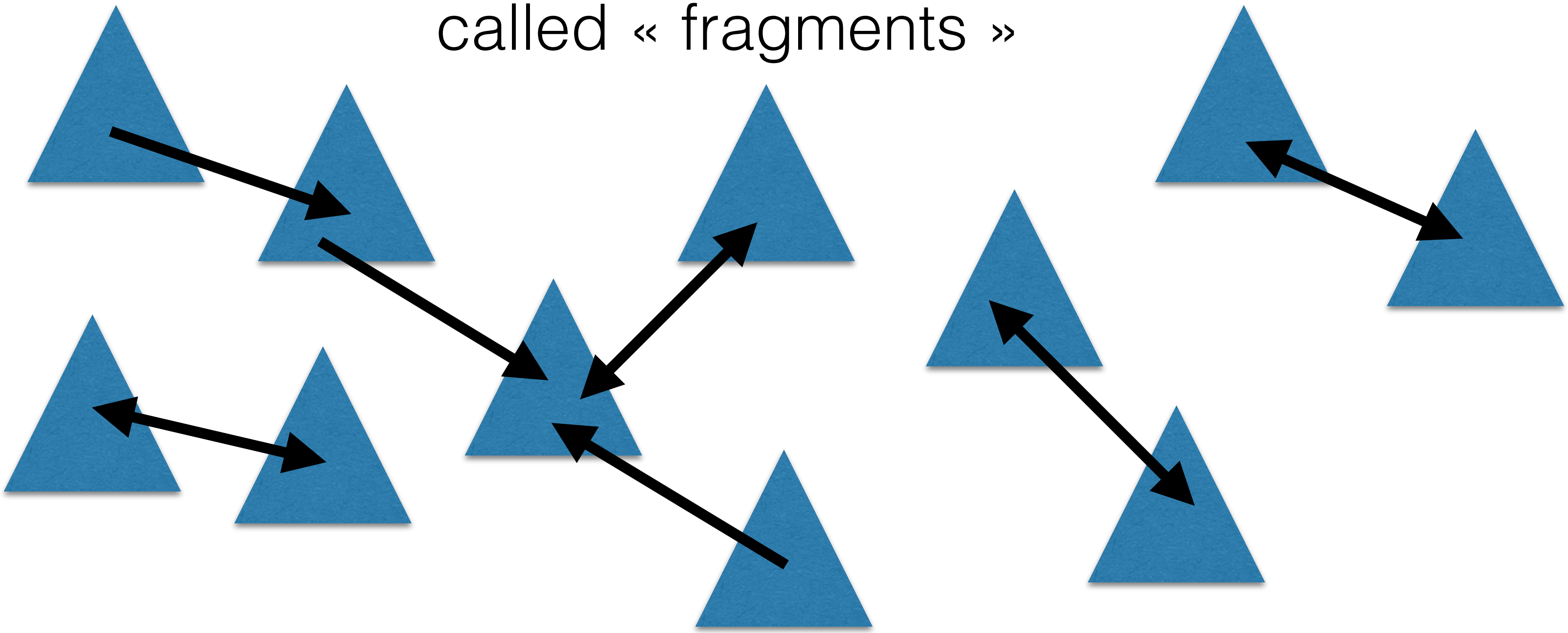
Algorithms with round-complexity O(f(n)+D)
in n-node graphs of diameter D.

Objective: minimizing f(n)

Borůvska's algorithm (1926)

