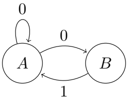
Recurrence function of Sturmian sequences. A probabilistic study

Pablo Rotondo[1], Brigitte Vallée[2]



ANALCO,

Barcelona, 16 January, 2017.

[1] IRIF, Paris 7 Diderot. Universidad de la República, Uruguay. GREYC, associate

[2] CNRS, GREYC Univ. de Caen

Study in combinatorics of words.

Objective: description of the finite factors of an infinite word $\boldsymbol{u}$

- How many factors of length n? $\longrightarrow$ Complexity
- What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Study in combinatorics of words.

Objective: description of the finite factors of an infinite word $\boldsymbol{u}$

- How many factors of length n? $\longrightarrow$ Complexity
- What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the "simplest" binary infinite words that are not eventually periodic

Study in combinatorics of words.

Objective: description of the finite factors of an infinite word $\boldsymbol{u}$

– How many factors of length n? $\longrightarrow$ Complexity

– What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the "simplest" binary infinite words that are not eventually periodic

The recurrence function is widely studied for Sturmian words.

Classical study : for each fixed Sturmian word,

what are the extreme bounds for the recurrence function?

Study in combinatorics of words.

Objective: description of the finite factors of an infinite word $\boldsymbol{u}$

– How many factors of length n? $\longrightarrow$ Complexity

– What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the "simplest" binary infinite words that are not eventually periodic

The recurrence function is widely studied for Sturmian words.

Classical study : for each fixed Sturmian word,

what are the extreme bounds for the recurrence function?

Here, in a natural model,

we perform a probabilistic study:

For a "random" sturmian word

– what is the mean value of the recurrence?

– what is the limit distribution of the recurrence?

Plan of the talk

Complexity, Recurrence, and Sturmian words

- Complexity and Recurrence
- Sturmian words
- Recurrence of Sturmian words

Our probabilistic point of view. Statement of the results

- Classical results
- Our point of view
- Our results for $R_\alpha(n)$

Methods for the proof

- General description

Complexity

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

Complexity

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

Simple facts:

$$p_{\boldsymbol{u}}(n) \le |\mathcal{A}|^n\,, \quad p_{\boldsymbol{u}}(n) \le p_{\boldsymbol{u}}(n+1)\,.$$

Complexity

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

Simple facts:

$$p_{\boldsymbol{u}}(n) \leq |\mathcal{A}|^n\,, \quad p_{\boldsymbol{u}}(n) \leq p_{\boldsymbol{u}}(n+1)\,.$$

Important property

$\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic

$\iff p_{\boldsymbol{u}}(n+1) > p_{\boldsymbol{u}}(n)$ for all $n \in \mathbb{N}$

Complexity

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

Simple facts:

$$p_{\boldsymbol{u}}(n) \leq |\mathcal{A}|^n\,, \quad p_{\boldsymbol{u}}(n) \leq p_{\boldsymbol{u}}(n+1)\,.$$

Important property

$\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic

$\iff p_{\boldsymbol{u}}(n+1) > p_{\boldsymbol{u}}(n)$ for all $n \in \mathbb{N}$

$\implies p_{\boldsymbol{u}}(n) \geq n+1$

Recurrence

Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

Recurrence

Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Consider $\boldsymbol{u}$ uniformly recurrent. Its recurrence function is:

$$R_{\boldsymbol{u}}(n) = \inf \{m \in \mathbb{N} : \text{every factor of length } m \text{ **contains** all the factors of length } n\} .$$

Recurrence

Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Consider $\boldsymbol{u}$ uniformly recurrent. Its recurrence function is:

$$R_{\boldsymbol{u}}(n) = \inf \{m \in \mathbb{N} : \text{every factor of length } m \text{ } \textbf{contains} \text{ all the factors of length } n\} \,.$$

- Cost we have to pay to discover the factors if we start from an arbitrary point in $\boldsymbol{u} = u_1 u_2 \ldots$

Recurrence

Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Consider $\boldsymbol{u}$ uniformly recurrent. Its recurrence function is:

$$R_{\boldsymbol{u}}(n) = \inf \{m \in \mathbb{N} : \text{every factor of length } m \text{ **contains** all the factors of length } n\}.$$

- Cost we have to pay to discover the factors if we start from an arbitrary point in $\boldsymbol{u} = u_1 u_2 \ldots$
- Related to the complexity function

$$R_{\boldsymbol{u}}(n) \geq n + p_{\boldsymbol{u}}(n) - 1.$$

Recurrence

Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Consider $\boldsymbol{u}$ uniformly recurrent. Its recurrence function is:

$$R_{\boldsymbol{u}}(n) = \inf \{m \in \mathbb{N} : \text{every factor of length } m \text{ **contains** all the factors of length } n\}.$$

