# Probabilistic studies in Number Theory and Word Combinatorics: instances of dynamical analysis

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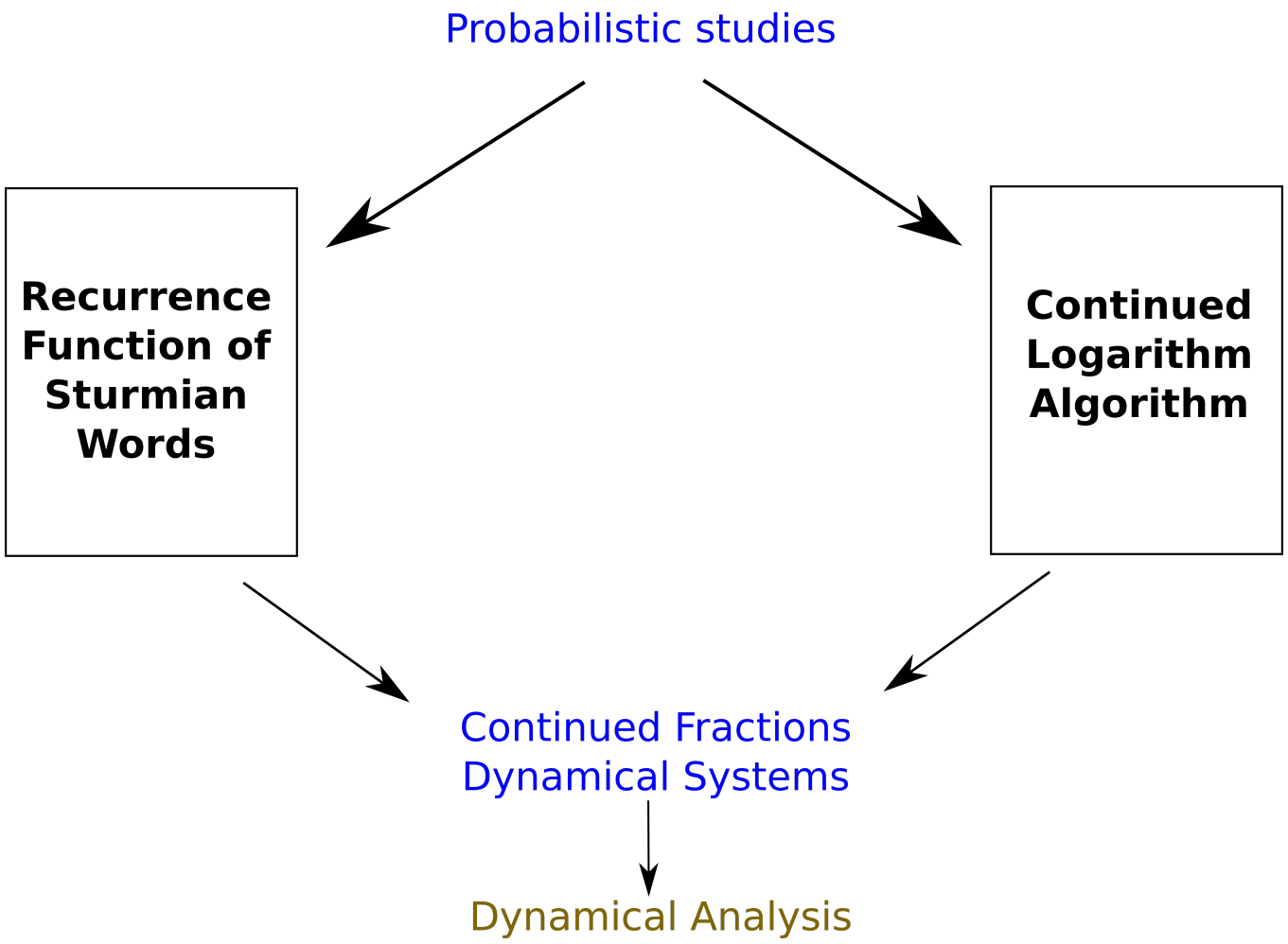
# Some key words



# This talk

1. General Introduction: continued fractions and dynamical systems
   - Continued Fractions
   - Euclidean dynamical system
2. The recurrence function of a random Sturmian word
   - Sturmian words and recurrence
   - Our models and results
3. The Continued Logarithm
   - Origins and algorithm
   - The CL dynamical system
   - Extended system and results
   - Conclusions and extensions

# Section

# Continued Fractions

Every irrational number $\alpha \in (0,1)$ has a unique representation

$$\alpha = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots}}$$

where $m_1, m_2, \ldots \geq 1$ are integers called the digits or quotients.

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where $m_1, m_2, \ldots \geq 1$ are integers called the digits or quotients.

Truncating the expansion at depth $k$ we get a convergent

$$\frac{p_k(\alpha)}{q_k(\alpha)} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots \frac{1}{m_k}}}.$$

The denominators $q_k(\alpha)$ are called the continuants of $\alpha$.

# Euclidean Algorithm and Continued Fractions

### Property

Given integers $x$ and $y$ with $x \geq y \geq 0$

$$\gcd(x, y) = \gcd(y, x \bmod y)\,.$$

In conjunction with $\gcd(x, 0) = x$, we get the Euclidean Algorithm.

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This algorithm is equivalent to the continued fraction expansion:

- given the integer division $y = mx + r$,

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# Euclidean dynamical system

To get the digits of the continued fraction expansion observe

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$$\Longrightarrow m_1 = \left\lfloor \frac{1}{\alpha} \right\rfloor , \qquad \cfrac{1}{m_2 + \cfrac{1}{m_3 + \ddots}} = \left\{ \frac{1}{\alpha} \right\} .$$