distributed version

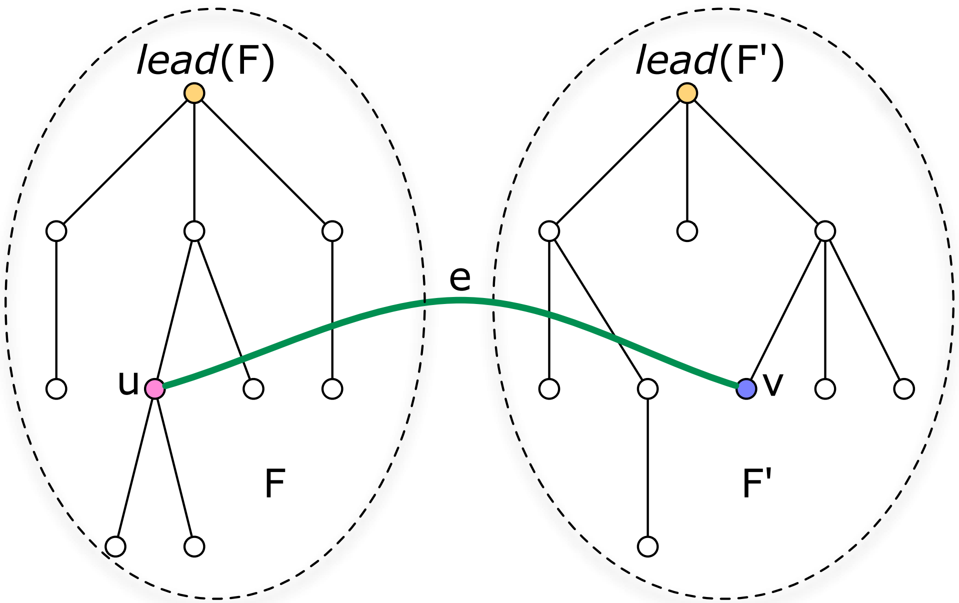
Collection of subtrees
called « fragments »