- Cost we have to pay to discover the factors if we start from an arbitrary point in $\boldsymbol{u} = u_1 u_2 \ldots$
- Related to the complexity function

$$R_{\boldsymbol{u}}(n) \geq \underbrace{n}_{\text{length of first factor}} + \underbrace{p_{\boldsymbol{u}}(n) - 1}_{\text{count } +1 \text{ for every other factor}}.$$

Sturmian words

These are the "simplest" words that are not eventually periodic.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

Explicit construction

Given $\alpha, \beta \in [0,1)$ we define

$$\underline{\mathfrak{S}}_{\alpha,\beta}(n) = \lfloor (n+1)\,\alpha + \beta \rfloor - \lfloor n\,\alpha + \beta \rfloor \;,$$
$$\overline{\mathfrak{S}}_{\alpha,\beta}(n) = \lceil (n+1)\,\alpha + \beta \rceil - \lceil n\,\alpha + \beta \rceil \;,$$

for $n \geq 0$.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

Explicit construction

Given $\alpha, \beta \in [0,1)$ we define

$$\underline{\mathfrak{S}}_{\alpha,\beta}(n) = \lfloor (n+1)\,\alpha + \beta \rfloor - \lfloor n\,\alpha + \beta \rfloor \ ,$$
$$\overline{\mathfrak{S}}_{\alpha,\beta}(n) = \lceil (n+1)\,\alpha + \beta \rceil - \lceil n\,\alpha + \beta \rceil \ ,$$

for $n \geq 0$.

- $\boldsymbol{u}$ is Sturmian $\iff$ there are $\alpha, \beta \in [0,1)$, α irrational, such that

$u_i = \underline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad$ for all $\quad i \geq 0\,,$ or $u_i = \overline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad$ for all $\quad i \geq 0\,.$

Digital lines

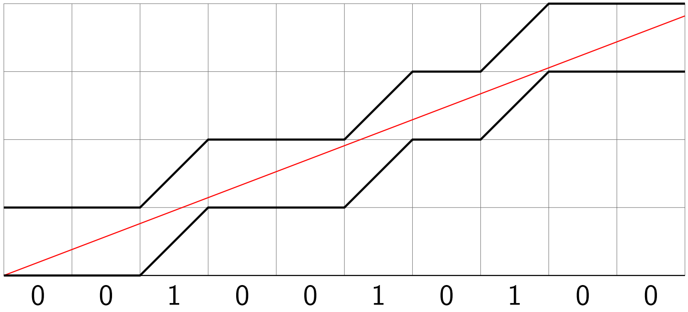


Figure : In digital geometry $\underline{\mathfrak{S}}$ and $\overline{\mathfrak{S}}$ code discrete lines. In the picture we see $\underline{\mathfrak{S}}(\alpha, 0)$ written below, where α is the slope.

Recurrence of Sturmian words

Properties

Let $\boldsymbol{u}$ be Sturmian of the form $\underline{\mathfrak{S}}(\alpha, \beta)$ or $\overline{\mathfrak{S}}(\alpha, \beta)$. Then

- $\boldsymbol{u}$ is uniformly recurrent

Recurrence of Sturmian words

Properties

Let $\boldsymbol{u}$ be Sturmian of the form $\underline{\mathfrak{S}}(\alpha, \beta)$ or $\overline{\mathfrak{S}}(\alpha, \beta)$. Then

- $\boldsymbol{u}$ is uniformly recurrent
- $R_{\boldsymbol{u}}(n)$ only depends on $\alpha \implies$ we write $R_{\alpha}(n)$.

Recurrence of Sturmian words

Properties

Let $\boldsymbol{u}$ be Sturmian of the form $\underline{\mathfrak{S}}(\alpha, \beta)$ or $\overline{\mathfrak{S}}(\alpha, \beta)$. Then

- $\boldsymbol{u}$ is uniformly recurrent
- $R_{\boldsymbol{u}}(n)$ only depends on $\alpha \implies$ we write $R_{\alpha}(n)$.
- Further $(R_{\alpha}(n))_{n \in \mathbb{N}}$ only depends on the **continuants** of α.

Recurrence of Sturmian words

Properties

Let $\boldsymbol{u}$ be Sturmian of the form $\underline{\mathfrak{S}}(\alpha,\beta)$ or $\overline{\mathfrak{S}}(\alpha,\beta)$. Then

- $\boldsymbol{u}$ is uniformly recurrent
- $R_{\boldsymbol{u}}(n)$ only depends on $\alpha \implies$ we write $R_\alpha(n)$.
- Further $(R_\alpha(n))_{n\in\mathbb{N}}$ only depends on the **continuants** of α.