The map

$$T \colon (0,1) \to (0,1) , \qquad x \mapsto \left\{ \frac{1}{x} \right\} ,$$

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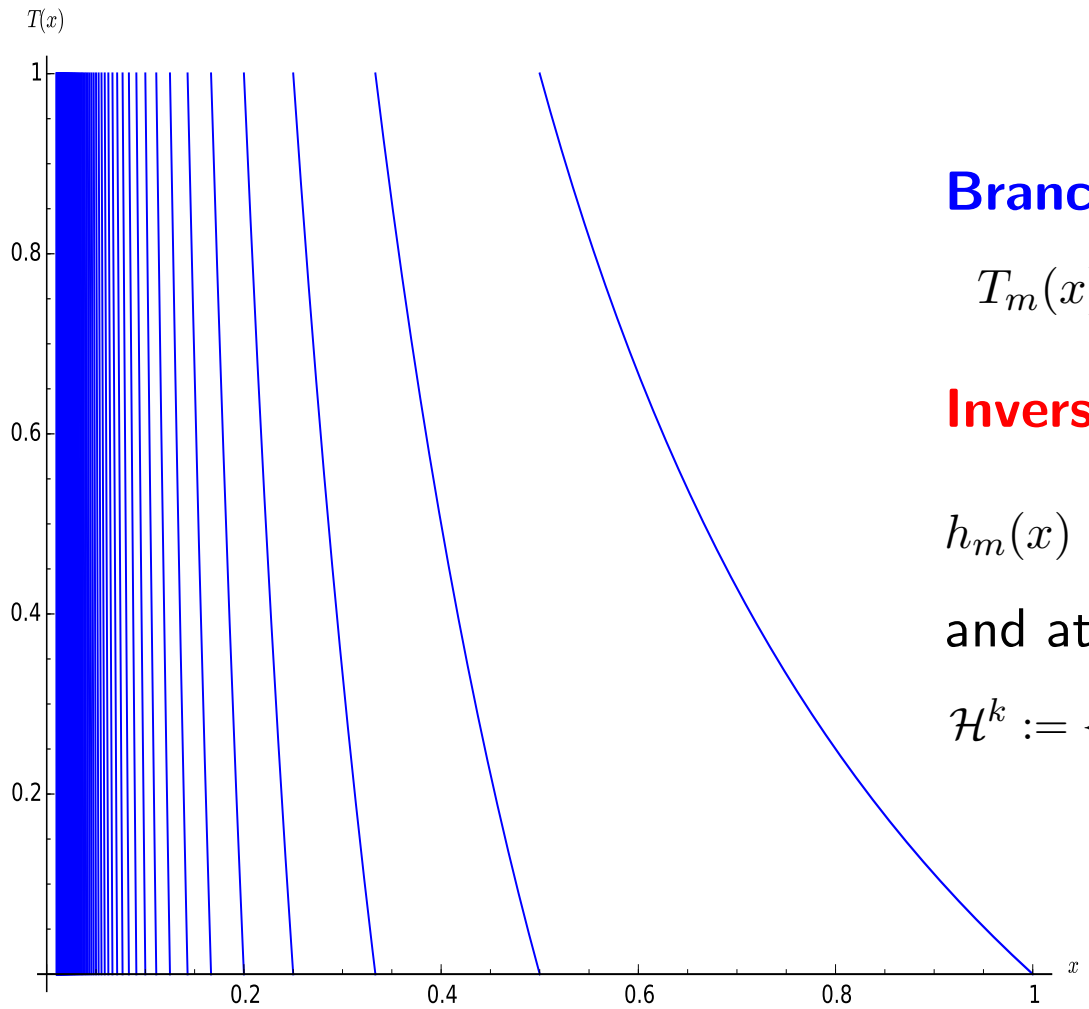


**Branches**

$$T_m(x) := \frac{1}{x} - m\,, \quad x \in \left(\frac{1}{m+1}, \frac{1}{m+0}\right].$$

**Inverse branches**

$$h_m(x) := \frac{1}{m+x}\,, \quad \mathcal{H} := \{h_m : m \in \mathbb{N}\}\,,$$

and at depth $k$

$$\mathcal{H}^k := \{h_{m_1} \circ \cdots \circ h_{m_k} : m_1, \ldots, m_k \in \mathbb{N}\}\,.$$

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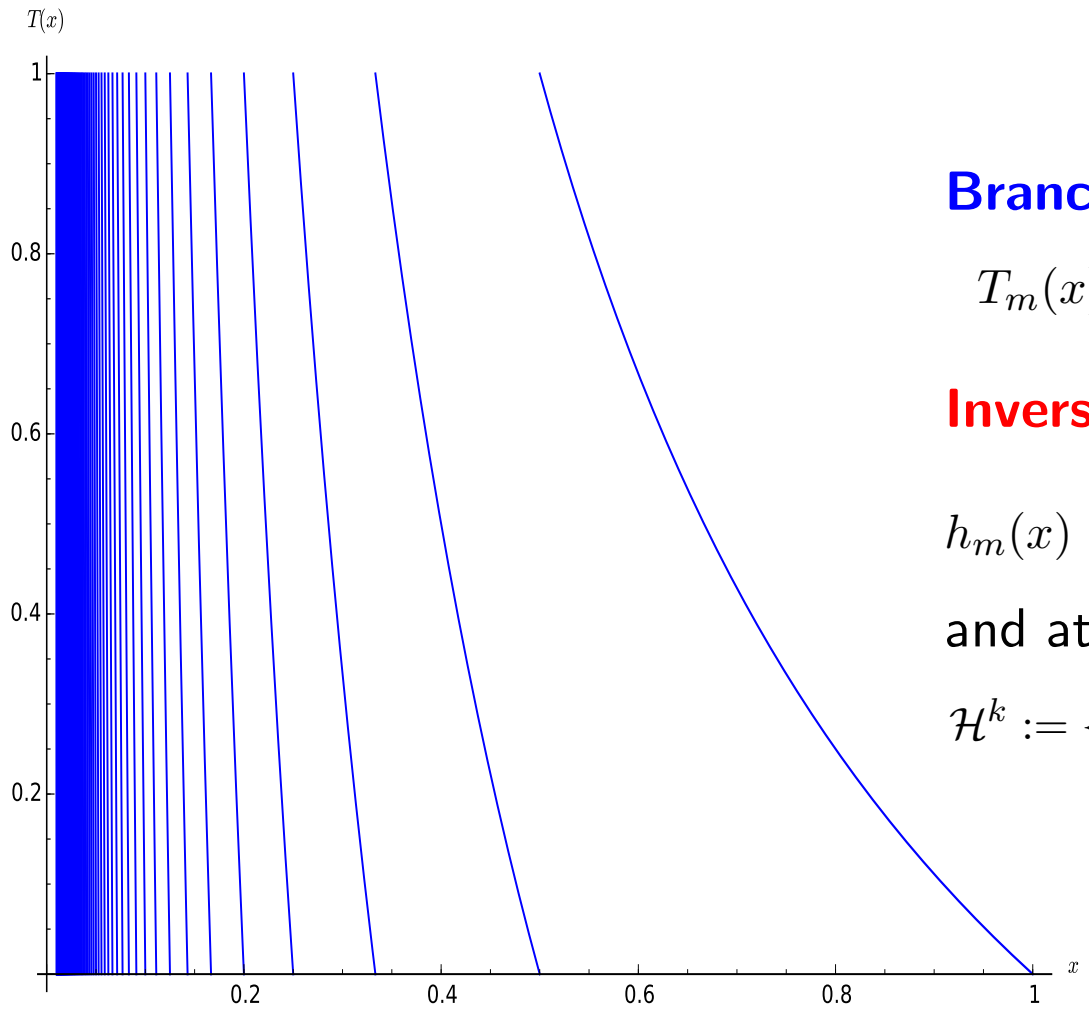


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**Property.** Let $h := h_{m_1} \circ \cdots h_{m_k}$, if $h(0) = \frac{p}{q} \Longrightarrow |h'(0)| = \frac{1}{q^2}$ .

# Density transformer

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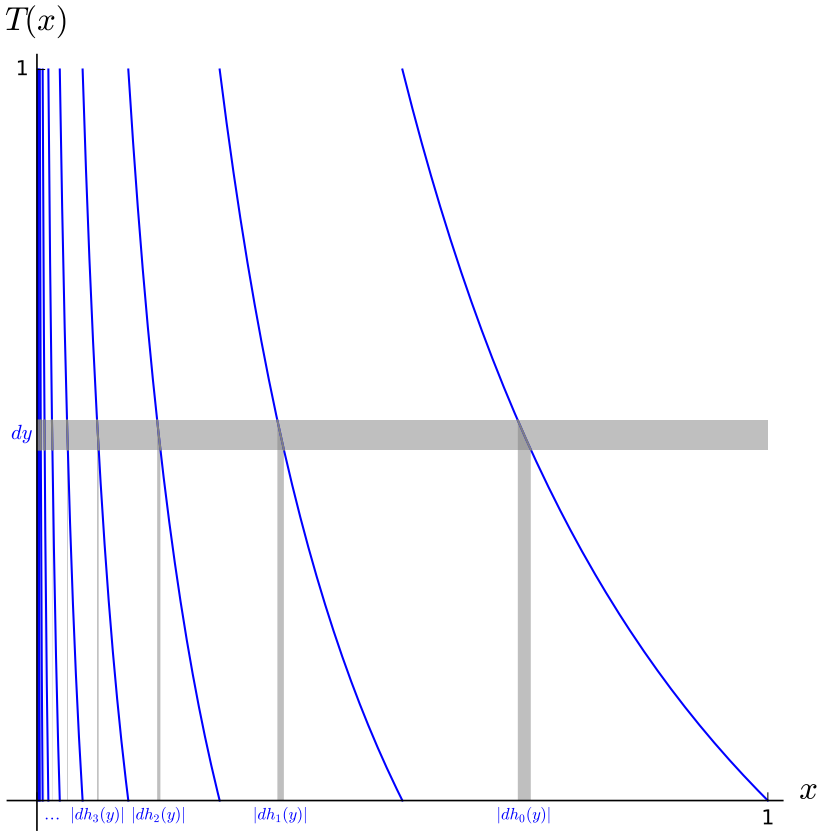