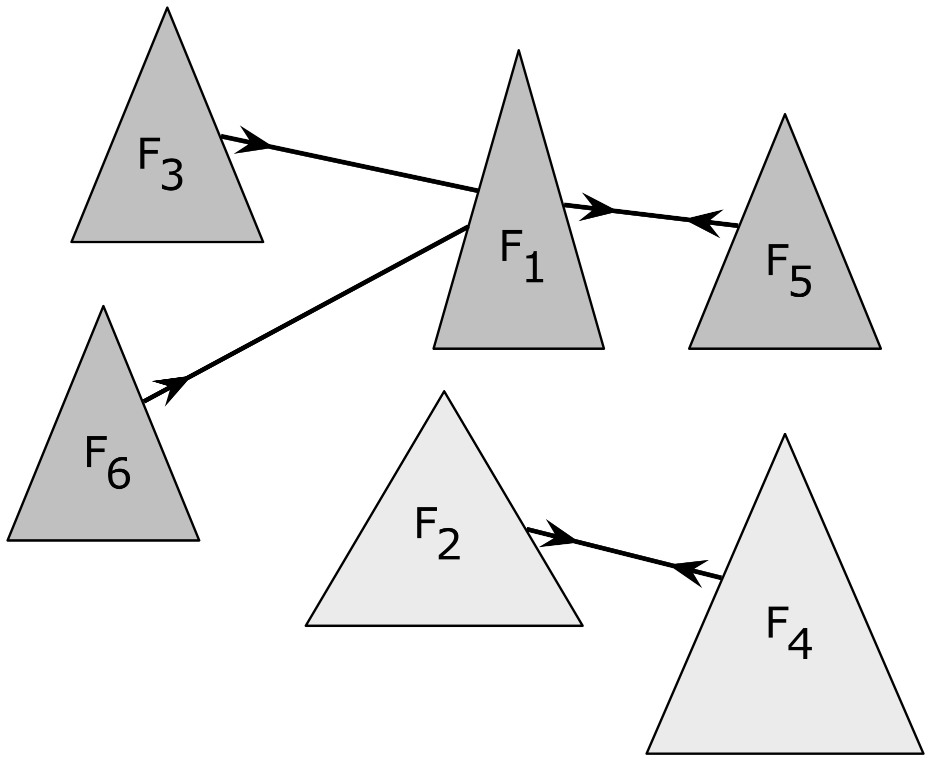


A phase = fragments
are merged

Merges use the edge of minimum
weight going out of each fragment

Fragments & Merging





$e = (ID(u), ID(v), w(e))$

$N(t)$ = #fragments after t phases

$N(0) = n$
$N(t+1) \leq N(t)/2$

$\Rightarrow$ at most $\lceil \log_2 n \rceil$ phases

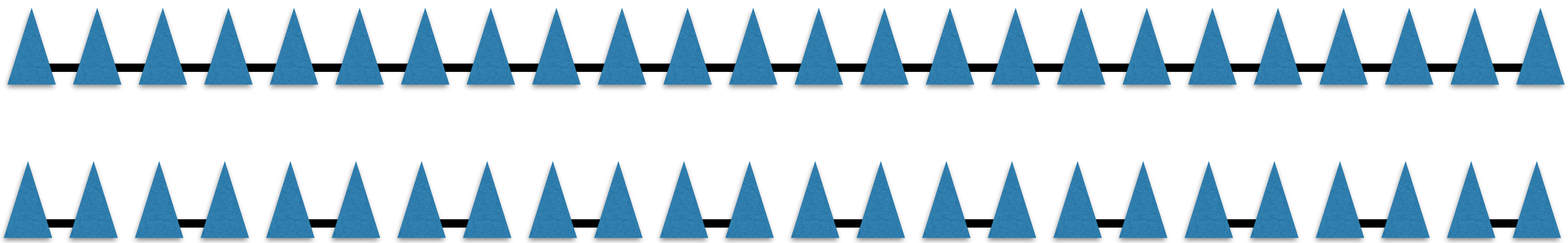
Round complexity

complexity of a phase = $O(\max_F \text{diam}(F))$

$\text{diam}(F) \leq n-1$

Theorem The distributed version of Borůvska's algorithm can be implemented in $O(n \log n)$ rounds in the CONGEST model.

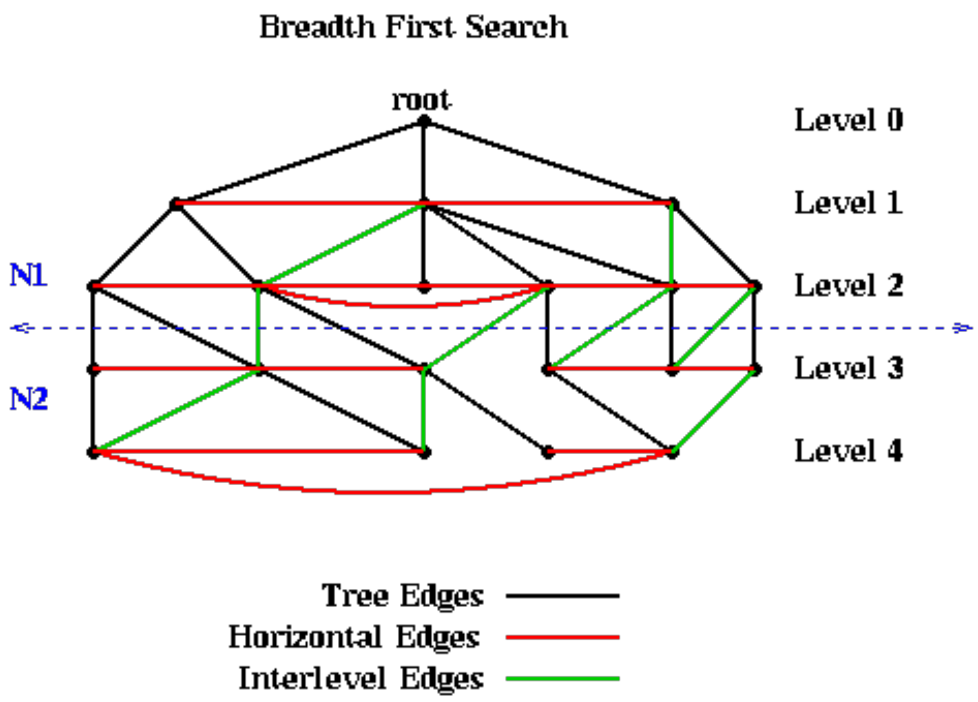
The bound is tight:



Another MST algorithm

Based on a **Breadth-First Search (BFS)** tree

Lemma BFS construction requires $O(D)$ rounds in the CONGEST model



Algorithm of node u

`id`$_{min}$ ← ID(u)
repeat
 send `id`$_{min}$ to neighbors, and receive IDs from neighbors
 if ∃ `id` ∈ {IDs sent by neighbors} : `id` < `id`$_{min}$ **then**
  `id`$_{min}$ ← `id`
  parent(u) ← ID(v) where v is the neighbor which sent `id`

Matroid Algorithm (1)

