Continued Logarithm Algorithm.
A probabilistic study

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$$\frac{13}{31} = \cfrac{2^{-1}}{1 + \cfrac{2^{-2}}{1 + \cfrac{2^{-1}}{1 + \cfrac{2^{0}}{1 + \cfrac{2^{-1}}{1}}}}}$$

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Table of Contents

The origins

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- let us see an example!

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- Powers of two pop up everywhere!
- We ended up with $(8, 0)$, what is the $\gcd$? $\Rightarrow$ odd $\gcd = 1$.

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Average number of steps K and shifts S satisfy

$$E_N[K] \sim k \log N\,, \qquad E_N[S] \sim \frac{\log 3 - \log 2}{2\log 2 - \log 3}\, E_N[K]$$

for an *explicit constant* $k \doteq 1.49283\ldots$ given by

$$k = \frac{2}{H}\,, \quad H = \frac{1}{\log(4/3)}\left(\frac{\pi^2}{6} + 2\mathrm{Li}_2(-1/2) - (\log 2)\frac{\log 27}{\log 16}\right)$$

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$\Longrightarrow$ proof turns out to be a bit unexpected.

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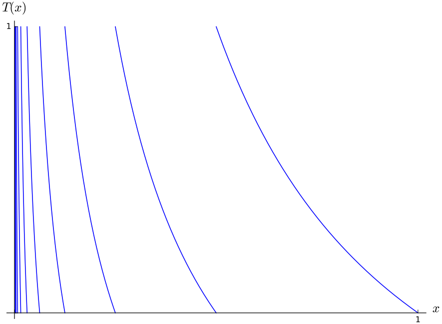
Dynamical system $(\mathcal{I}, T)$



The map $T\colon \mathcal{I} \to \mathcal{I}$

Branches

For $x \in \mathcal{I}_a := [2^{-a-1}, 2^{-a}]$

$$x \mapsto T(x) := \frac{2^{-a}}{x} - 1\,.$$

where $a(x) := \lfloor \log_2(1/x) \rfloor\,.$

Inverse branches

$$h_a(x) := \frac{2^{-a}}{1+x}\,, \quad \mathcal{H} := \{h_a : a \in \mathbb{N}\}\,,$$

and at depth k

$$\mathcal{H}^k := \{h_{a_1} \circ \cdots \circ h_{a_k} : a_1, \ldots, a_k \in \mathbb{N}\}\,.$$

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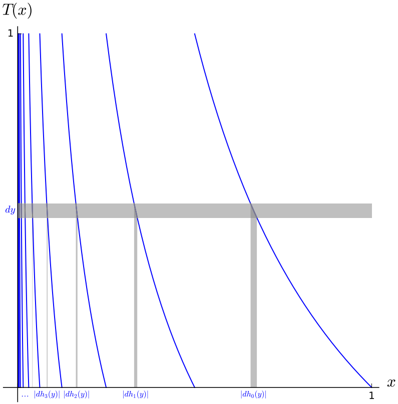

Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| \, g\left(h(x)\right)$$

$$= \frac{1}{(1+x)^2} \sum_{a \geq 0} 2^{-a} \, g\left(\frac{2^{-a}}{1+x}\right) .$$

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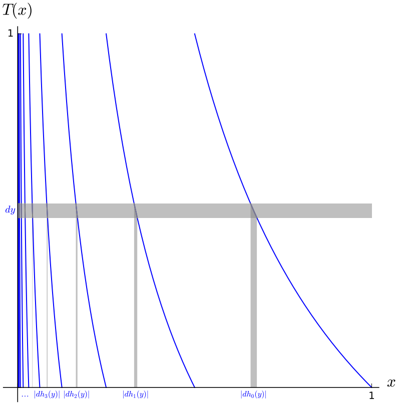

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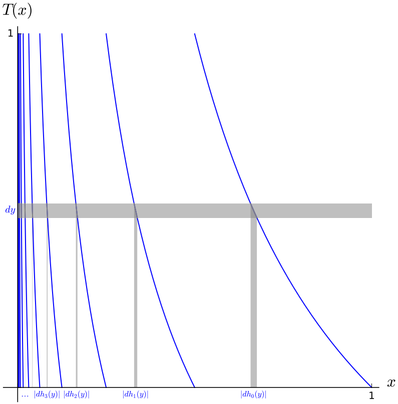

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$$\mathbf{H}^k[g](x) = \sum_{h \in \mathcal{H}^k} \left| h'(x) \right| g\left(h(x)\right) .$$

$\implies$ Transfer operator $\mathbf{H}_s$ extends $\mathbf{H}$, introducing a variable s

$$\mathbf{H}_s[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right|^s g\left(h(x)\right) .$$