Reminder: Consider the continued fraction expansion (CFE) of α

$$\alpha = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + \cfrac{1}{\ddots}}}},$$

The continuant $q_n(\alpha)$ is the denominator of the truncated CFE

$$\cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_n}}}.$$

Recurrence of Sturmian words: Morse, Hedlund

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

Recurrence of Sturmian words: Morse, Hedlund

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \textit{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

Remark

- (α, n) determines a unique k with $n \in [q_{k-1}(\alpha), q_k(\alpha)[$.

Recurrence of Sturmian words: Morse, Hedlund

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \textit{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

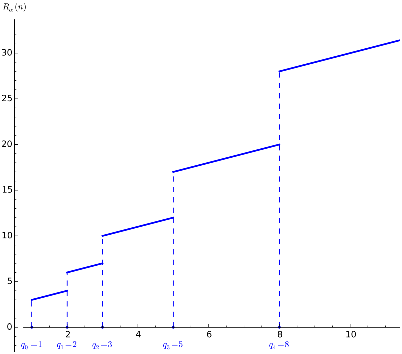
Remark

- (α, n) determines a unique k with $n \in [q_{k-1}(\alpha), q_k(\alpha)[$.

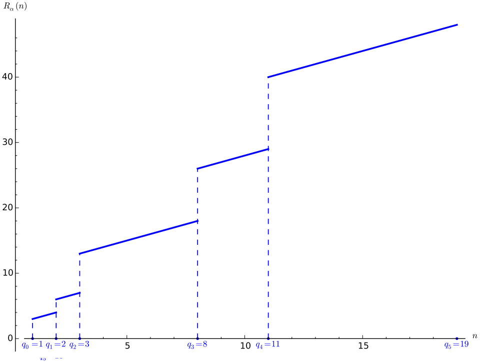
Let us see what they look like...

Recurrence function for two Sturmian words

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$



Recurrence function for $\alpha = \varphi^2$,
with $\varphi = (\sqrt{5}-1)/2$.



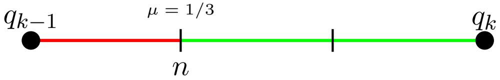
Recurrence function for $\alpha = 1/e$.

Position parameters

Consider k such that $n \in [q_{k-1}(\alpha), q_k(\alpha))$, introduce

- the relative position of n within the interval

$$\mu(\alpha, n) = \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)} \,,$$



- the quotient between the ends of the interval

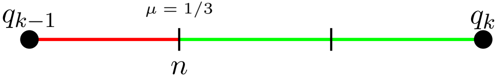
$$\rho(\alpha, n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)} \,.$$

Position parameters

Consider k such that $n \in [q_{k-1}(\alpha), q_k(\alpha))$, introduce

- the relative position of n within the interval

$$\mu(\alpha, n) = \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)} \,,$$



- the quotient between the ends of the interval

$$\rho(\alpha, n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)} \,.$$

Morse-Hedlund reads

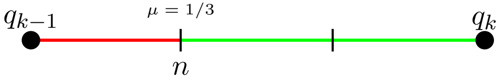
$$\boxed{\frac{R_\alpha(n) + 1}{n} = 1 + \frac{1 + \rho}{\mu + \rho - \rho\mu} \,.}$$

Position parameters

Consider k such that $n \in [q_{k-1}(\alpha), q_k(\alpha))$, introduce

- the relative position of n within the interval

$$\mu(\alpha, n) = \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)},$$



- the quotient between the ends of the interval

$$\rho(\alpha, n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)}.$$

Morse-Hedlund reads

$$\boxed{\frac{R_\alpha(n) + 1}{n} = 1 + \frac{1 + \rho}{\mu + \rho - \rho\mu}}.$$

Observe. $\frac{R_\alpha(n)+1}{n}$ large $\Longrightarrow (\mu, \rho) \approx (0, 0)$.

Recurrence function of Sturmian words: classical results.

Theorem (Morse, Hedlund, 1940)

For almost every irrational α, one has

$$\limsup_{n\to\infty} \frac{R_\alpha(n)}{n\log n} = \infty, \qquad \lim_{n\to\infty} \frac{R_\alpha(n)}{n\,(\log n)^{1+\varepsilon}} = 0 \quad \textit{for any } \varepsilon > 0.$$

Recurrence function of Sturmian words: classical results.