Algorithm for a node u

$K \leftarrow E(u)$ edges incident to node u

wait until having received an edge from each child

repeat

know $K \leftarrow K \cup \{\text{received edges}\}$

up $U \leftarrow \{\text{edges previously sent to parent(u)}\}$

remove $R \leftarrow \{e \in K \setminus U : U \cup \{e\} \text{ contains a cycle}\}$

candidate $C \leftarrow K \setminus (U \cup R)$

if $C \neq \varnothing$ **then**

send $e \in C$ with minimum weight to parent

receive edges from children

else terminate

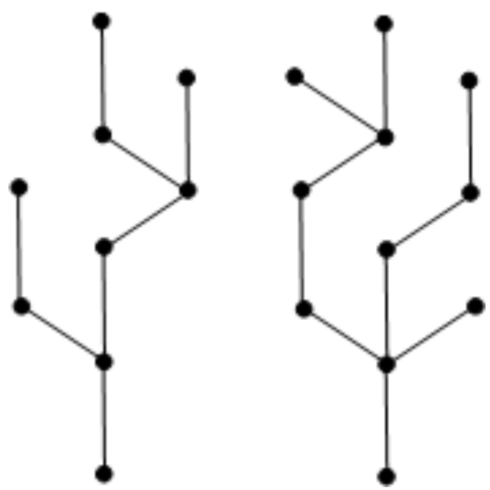
Proof of correctness

Theorem The Matroid algorithm performs in $O(n + D)$ rounds in the CONGEST model, and enables the root of the tree to construct a MST.

Lemma 0 Let A and B be acyclic subsets of edges. If $|A|>|B|$ then there exists $e \in A \setminus B$ such that $B \cup \{e\}$ is acyclic.

This is a matroid axiom

Proof B is a forest $\{T_1,\ldots,T_k\}$. Let $n_i = |V(T_i)|$. We have $|E(T_i)|=n_i-1$.



For every i, there are at most n_i-1 edges of A connecting nodes in T_i.

➥ There is an edge in A whose extremities do not belong to a same tree T_i. ❑

A node u is said **active** at phase t if it has not terminated at phase $t - 1$.

Let $h(u)$ = height of u = length of longest path to a leaf of the subtree T_u rooted at u.

Lemma 1 For every active child v of a node u, the set C of candidates for u at time t contains at least one edge sent by v to u before time t. ➪ **no premature termination**

Proof Induction on $h(u)$. Lemma holds for $h(u)=0$.

Assume lemma hold for all nodes at height $\leq k$.

Let u with $h(u)=k+1$, and v active child of u. Note $h(v) \leq k$.

Let E_u and E_v be edges sent by u to $p(u)$, and by v to $u=p(v)$ before phase t.

Since $h(v)<h(u)$ we have $|E_v| > |E_u|$.

By Lemma 0, $\exists\, e \in E_v \setminus E_u$ such that $E_u \cup \{e\}$ is acyclic ➡ $e \in C$. ❑

Lemma 2

(a) If u sends e to $p(u)$ at phase t then

1. all edges received by u at phase $t-1$ from its active children were of weight $\geq w(e)$, and
2. all edges to be received by u at phases $\geq t$ will be of weight $\geq w(e)$.

(b) The weights of the edges sent by u to its parent are ↗

➜ it is legitimate to remove edges creating cycles with previously sent edges.

Proof True for height 0. Assume holds for height k.

(a.1) Let u with $h(u) = k+1$.

Let e' be edge sent by child v at phase $t-1$.

Let $e'' \in C$ whose existence follows from Lemma 1.

By induction, property (b) implies $w(e'') \leq w(e')$.

