Circular and non-wellfounded proofs: expressiveness and semantics I

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Outline

1. Types with fixed points
2. Circular proofs
3. Some results on expressivity
4. Conclusions

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Simple types

Types:

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In $\mu X\, \sigma$ and $\nu X\, \sigma$, the variable X must occur positively in σ.

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Example

- $B := 1 + 1$ represents the *Booleans*.
- $N := \mu X(1 + X)$ represents the *natural numbers*.
- $S_\tau := \nu X(\tau \times X)$ represents *infinite streams* over τ.
- $W := \mu X(1 + \nu Y(X \times Y))$ represents the (ω-branching) *well-founded trees*.

Sequents: $\sigma_1, \ldots, \sigma_n \Rightarrow \tau$ (interpret as $\sigma_1 \times \cdots \times \sigma_n \to \tau$)

Each type can be constructed and destructed. E.g.

$$\to_r \frac{\sigma \Rightarrow \tau}{\Rightarrow \sigma \to \tau} \qquad \to_l \frac{\Rightarrow \rho \quad \sigma \Rightarrow \tau}{\rho \to \sigma \Rightarrow \tau}$$

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Fixed point rules:

$$\mu_r \frac{\Rightarrow \sigma(\mu X\, \sigma(X))}{\Rightarrow \mu X\, \sigma(X)} \quad \mu_l \frac{\sigma(\tau) \Rightarrow \tau}{\mu X\, \sigma(X) \Rightarrow \tau} \quad \nu_r \frac{\tau \Rightarrow \sigma(\tau)}{\tau \Rightarrow \nu X\, \sigma(X)} \quad \nu_l \frac{\sigma(\nu X\, \sigma(X)) \Rightarrow \tau}{\nu X\, \sigma(X) \Rightarrow \tau}$$

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Definition ([Cla09])

μLJ is the extension of usual LJ by the fixed point rules above.

Computational theory given by cut-reduction.

$$\underline{N := \mu X(1 + X)}$$

$$\underline{0} := \;\; \mu_r\,\dfrac{\dfrac{\overline{\Rightarrow 1}}{\Rightarrow 1 + N}}{\Rightarrow N} \qquad\qquad \underline{n + 1} := \;\; \mu_r\,\dfrac{\dfrac{\begin{array}{c}\underline{n}\\ \vdots \\ \Rightarrow N\end{array}}{\Rightarrow 1 + N}}{\Rightarrow N}$$

Examples: natural numbers and streams

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$$\underline{\mathsf{add} : N \times N \to N}$$

$$\mu_l \dfrac{\dfrac{\dfrac{\mathsf{id}\,\overline{N \Rightarrow N}}{1, N \Rightarrow N} \qquad \mu_r \dfrac{\dfrac{\mathsf{id}\,\overline{N \Rightarrow N}}{N \Rightarrow 1 + N}}{N \Rightarrow N}}{1 + N, N \Rightarrow N}}{N, N \Rightarrow N}$$

$$\left(\begin{array}{c} \mathsf{add}(0, n) = n \\ \mathsf{add}(m + 1, n) = \mathsf{add}(m, n) + 1 \end{array} \right)$$

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$$\underline{S := \nu Y(N \times Y)}$$

$$\mathtt{hd} := \quad \nu_l \,\cfrac{\cfrac{\mathsf{id}\;\overline{N \Rightarrow N}}{N \times S \Rightarrow N}}{S \Rightarrow N} \qquad \mathtt{tl} := \quad \nu_l \,\cfrac{\cfrac{\mathsf{id}\;\overline{S \Rightarrow S}}{N \times S \Rightarrow S}}{S \Rightarrow S}$$

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$$\underline{f : n \mapsto [n, n+1, \ldots]}$$

$$\dfrac{\dfrac{\overline{N \Rightarrow N}\ \text{id} \qquad \dfrac{\dfrac{\overline{N \Rightarrow N}\ \text{id}}{N \Rightarrow 1 + N}}{N \Rightarrow N}\,\mu_r}{N \Rightarrow N \times N}}{N \Rightarrow S}\,\nu_r$$

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$$\left(\ f(n) = n :: f(n+1)\ \right)$$

Aside: interpreting types with fixed points

A set theoretic model

Interpret τ as a set $\tau^{\mathfrak{S}}$:

$$
\begin{aligned}
\bot^{\mathfrak{S}} &:= \varnothing \\
1^{\mathfrak{S}} &:= \varnothing \\
(\sigma + \tau)^{\mathfrak{S}} &:= \sigma^{\mathfrak{S}} \uplus \tau^{\mathfrak{S}} \\
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Q: what do B, N, S_τ, W denote in $\mathfrak{K}$?