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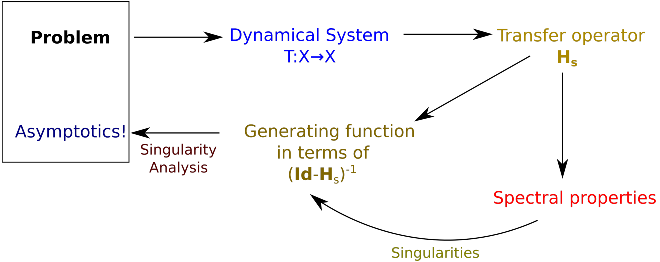

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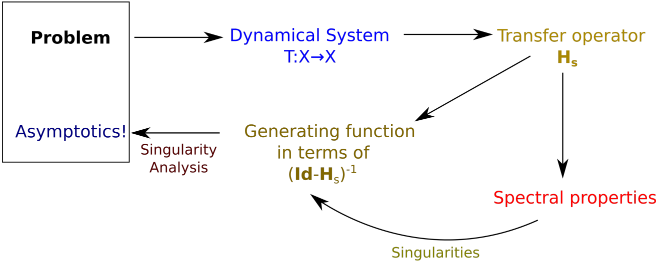

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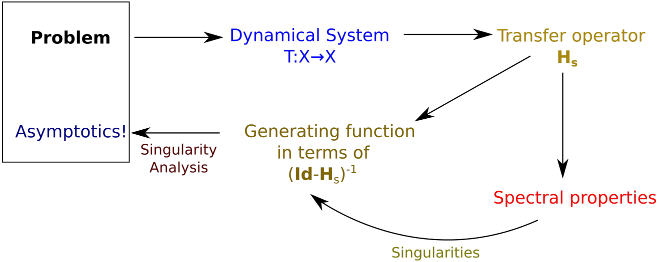

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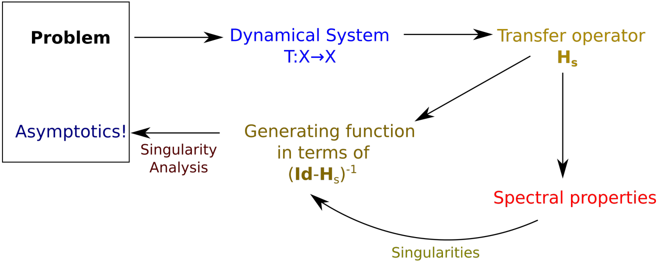

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$\Longrightarrow$ Essential to keep track of this “dyadic behaviour”.

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  $\Rightarrow$ yet the real component that "leads".

Idea works!

Average behaviour of the CL algorithm

Input model:

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We have explicit constants

$$E_N[K] \sim \frac{2}{H} \log N\,, \qquad E_N[S] \sim \frac{\log 3 - \log 2}{2\log 2 - \log 3} E_N[K]\,,$$

here H is known as the entropy of the system,

$$H = \frac{1}{\log(4/3)} \left(\frac{\pi^2}{6} + 2\mathrm{Li}_2\left(-\tfrac{1}{2}\right) - (\log 2)\frac{\log 27}{\log 16} \right)\,,$$

numerically $H \doteq 1.33973\ldots$

The extended dynamical system

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$\Longrightarrow$ Consider related measure $\tilde{\nu}$ on $\mathbb{Q}_2$!

⇒ *extended* transfer operator $\underline{\mathbf{H}}_s$.

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Several remarks

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- Ergodic.
- Invariant density $\psi(x) = \frac{1}{\log(4/3)} \frac{1}{(x+1)(x+2)}$.
- Geometrical properties that make it act nicely on $C^1(\mathcal{I})$

  $\Longrightarrow$ Dominant eigenvalue and spectral gap!

Functional space $\mathcal{F}$ for the extended operator $\underline{\mathbf{H}}_s$

Real component directs the dynamical system:

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We can finish the dynamical analysis!

Conclusion and further questions

We have analysed

1. the number of **steps**
2. the number of **binary shifts**

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In particular when dividing $(p, q) \mapsto (p/2, q/2)$ as soon as possible!

$\Rightarrow$ may have competitive bit complexity.

The Continued Logarithm vs. other binary algorithms

Other well-known binary algorithms include

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Unify the analysis to better understand the role of the dyadics?

Continued Logarithm expansion over the reals

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- the averages of the exponents $(a_1(x) + \ldots + a_M(x))/M$.
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$\rightarrow$ related to growth of $\gcd(p, q)$ in the algorithm!

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