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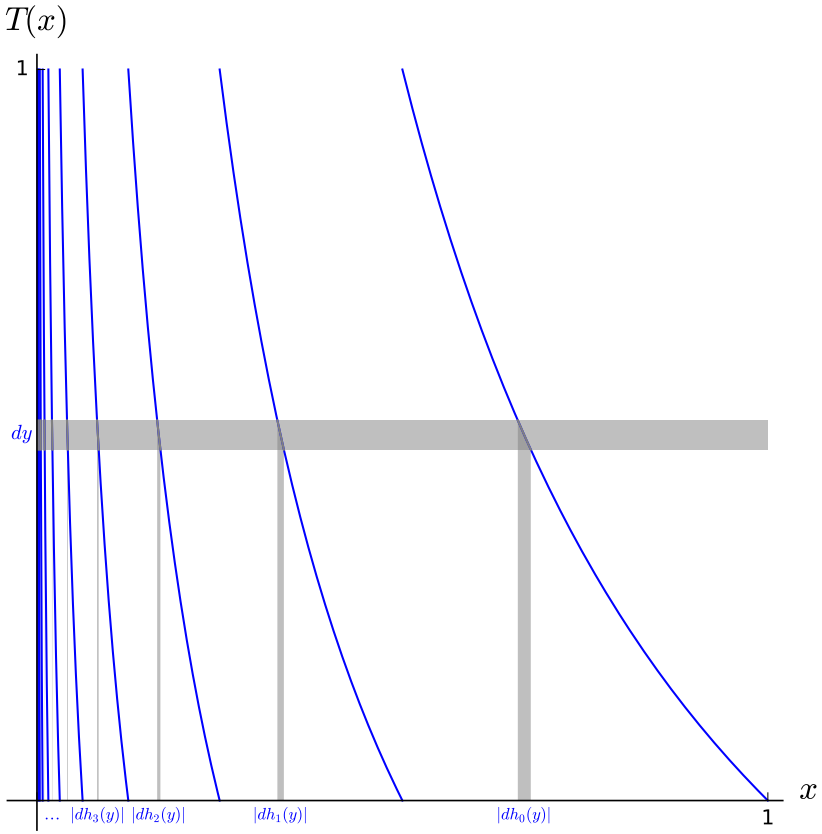


Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left|h'(x)\right| \, g\left(h(x)\right)$$

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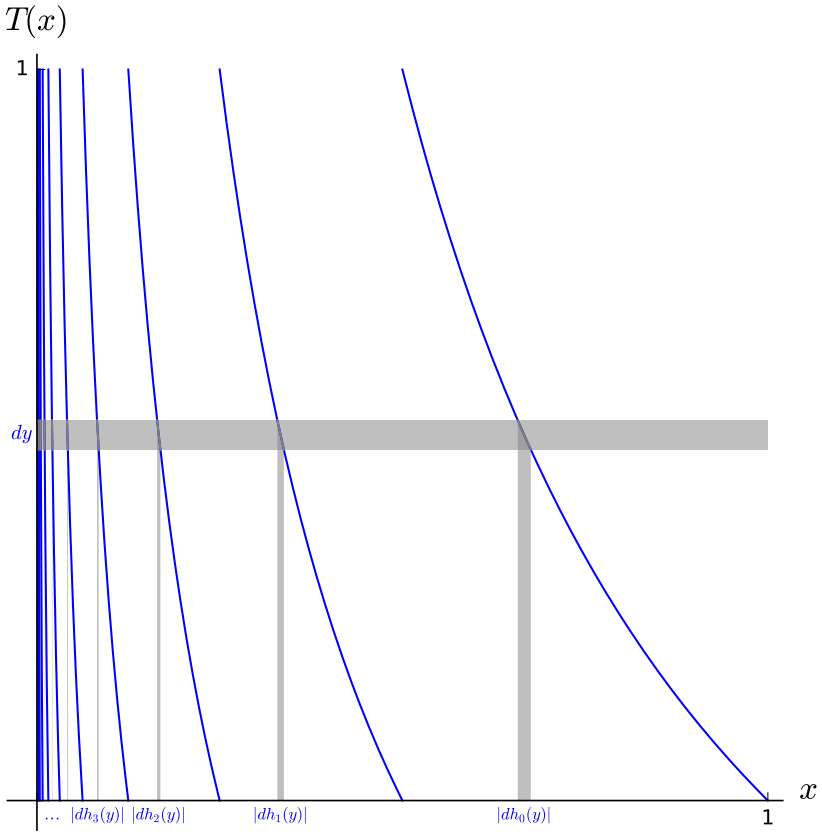


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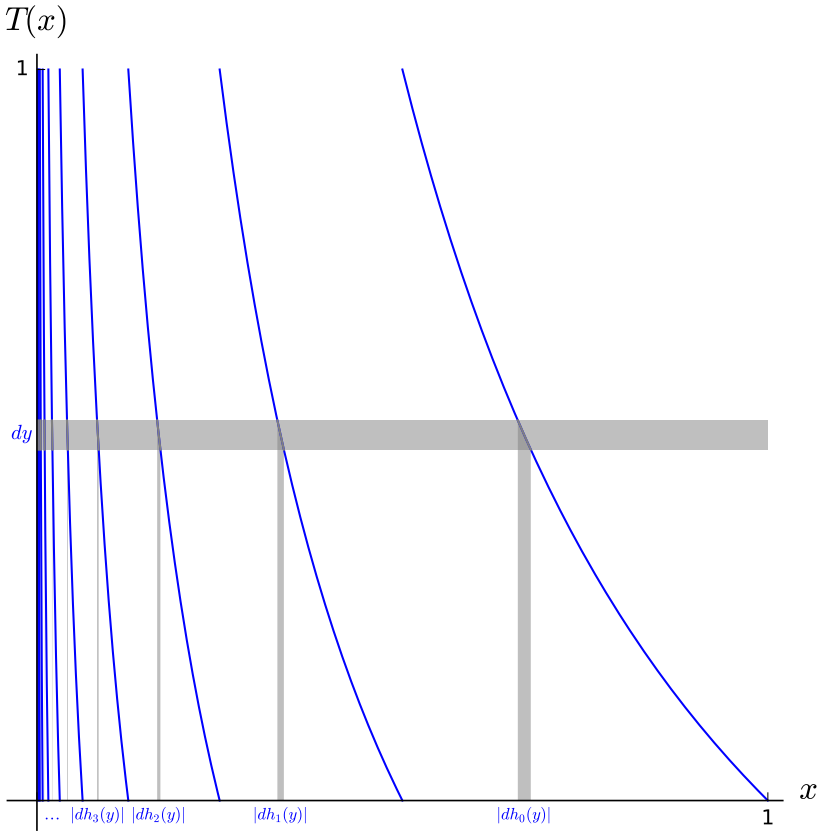