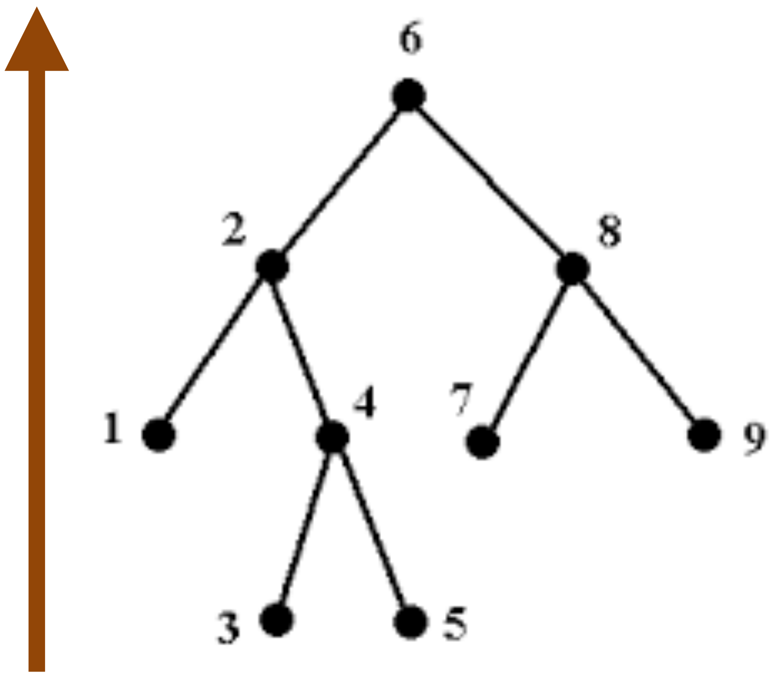
By the choice of the edge in C, we have $w(e) \leq w(e'')$.

➥ $w(e') \geq w(e)$.

(a.2) follows from (a.1) and by induction from (b).