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System	Computation	Logic
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+ N	HO Primitive Recursion (T)	FO Arithmetic (PA)
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What do fixed point type systems compute?

This may be model-sensitive, but is *robust* for type 1 functions.

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Non-wellfounded typing

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- It is **progressing** if every infinite branch has an infinite progressing thread.
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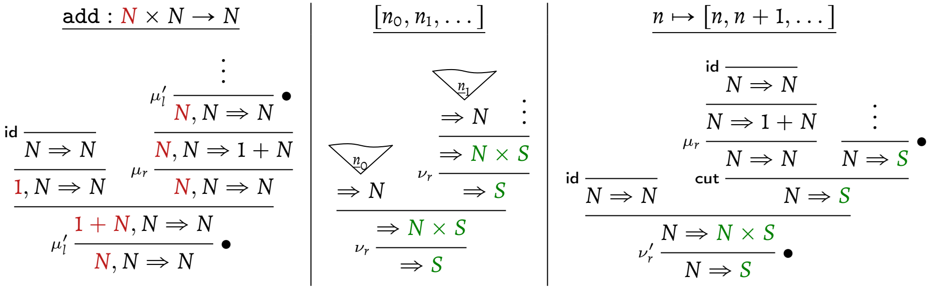
Computational theory again given by cut-reduction.

NB: cyclic proof checking is decidable, reducing to universality of Büchi automata.


add : N × N → N
id
N ⇒ N
1, N ⇒ N
μ'_l
N, N ⇒ N
μ_r
N, N ⇒ 1 + N
N, N ⇒ N
1 + N, N ⇒ N
μ'_l
N, N ⇒ N
[n_0, n_1, . . .]
n_1
⇒ N
⇒ N × S
ν_r
⇒ S
n_0
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ν_r
⇒ S
n ↦ [n, n + 1, . . .]
id
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$$\underline{[n_0, n_1, \dots]}$$

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Iteration to cycles:

$$\mu_l \; \dfrac{\sigma(\tau) \Rightarrow \tau}{\mu X \sigma(X) \Rightarrow \tau} \qquad \rightsquigarrow \qquad \mu_l' \; \dfrac{\mathsf{cut}\;\dfrac{\sigma\;\dfrac{\mu_l'\;\dfrac{\vdots}{\mu X \sigma(X) \Rightarrow \tau}\;\bullet}{\sigma(\mu X \sigma(X)) \Rightarrow \sigma(\tau)} \qquad \sigma(\tau) \Rightarrow \tau}{\sigma(\mu X \sigma(X)) \Rightarrow \tau}}{\mu X \sigma(X) \Rightarrow \tau}\;\bullet$$

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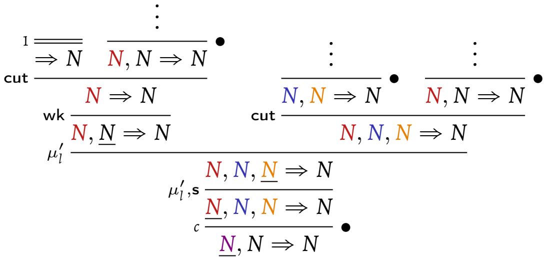

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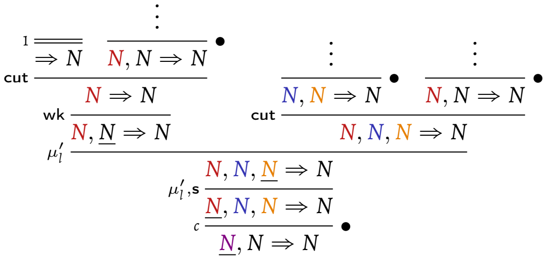

NB: The function A requires iteration at type 1 in finitary derivations.