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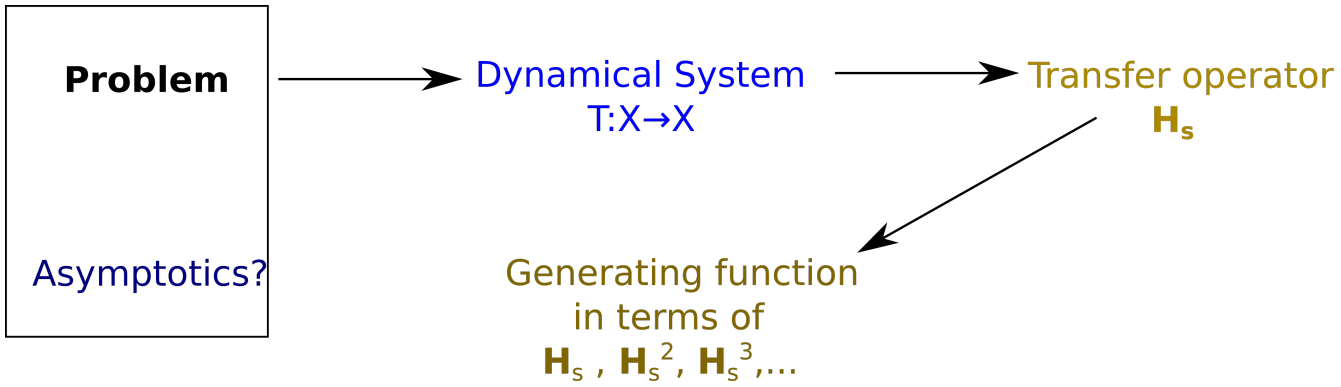
$\Longrightarrow$ Transfer operator $\mathbf{H}_s$ extends $\mathbf{H}$, introducing a variable $s$

$$\mathbf{H}_s[g](x) = \sum_{h \in \mathcal{H}} \left|h'(x)\right|^s \, g\left(h(x)\right) .$$

**Principles of dynamical analysis** [Vallée,Flajolet,Baladi,...]:

- Transfer operator $\mathbf{H}_s \Rightarrow$ expressions for generating functions.



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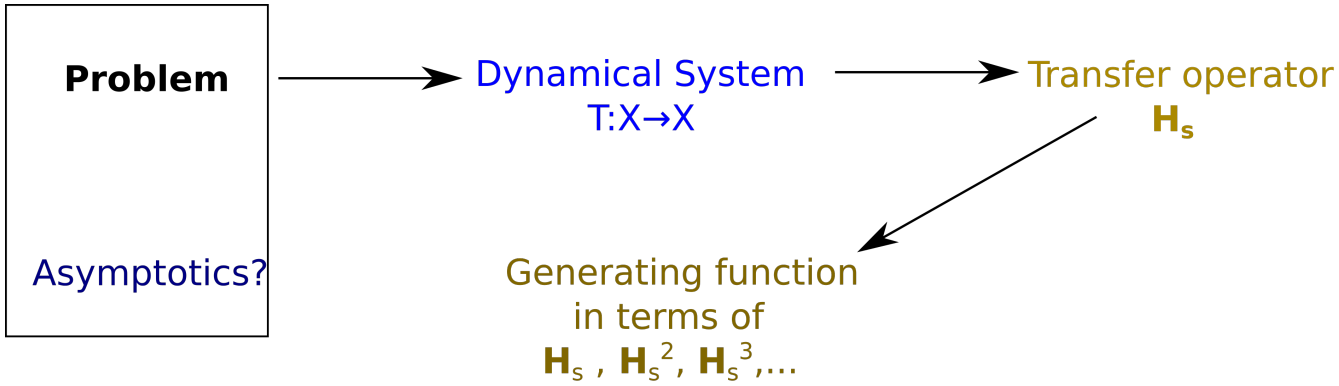


## Note

- $\mathbf{H}_s$ describes all execution of depth $1$.
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- $\vdots$
- and $(\mathbf{I} - \mathbf{H}_s)^{-1} = \mathbf{I} + \mathbf{H}_s + \mathbf{H}_s^2 + \ldots$ describes *all* executions.

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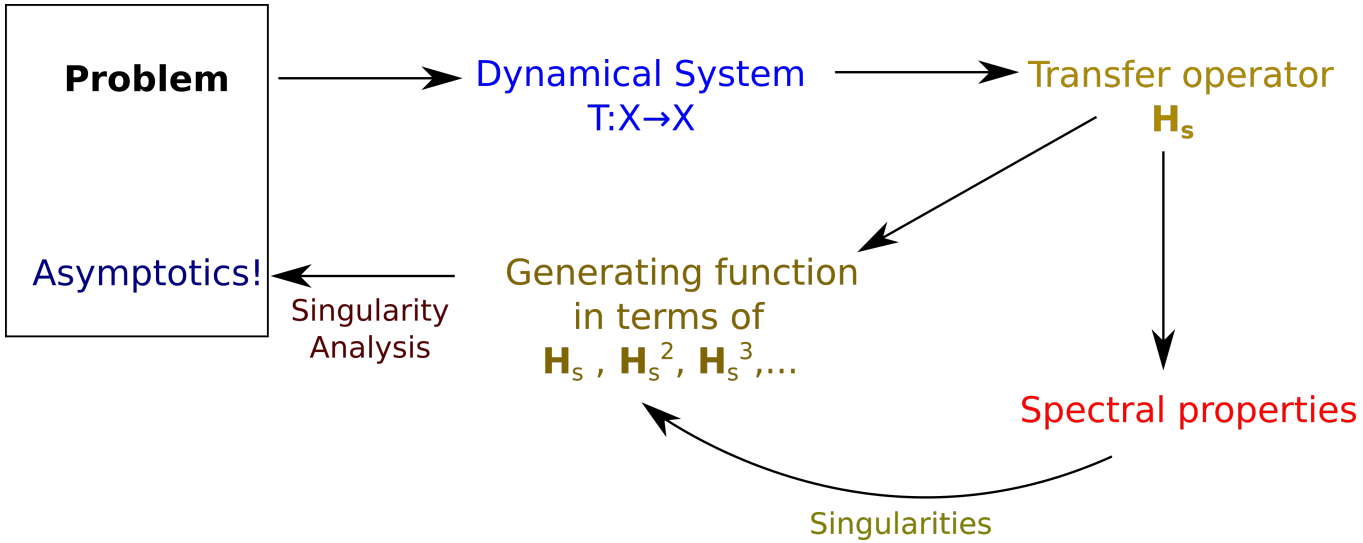
- Transfer operator $\mathbf{H}_s \Rightarrow$ expressions for generating functions.
- Dominant eigenvalue of $\mathbf{H}_s \Rightarrow$ dominant singularities.