Theorem (Morse, Hedlund, 1940)

For almost every irrational α, one has

$$\limsup_{n\to\infty} \frac{R_\alpha(n)}{n \log n} = \infty, \qquad \lim_{n\to\infty} \frac{R_\alpha(n)}{n\,(\log n)^{1+\varepsilon}} = 0 \quad \textit{for any } \varepsilon > 0.$$

But from below

$$\liminf_{n\to\infty} \frac{R_\alpha(n)}{n} \leq 3\,,$$

consider $n \approx \frac{1}{2}\left(q_{k-1}(\alpha) + q_k(\alpha)\right)$.

Our point of view

Usual studies of $R_\alpha(n)$

- give information about extreme cases.
- give results for almost all α.

Our point of view

Usual studies of $R_\alpha(n)$

- give information about extreme cases.
- give results for almost all α.

In our probabilistic setting we

- fix an integer n (we want $n \to \infty$ ...)
- pick an irrational α uniformly from $[0, 1]$.

$\Longrightarrow$ we perform the probabilistic study
of the normalised recurrence quotient

$$S(\alpha, n) = \frac{R_\alpha(n) + 1}{n},$$

as $n \to \infty$.

We consider the recurrence quotient

$$S_n(\alpha) := S(\alpha, n) = \frac{R_\alpha(n) + 1}{n}\,.$$

We perform a probabilistic study

- for expected values: $\mathbb{E}[S_n]$
- for distributions : $\mathbb{P}\left(S_n \in J\right)$

as $n \to \infty$.

We consider the recurrence quotient

$$S_n(\alpha) := S(\alpha, n) = \frac{R_\alpha(n) + 1}{n}\,.$$

We perform a probabilistic study

- for expected values: $\mathbb{E}[S_n]$
- for distributions : $\mathbb{P}\left(S_n \in J\right)$

as $n \to \infty$.

Worst case of $S(\alpha, n)$ is roughly $\log n$ (Morse-Hedlund).

$\Longrightarrow$ We wish to obtain this $\log n$ behaviour in our study of $S(\alpha, n)$.

Study of the recurrence quotient S

Theorem

The random variable $S_n(\alpha) := S(\alpha, n)$ admits a limiting distribution when $n \to \infty$, which is given by

$$\lim_{n\to\infty} \mathbb{P}(\alpha : S_n(\alpha) \leq \lambda) = \int_{[2,\lambda]} g(y)dy\,,$$

for $t \geq 2$ (and 0 otherwise), where the density g equals

$$g(\lambda) = \begin{cases} \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log(\lambda-1) & \textit{if } \lambda \in [2,3] \\ \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log\left(1+\frac{1}{\lambda-2}\right) & \textit{if } \lambda \in [3,\infty) \end{cases}.$$

Study of the recurrence quotient S

Theorem

The random variable $S_n(\alpha) := S(\alpha, n)$ admits a limiting distribution when $n \to \infty$, which is given by

$$\lim_{n\to\infty} \mathbb{P}(\alpha : S_n(\alpha) \le \lambda) = \int_{[2,\lambda]} g(y)dy\,,$$

for $t \ge 2$ (and 0 otherwise), where the density g equals

$$g(\lambda) = \begin{cases} \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log(\lambda-1) & \textit{if } \lambda \in [2,3] \\ \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log\left(1+\frac{1}{\lambda-2}\right) & \textit{if } \lambda \in [3,\infty) \end{cases}.$$

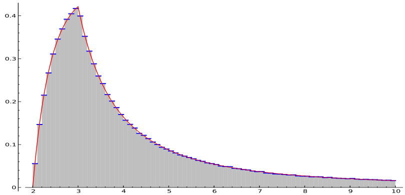
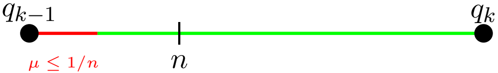


Figure : The limit density $g(x)$ in red and a scaled experimental histogram for $S(\alpha, n)$ in blue, produced with $N = 10^6$.

Conditional Expectation

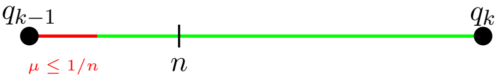
We seek to characterize the $\log n$ behaviour of $S(\alpha, n)$.

To do this we exclude the cases in which μ is small.



Conditional Expectation

We seek to characterize the $\log n$ behaviour of $S(\alpha, n)$.
To do this we exclude the cases in which μ is small.



Theorem

The conditional expectation of S_n with respect to $\mu_n \geq \frac{1}{n}$ satisfies

$$\mathbb{E}\left[S_n \middle| \mu_n \geq \frac{1}{n}\right] = \frac{12}{\pi^2}\log n + O(1)\,.$$

An analogous result holds when conditioning w.r.t. ρ_n.