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A metamathematical approach

- Second-order arithmetic can formalise *metatheory* of infinite proofs.
- Computational interpretations allow us to extract finitary proofs thence.

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- $N^{\mathfrak{S}} \cong \mathbb{N}$: so $\mathfrak{S}$ reduces to a model of higher-order functionals over $\mathbb{N}$.
- A circular preproof $P : \Gamma \Rightarrow \tau$ represents a partial functional $P^{\mathfrak{S}} : \Gamma^{\mathfrak{S}} \rightarrow \tau^{\mathfrak{S}}$.

Theorem ([KPP21])

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Write (C)T_n for the restriction of (C)T to just level n types.

Theorem ([Das21])

T_{n+1} *and* CT_n *interpret each other.*
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Proposition ([Das21])
T *and* CT *represent the same functionals in* $\mathfrak{S}$.

Proposition (Totality)

If P is progressing, then $P^{\mathfrak{S}}$ is total.

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Proof sketch.

Let P_0 be a coderivation of $\Gamma_0 \Rightarrow \tau_0$.

- Suppose not P_0 total, and let $\vec{a}_0 \in \Gamma_0^{\mathfrak{S}}$ s.t. $P_0^{\mathfrak{S}} \vec{a}_0 \uparrow$.
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If P is progressing, then $P^{\mathfrak{S}}$ is total.

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NB: this proof is highly non-constructive! How can we extract induction invariants?

Before dissecting these results, let us set up our toolbox:

Proof theoretic ingredients:

- Totality at type 1 is a Π^0_2 property $(\forall m \exists n\, Pm \downarrow n)$.
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  $\rightsquigarrow$ formalisation in subsystems of SO arithmetic.
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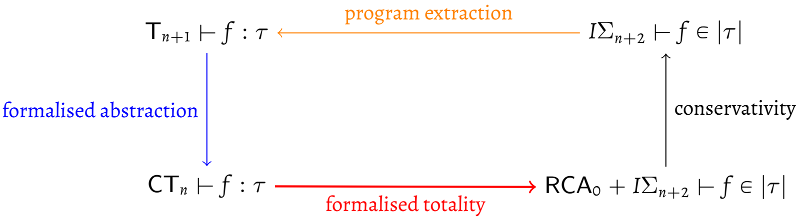
Textbook interpretations:

- SO theories are conservative extensions of appropriate FO ones.
- *Computational interpretations* allow T to realise FO theorems.

For f of type 1:



- formalised abstraction requires a careful *partial evaluation* result.
- Confluence of CT in $\mathsf{RCA}_0 \implies$ determinism.
- formalised totality arithmetises the totality argument for CT.
    - *Reverse mathematics* of ω-automaton theory [KMPS19, Das20].
    - Must formalise totality argument in a HO computability model $|\cdot|$.
- program extraction is textbook.

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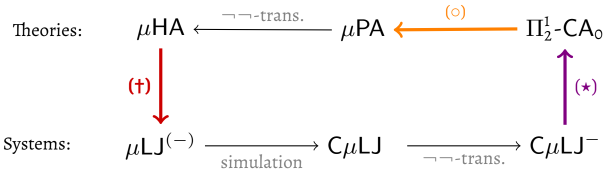

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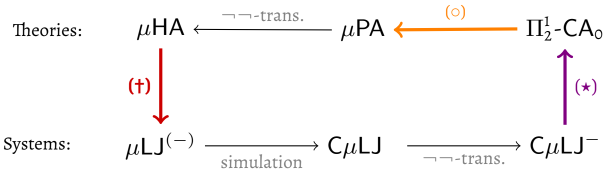

(⋆) Formalisation of semantics by *fixed points as fixed points*:

- Novel reverse mathematics of ordinal and fixed point theory, building on [Das21, DM23].

(○) A complex black box result due to [Mö02].

(†) Realisability interpretation by *fixed points as SO types*.

- (Considerable) specialisation of HA2 $\to$ F.

Outline

1. Types with fixed points
2. Circular proofs
3. Some results on expressivity
4. Conclusions

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Thank you.

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