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# Section

# Definition of Sturmian words

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Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

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## Important property

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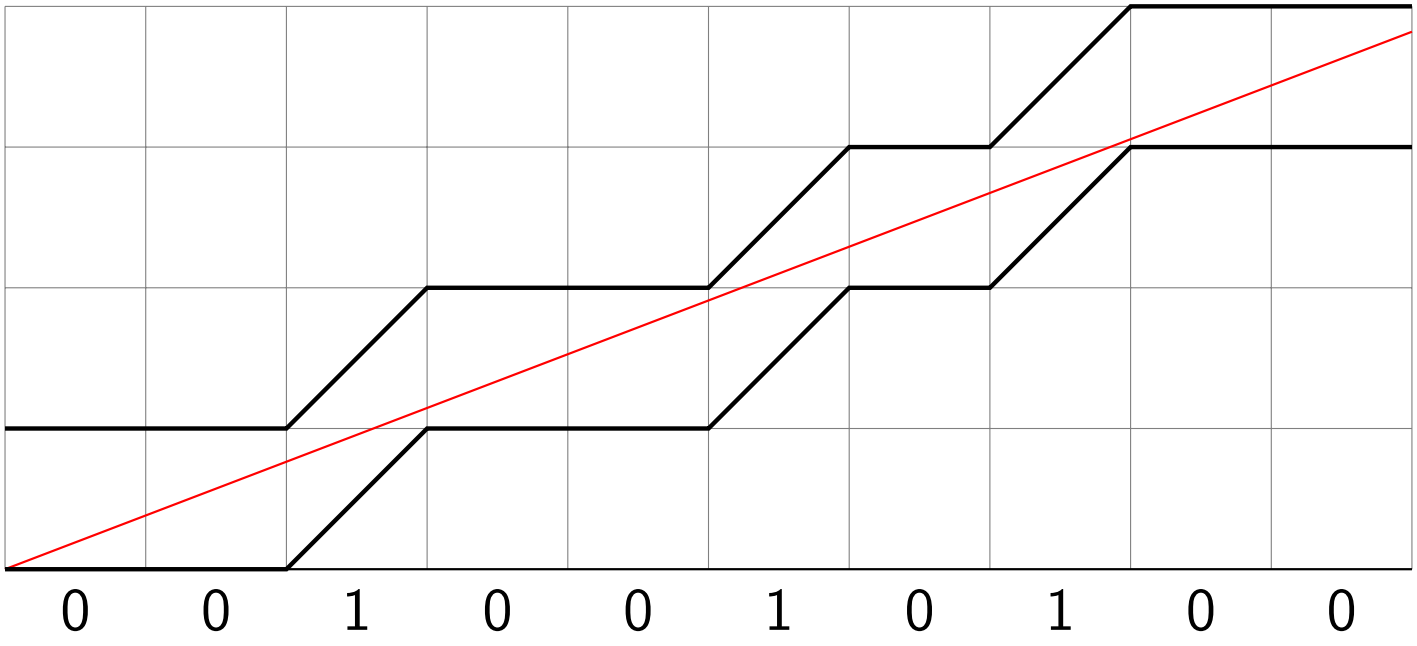
## Definition

$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

# Sturmian words and digital lines

Sturmian words correspond to discrete codings of lines, from below or above, by horizontal lines and diagonals.



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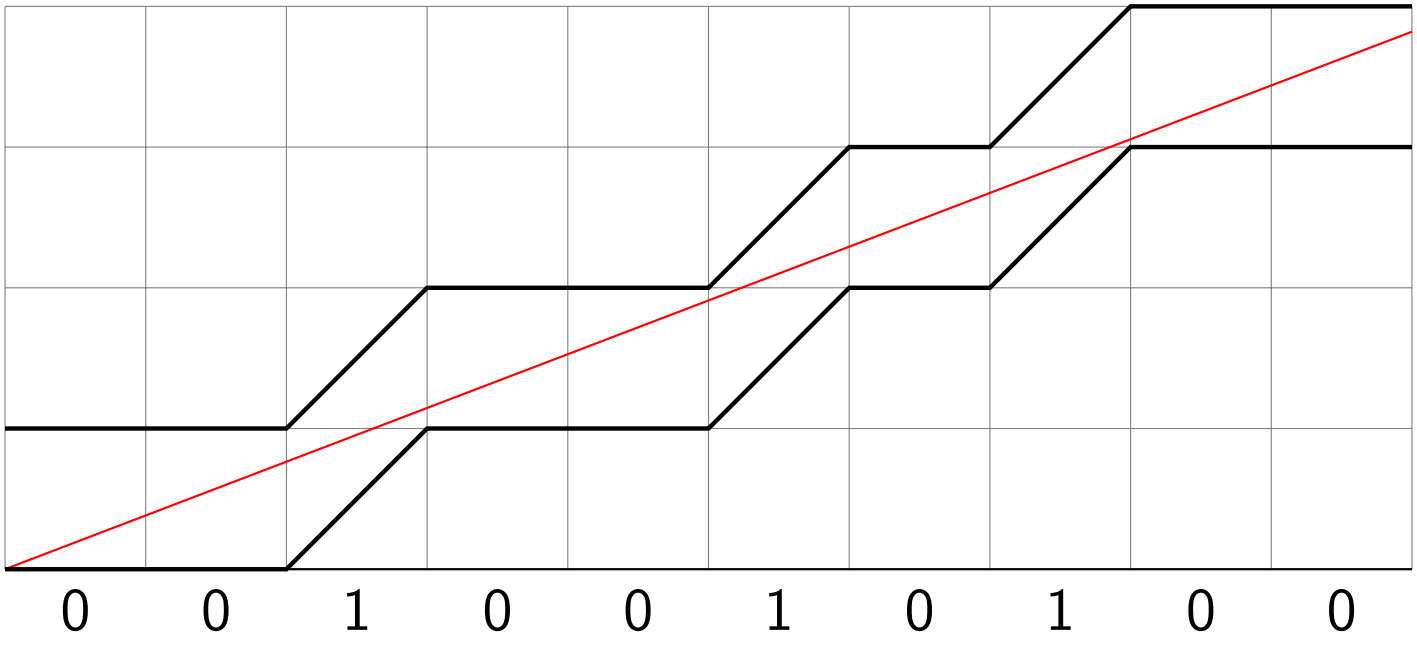


The *slope* $\alpha$ plays a key role:

the finite factors are determined exclusively by $\alpha$.

# Recurrence of Sturmian words

## Definition (Recurrence function)

Consider an infinite word $\boldsymbol{u}$. Its recurrence function is:

$$R_{\boldsymbol{u}}(n) = \inf \{m \in \mathbb{N} : \text{every factor of length } m \\ \textbf{contains} \text{ all the factors of length } n\}.$$

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## Theorem (Morse, Hedlund, 1940)

*The recurrence function is piecewise affine and satisfies*

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha), \qquad \text{for } q_{k-1}(\alpha) \leq n < q_k(\alpha).$$