Principles of the proof

For $n \in [q_{k-1}(\alpha), q_k(\alpha))$, let $x(\alpha, n) = \frac{q_{k-1}(\alpha)}{n}, y(\alpha, n) = \frac{q_k(\alpha)}{n}$. Then

$$S_n(\alpha) = f(x, y) := 1 + x + y$$

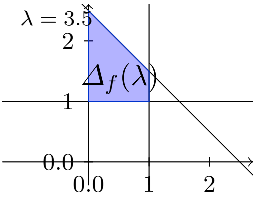
Principles of the proof

For $n \in [q_{k-1}(\alpha), q_k(\alpha))$, let $x(\alpha, n) = \frac{q_{k-1}(\alpha)}{n}, y(\alpha, n) = \frac{q_k(\alpha)}{n}$. Then

$$S_n(\alpha) = f(x, y) := 1 + x + y$$

Distribution $\mathbb{P}\left(S_n \leq \lambda\right)$ is expressed as the *coprime* Riemann sum of step $\frac{1}{n}$ of

$$\omega(x, y) = \frac{2}{y(x+y)}\,, \text{ over } \Delta_f(\lambda) := \{(x, y) : 0 < x \leq 1 < y,\ f(x, y) \leq \lambda\}\,.$$



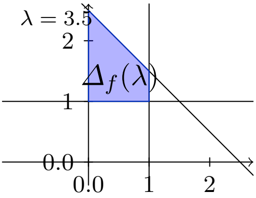
Principles of the proof

For $n \in [q_{k-1}(\alpha), q_k(\alpha))$, let $x(\alpha, n) = \frac{q_{k-1}(\alpha)}{n}, y(\alpha, n) = \frac{q_k(\alpha)}{n}$. Then

$$S_n(\alpha) = f(x, y) := 1 + x + y$$

Distribution $\mathbb{P}(S_n \leq \lambda)$ is expressed as the *coprime* Riemann sum of step $\frac{1}{n}$ of

$$\omega(x, y) = \frac{2}{y(x+y)}, \text{ over } \Delta_f(\lambda) := \{(x, y) : 0 < x \leq 1 < y,\ f(x, y) \leq \lambda\}.$$



These converge to the integral

$$\lim_{n\to\infty} \mathbb{P}(S_n \leq \lambda) = \frac{6}{\pi^2} \iint_{\Delta_f(\lambda)} \omega(x, y) dx dy$$

Concluding remarks

Interesting subfamilies of $\alpha \in (0, 1)$ include

Concluding remarks

Interesting subfamilies of $\alpha \in (0, 1)$ include

- The rationals

Concluding remarks

Interesting subfamilies of $\alpha \in (0, 1)$ include

- The rationals
  - $\circledast$ Give rise to Christoffel words.

Concluding remarks

Interesting subfamilies of $\alpha \in (0,1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.

Concluding remarks

Interesting subfamilies of $\alpha \in (0, 1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]

Concluding remarks

Interesting subfamilies of $\alpha \in (0,1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]
- The quadratic irrationals

Concluding remarks

Interesting subfamilies of $\alpha \in (0, 1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]
- The quadratic irrationals
  - ⊛ Give rise to substitutive words.

Concluding remarks

Interesting subfamilies of $\alpha \in (0,1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]
- The quadratic irrationals
  - ⊛ Give rise to substitutive words.
    - ⇒ Produced naturally by substitutions like $\tau: 0 \mapsto 01 \; 1 \mapsto 0$.

Concluding remarks

Interesting subfamilies of $\alpha \in (0,1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]
- The quadratic irrationals
  - ⊛ Give rise to substitutive words.

    $\Rightarrow$ Produced naturally by substitutions like $\tau : 0 \mapsto 01 \; 1 \mapsto 0$.

    $$\Longrightarrow \lim_{n\to\infty} \tau^n(0) = 0100101001001\ldots$$

Concluding remarks

Interesting subfamilies of $\alpha \in (0,1)$ include

- The rationals
  - ⊛ Give rise to Christoffel words.
  - ⊛ Finite continued fraction expansion.
  - ⊛ Possible to apply similar techniques. [Ustinov?]
- The quadratic irrationals
  - ⊛ Give rise to substitutive words.

    $\Rightarrow$ Produced naturally by substitutions like $\tau\colon 0 \mapsto 01 \; 1 \mapsto 0$.

    $$\Longrightarrow \lim_{n\to\infty} \tau^n(0) = 0100101001001\ldots$$
  - ⊛ Eventually periodic continued fraction expansion.