(b) follows from (a.2) by the choice of the edge in C. ❑

Complexity

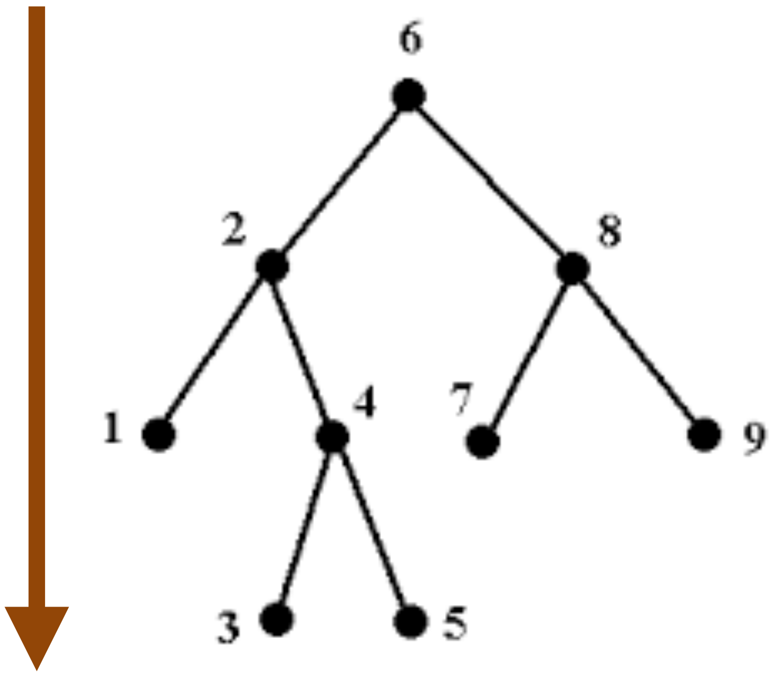


In n-node graphs, any set of n edges includes a cycle

➥ every node sends ≤ n-1 edges

➥ #rounds ≤ D + n - 1

Broadcasting the MST from the root to all nodes



Pipelining the edges of $T = \{e_1, e_2, \ldots, e_{n-1}\}$ down the BFS tree

➥ #rounds $\leq D + n - 1$

Wrap Up

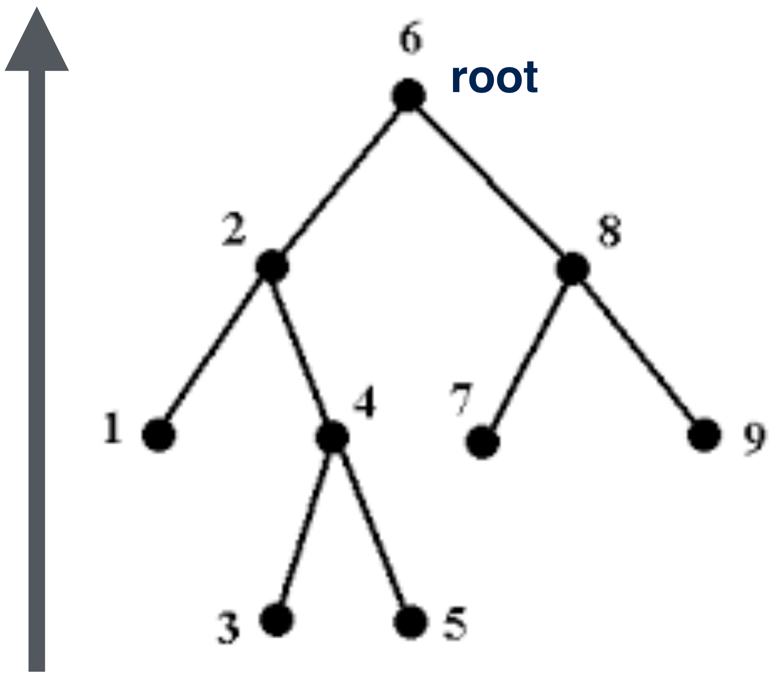
- **Borůvska:** O(n log n) rounds — this is because fragments can have arbitrarily large diameter
- **Matroid:** O(D+n) rounds — this is because too many edges are gathered at a single node.
- **Combining Borůvska and Matroid:**
  - control the diameter of the fragment, and stops when fragments have too large diameter
  - carry on with matroid for computing the (few) edges connecting the fragments already computed by Borůvska

Tool

- $D \subseteq V$ is a dominating set if every $u \notin D$ has a neighbor in D.
- Remarks:
  - Every maximal independent set is a dominating set.
  - Every tree has a dominating set of size $\leq n/2$
- **Objective:** Distributed computing of a dominating set of size $\leq n/2$ in consistently oriented trees.

MIS in Rooted Trees

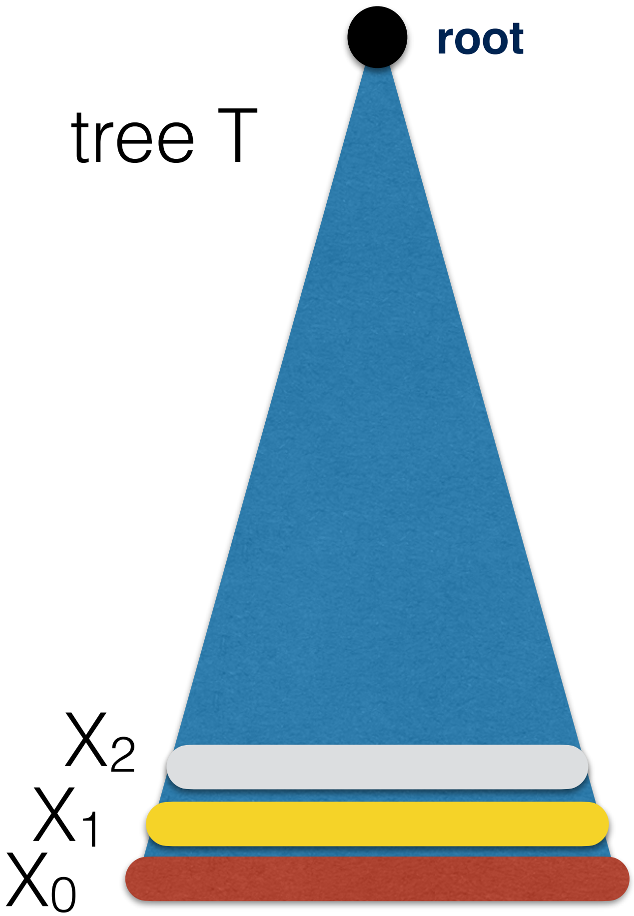
Every node has pointer to its parent



- Perform Cole and Vishkin algorithm with parent
- When colors are on 3 bits, every node pushes down its color
- Performs 5 rounds to get all colors in {1,2,3}.