# Recurrence quotient and its parameters

$$S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha).$$

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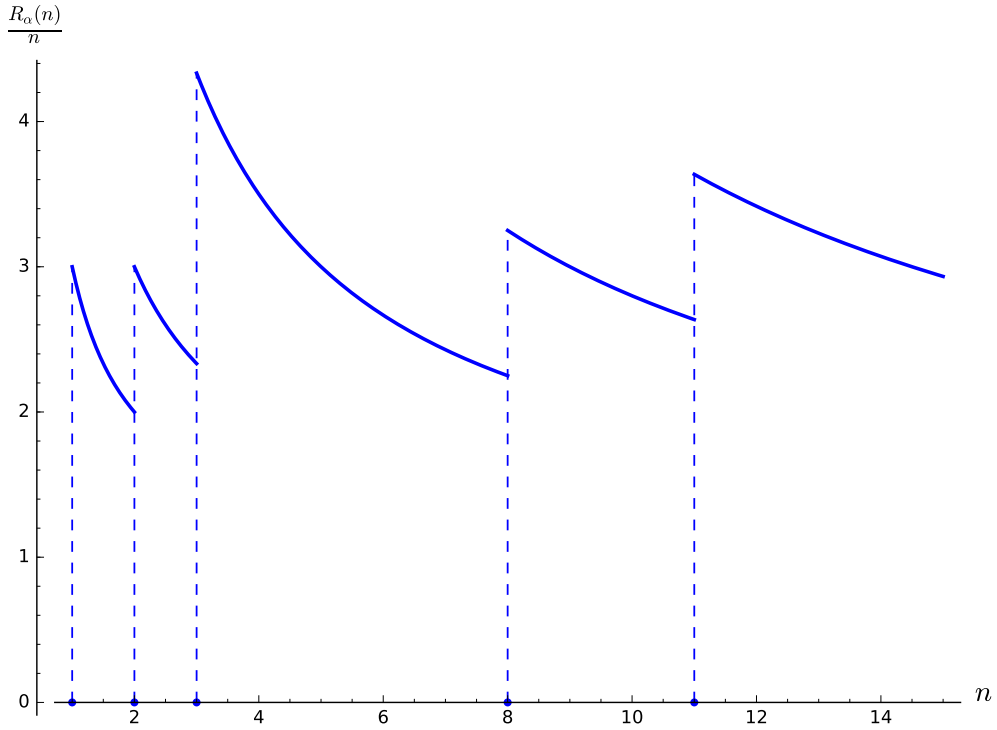


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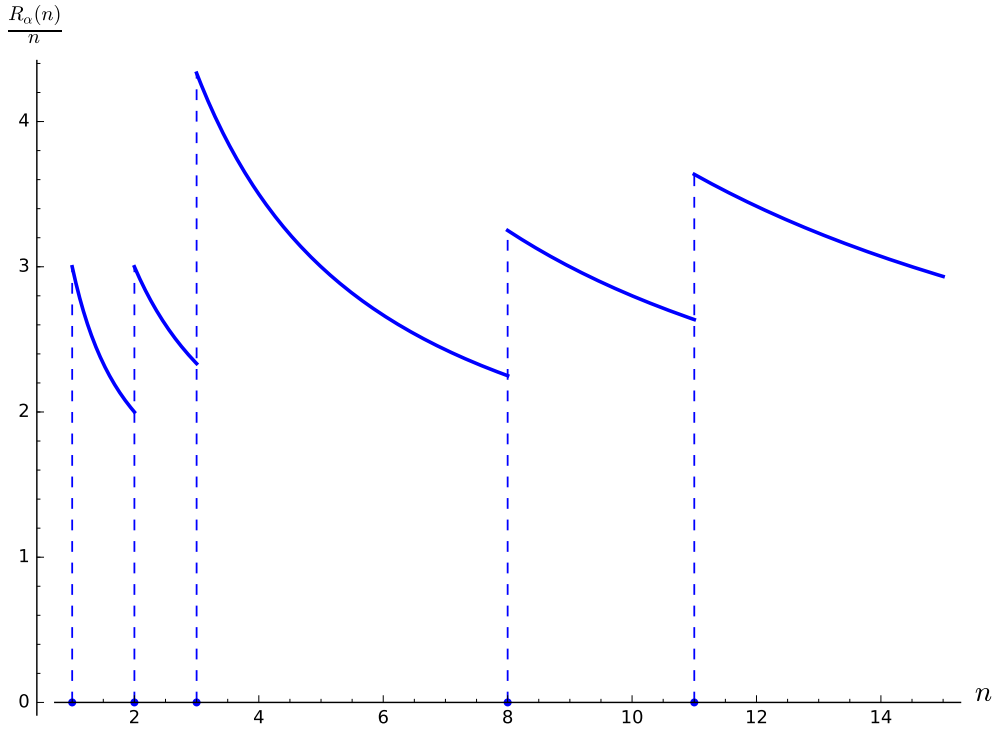


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*Size* of $S(\alpha, n)$ dictated by

- the relative position of $n$ within the interval

$$\mu(\alpha, n) = \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)},$$

- the quotient between the ends of the interval

$$\rho(\alpha, n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)}.$$

# Studies of the recurrence function

Classical results concern the worst case scenarios for *fixed* $\alpha$: let $\epsilon > 0$, for almost every $\alpha$

$$\limsup_{n\to\infty} \frac{S(\alpha, n)}{\log n} = \infty\,, \quad \lim_{n\to\infty} \frac{S(\alpha, n)}{(\log n)^{1+\epsilon}} = 0\,.$$

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We define two probabilistic models

in both cases $\alpha$ is drawn *uniformly at random*

1) fix the length $n$ $\Rightarrow$ random variables $S_n(\alpha) := S(\alpha, n)$.
   in distribution and expectation as $n \to \infty$.
2) fix index $k$ of interval $[q_{k-1}(\alpha), q_k(\alpha))$ and
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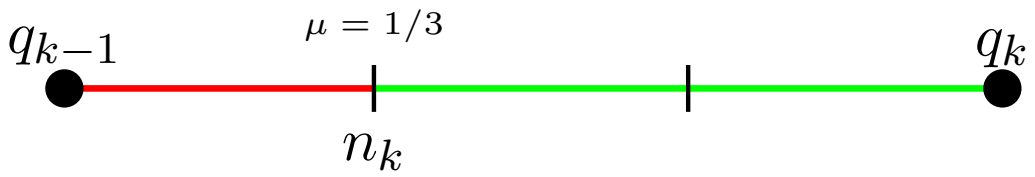


Figure : Sequence of indices $(n_k(\alpha))_k$ for $\mu = 1/3$.

# Results

## Model (1): fixed $n$. [RV17]

- Limit distribution for

$$\mu_n, \quad \rho_n, \quad S_n \, .$$

- Convergence of histograms to limit density.
- Conditional expectations

$$\mathbb{E}[S_n | \mu_n \geq \epsilon(n)] \sim \left| \log \epsilon(n) \right| .$$

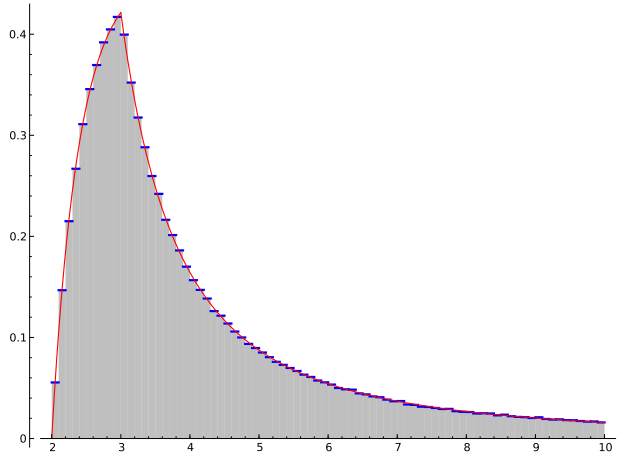


Figure : Limit density of $S_n$.

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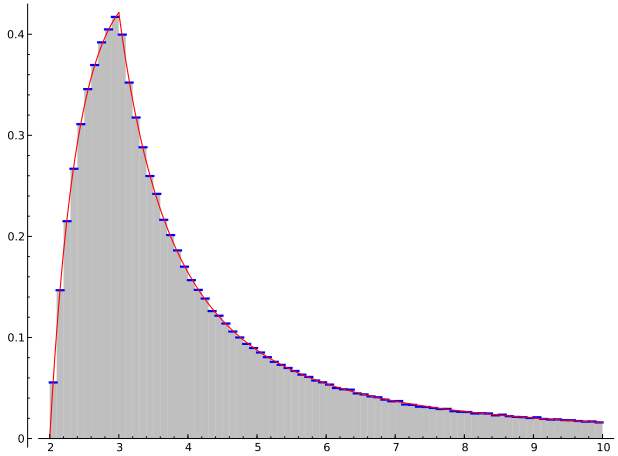
## Model (2): fixed $\mu$. [BCRVV15]

- Limit distribution/expectation of
$$S_k(\alpha) := S(\alpha, n_k)$$
depending on $\mu$.
- Study of $\mathbb{E}[S_k(\alpha)]$ as
$$\mu := \mu_k \to 0\,.$$

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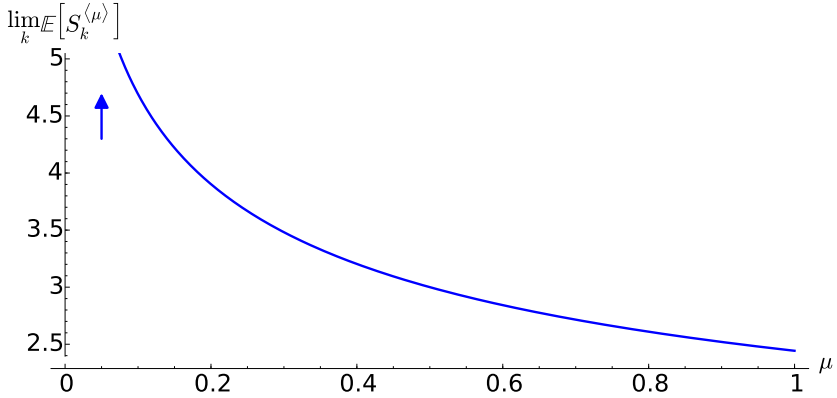


Figure : Limit of $\mathbb{E}[S_k]$ versus $\mu$.

# Elements of the proofs

## Model (1): fixed $n$

- Distribution at $\lambda$ given by *coprime* Riemann sum of step $\frac{1}{n}$ of

$$\omega(x,y) = \frac{2}{y(x+y)}\,,$$

over

$$\Delta_\lambda = \{(x,y) : 0 < x \leq 1 < y,\\ 1 + x + y \leq \lambda\}\,.$$

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## Model (2): fixed $\mu$

- Expected values given by Perron-Frobenius operator:

$$\mathbb{E}[S_k] \sim 1 + \mathbf{H}^k[g](0)\,,$$

where

$$g(\rho) = \left(\frac{1+\rho}{\mu+\rho-\mu\rho}\right) / (1+\rho)\,.$$

# Section

# The origins

Introduced by Gosper as a mutation of continued fractions:

- gives rise to a $\gcd$ algorithm akin to Euclid's.
- quotients are powers of two:
  - small information parcel.
  - employs only shifts and substractions.
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- appears to be simple and efficient.

More recently:

- Shallit studied its worst-case performance in 2016.
- We consider its average performance!

# Continued Logarithm Algorithm

A sequence of binary "divisions" beginning from $(p, q)$:

$$q = \mathbf{2^a} p + r\,, \qquad 0 \le r < 2^a p\,.$$

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- Ended with $(0, 8)$, what is the $\gcd$?
  $\Rightarrow$ odd $\gcd \times$ parasitic powers of $2$.

Consider

$$\Omega_N = \{(p,q) \in \mathbb{N} \times \mathbb{N} : p \leq q \leq N\}.$$

Worst-case studied by Shallit (2016): $2\log_2 N + O(1)$ steps.

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**Main result (R., Vallée, '18).**

Mean number of **steps** $E_N[K]$ and **shifts** $E_N[S]$ are $\Theta(\log N)$.

More precisely

$$E_N[K] \sim k\,\log N\,, \qquad E_N[S] \sim \tfrac{\log 3 - \log 2}{2\log 2 - \log 3}\,E_N[K]$$

for an *explicit constant* $k \doteq 1.49283\ldots$ given by

$$k = \frac{2}{H}\,, \quad H = \tfrac{1}{\log(4/3)}\left(\tfrac{\pi^2}{6} + 2\sum_j \tfrac{(-1)^j}{2^j j^2} - (\log 2)\tfrac{\log 27}{\log 16}\right)$$

Process depends **only** on $p/q$ rather than $(p, q)$.

- Map $p/q \mapsto p'/q'$ can be extended to $\mathcal{I} = (0, 1)$

$$T \colon \mathcal{I} \to \mathcal{I}\,, \quad T(x) = \frac{1}{2^a x} - 1\,,$$

where $a = \lfloor \log_2(1/x) \rfloor$.

- Iteration gives a special continued fraction

$$\frac{p}{q} = \frac{1}{2^a \left(1 + \frac{p'}{q'}\right)}\,.$$

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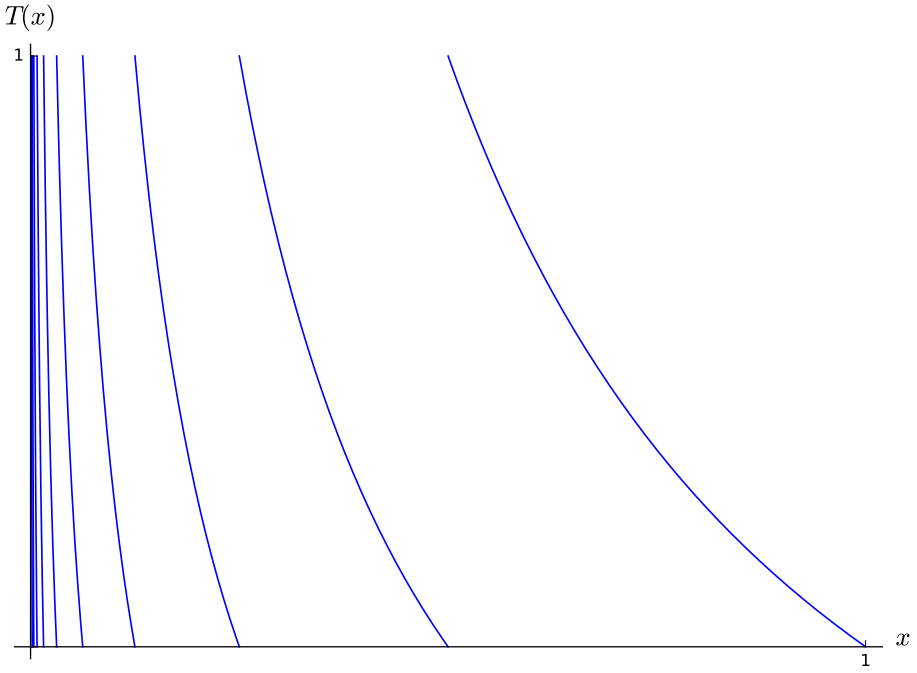
The continued fraction expansion ends (is finite) when we get $0$.