Complexity : O(log*n) rounds

Computing small dominating sets in rooted trees

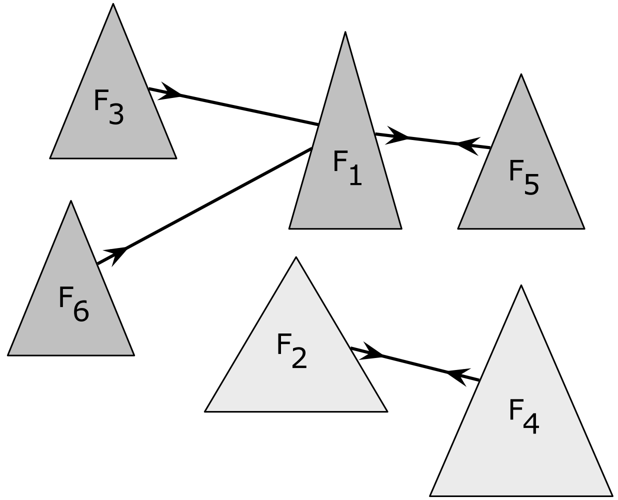


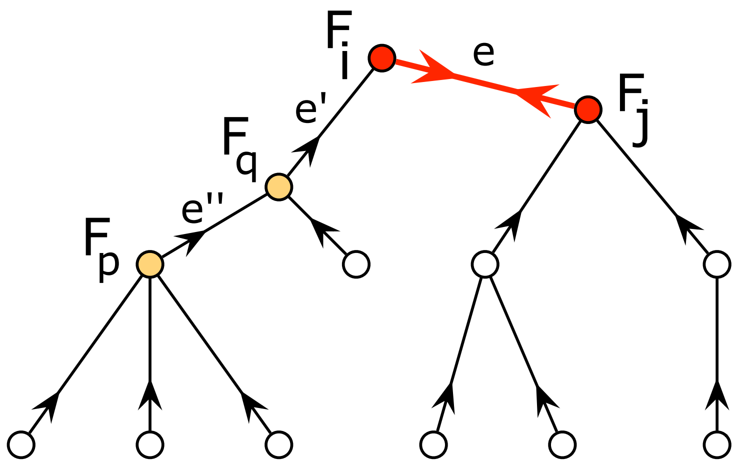
- X_d = {nodes at distance d from a leaf}
- $Y = V(T) \setminus (X_0 \cup X_1 \cup X_2)$
- Let J be MIS in Y (comput. in $O(\log^* n)$ rounds)
- Let $D = J \cup X_1$

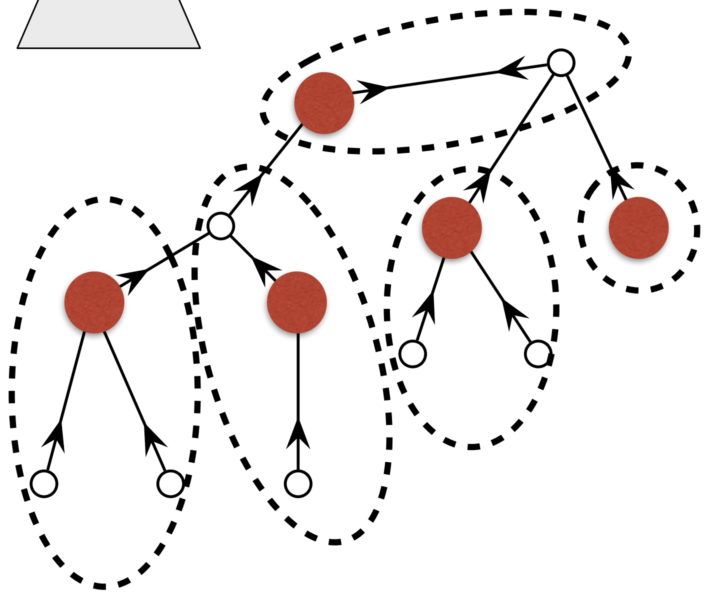
- D is a dominating set
- $|X_1| \leq |X_0| \;\Rightarrow\; |X_1| \leq \frac{1}{2}\,|X_0 \cup X_1|$
- $|J| \leq |(Y \cup X_2) \setminus J| \Rightarrow |J| \leq \frac{1}{2}\,|Y \cup X_2|$

➡ $|D| \leq n/2$

Bounding the diameter of fragments







Fast MST algorithm

Two stages:

1. Few phases of Borůvska
2. Completed by Matroid

$N(t) \leq N(t-1)/2$

$\Longrightarrow N(t) \leq n/2^t$

$\mathrm{diam}(t) \leq 3\ \mathrm{diam}(t-1) + 2$

$\Longrightarrow \mathrm{diam}(t) \leq 3^t - 1$

N(t) = #frags after t phases

diam(t) = max diameter frags

Phase t costs

$O(\mathrm{diam}(t) \log^* n)$ rounds

τ phases Borůvska costs

$\tilde{O}(3^\tau)$ rounds

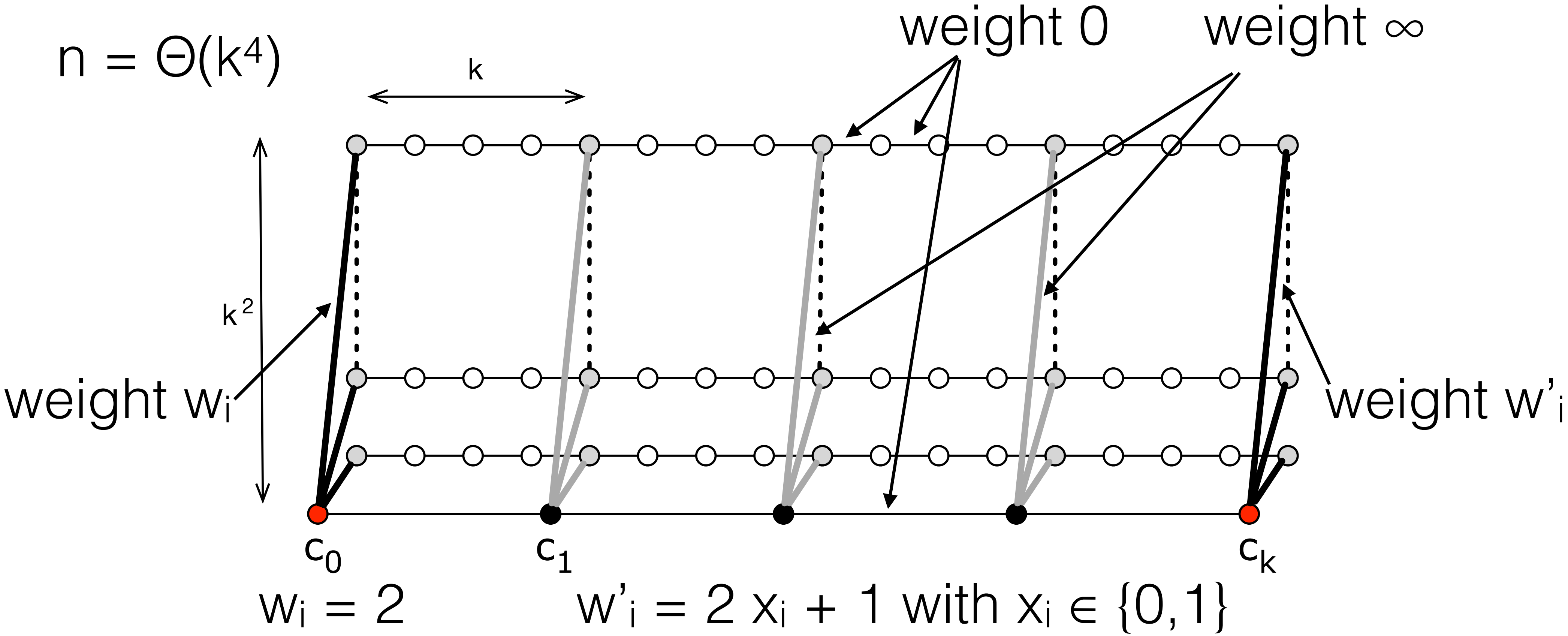
Matroid completes in

$O(D+N(\tau))$ rounds

$3^\tau = n/2^\tau \Longrightarrow \#\text{rounds} = \tilde{O}(D + n^{0.6131})$

Theorem MST construction can be achieved in $\tilde{O}(D + \sqrt{n})$ rounds in the CONGEST model.

$\tilde{\Omega}(\sqrt{n})$ lower bound for MST



Lemma Transmitting k^2 bits from c_k to c_1 takes $\Omega(k^2)$ rounds

Proof (simplified: no recombination)

- $\exists$ i, x_i uses $\leq$ k/2 of highway ➪ $\Omega(k \cdot k/2)$ rounds
- $\forall$ i, x_i uses > k/2 of highway ➪ $\Omega((k^2 \cdot k/2)/(k \log n)$ rounds □

End Lecture 8