# Dynamical system $(\mathcal{I}, T)$




The map $T\colon \mathcal{I} \to \mathcal{I}$

**Branches**

For $x \in \mathcal{I}_a := [2^{-a-1}, 2^{-a}]$

$$x \mapsto T_a(x) := \frac{2^{-a}}{x} - 1\,.$$

where $a(x) := \lfloor \log_2(1/x) \rfloor\,.$

**Inverse branches**

$$h_a(x) := \frac{2^{-a}}{1+x}\,, \quad \mathcal{H} := \{h_a : a \in \mathbb{N}\}\,,$$

and at depth $k$

$$\mathcal{H}^k := \{h_{a_1} \circ \cdots \circ h_{a_k} : a_1, \ldots, a_k \in \mathbb{N}\}\,.$$

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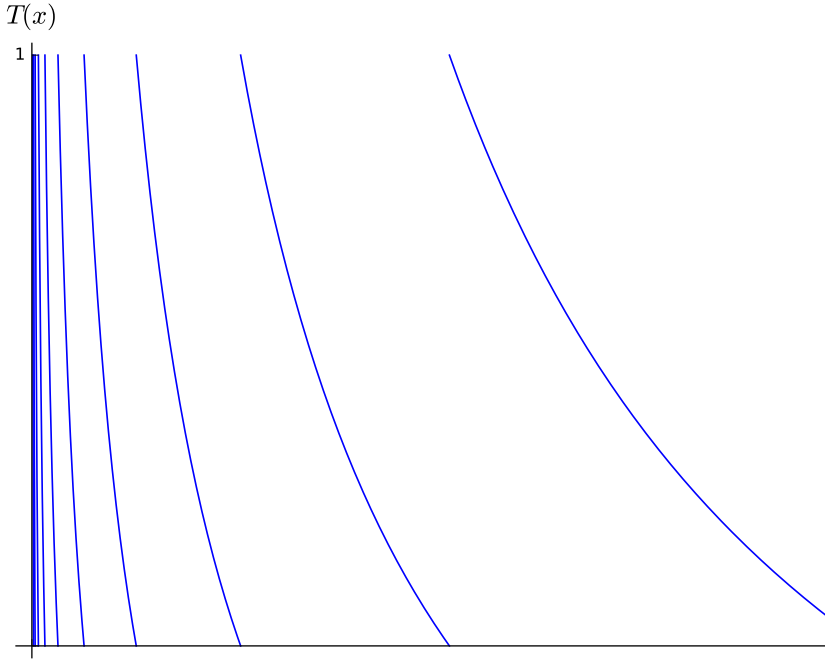


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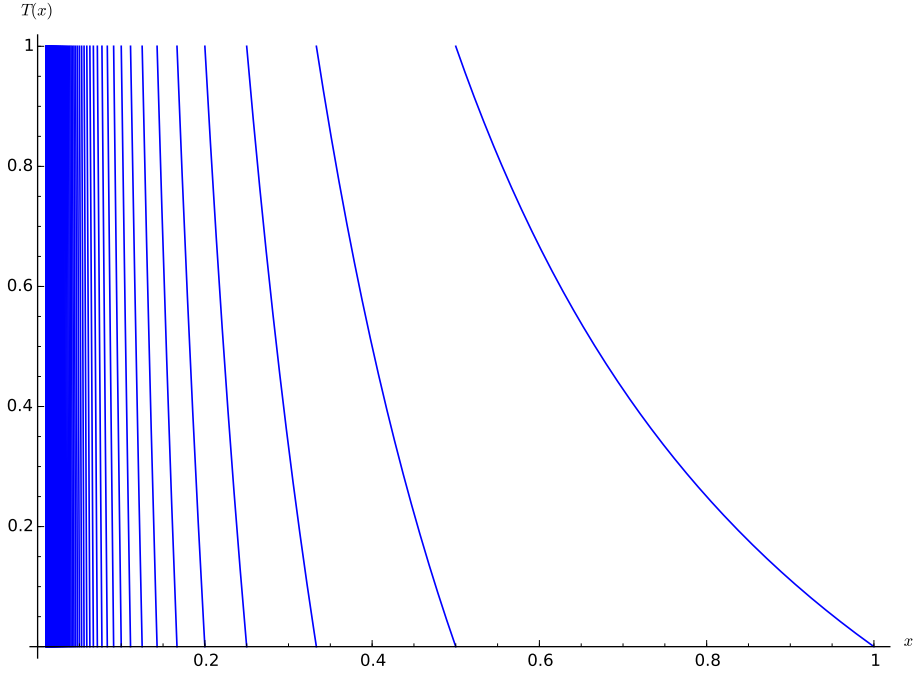


The map for Euclid's algorithm.

# Reduced denominators and inverse branches

Observe for $f(x) = \frac{ax+b}{cx+d}$ :

$$f'(x) = \frac{\det f}{(cx+d)^2}.$$

Euclidean algorithm:

- Homographies

$$h_m(x) = \frac{1}{m+x},$$

with $\det h_m = -1$.

- For $h = h_{m_1} \circ \ddots \circ h_{m_k}$

$$h(0) = \frac{p}{q} \Rightarrow |h'(0)| = \frac{1}{q^2},$$

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**Problem:** Denominator retrieved is engorged by powers of two.

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**Idea works!**

# The extended dynamical system

⊛ Introduce $\underline{\mathcal{I}} := \mathcal{I} \times \mathbb{Q}_2$ and $\underline{T} \colon \underline{\mathcal{I}} \to \underline{\mathcal{I}}$ as follows

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Haar (translation invariant) measure $\nu$ on $\mathbb{Q}_2$ has one!

# Functional space $\mathcal{F}$ for the extended operator $\underline{\mathbf{H}}_s$

Real component directs the dynamical system:

- *sections* $F_y$ fixing $y \in \mathbb{Q}_2$ asked to be $C^1(\mathcal{I})$.
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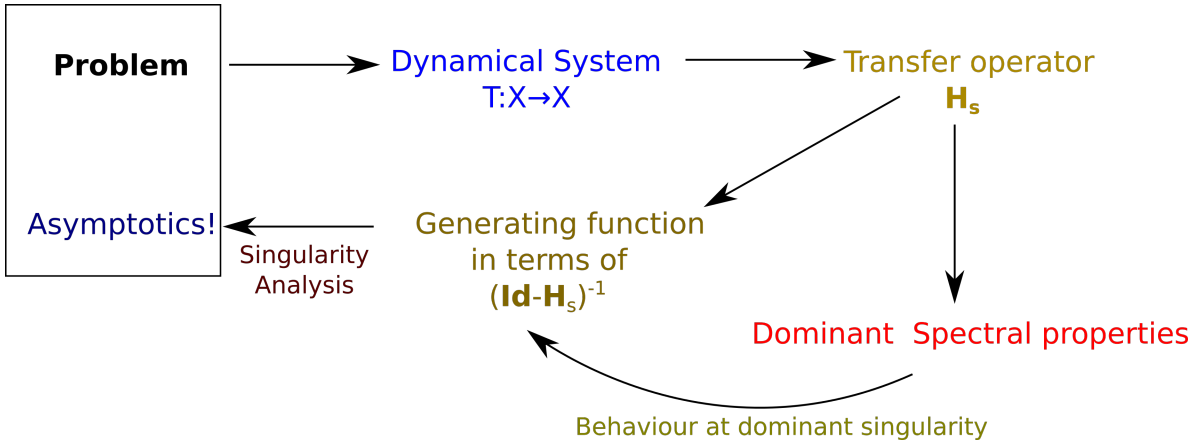
**We can finish the dynamical analysis!**



# Conclusion and further questions

⊛ We have studied the average number of shifts and substractions for the CL algorithm.

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Back to $(13, 31)$

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