Semantics of recursive types and logic with fixpoints

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Categorical Semantics of Fixpoint Logics (finitary system)

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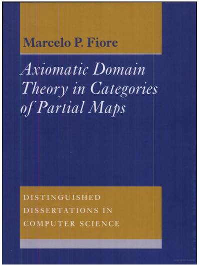
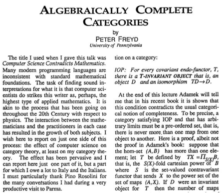




ALGEBRAICALLY COMPLETE CATEGORIES

by

PETER FREYD

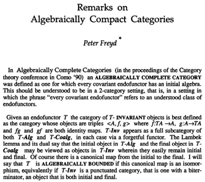
University of Pennsylvania

The title I used when I gave this talk was *Computer Science Contradicts Mathematics.* Many modern programming languages are inconsistent with standard mathematical foundations. The task of finding sound interpretations for what it is that computer scientists do strikes this writer as, perhaps, the highest type of applied mathematics. It is akin to the process that has been going on throughout the 20th Century with respect to physics. The interaction between the mathematicians and the practitioners in each case has resulted in the growth of both subjects. I wish here to report on just one side of this process: the effect of computer science on category theory, at least on my category theory. The effect has been pervasive and I can report here just one part of it, but a part for which I owe a lot to Italy and the Italians. I must particularly thank Pino Rosolini for the many conversations I had during a very productive visit to Parma.

tion on a category:

IOP: For every covariant endo-functor, T, there is a ***T-INVARIANT OBJECT*** *that is, an object D and an isomorphism $TD \to D$.*

At the end of this lecture Adamek will tell me that in his recent book it is shown that this condition contradicts the usual categorical notion of completeness. To be precise, a category satisfying IOP and that has arbitrary limits must be a pre-ordered set, that is, there is never more than one map from one object to another. Here is a proof, albeit not the proof in Adamek's book: suppose that the hom-set (A,B) has more than one element; let T be defined by $TX = \Pi_{S(X)} B$, that is, the $S(X)$-fold cartesian power of B where S is the set-valued contravariant functor that sends X to the power set of the set of maps (A,X). If D were an invariant object for T then the number of maps

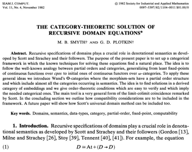


Remarks on Algebraically Compact Categories

Peter Freyd *

In Algebraically Complete Categories (in the proceedings of the Category theory conference in Como '90) an **ALGEBRAICALLY COMPLETE CATEGORY** was defined as one for which every covariant endofunctor has an initial algebra. This should be understood to be in a 2-category setting, that is, in a setting in which the phrase "every covariant endofunctor" refers to an understood class of endofunctors.

Given an endofunctor T the category of T- **INVARIANT** objects is best defined as the category whose objects are triples $\langle A, f, g\rangle$ where $f{:}TA \to A$, $g{:}A \to TA$ and fg and gf are both identity maps. ***T-Inv*** appears as a full subcategory of both ***T-Alg*** and ***T-Coalg***, in each case via a forgetful functor. The Lambek lemma and its dual say that the initial object in ***T-Alg*** and the final object in ***T-Coalg*** may be viewed as objects in ***T-Inv*** wherein they easily remain initial and final. Of course there is a canonical map from the initial to the final. I will say that T is **ALGEBRAICALLY BOUNDED** if this canonical map is an isomorphism, equivalently if ***T-Inv*** is a punctuated category, that is one with a biterminator, an object that is both initial and final.



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THE CATEGORY-THEORETIC SOLUTION OF RECURSIVE DOMAIN EQUATIONS*

M. B. SMYTH† AND G. D. PLOTKIN†

Abstract. Recursive specifications of domains plays a crucial role in denotational semantics as developed by Scott and Strachey and their followers. The purpose of the present paper is to set up a categorical framework in which the known techniques for solving these equations find a natural place. The idea is to follow the well-known analogy between partial orders and categories, generalizing from least fixed-points of continuous functions over *cpos* to initial ones of continuous functors over ω-categories. To apply these general ideas we introduce Wand's **O**-categories where the morphism-sets have a partial order structure and which include almost all the categories occurring in semantics. The idea is to find solutions in a derived category of embeddings and we give order-theoretic conditions which are easy to verify and which imply the needed categorical ones. The main tool is a very general form of the limit-colimit coincidence remarked by Scott. In the concluding section we outline how compatibility considerations are to be included in the framework. A future paper will show how Scott's universal domain method can be included too.

Key words. Domains, semantics, data-types, category, partial-order, fixed-point, computability

1. Introduction. Recursive specifications of domains play a crucial role in denotational semantics as developed by Scott and Strachey and their followers (Gordon [13], Milne and Strachey [26], Stoy [39], Tennent [40], [41]). For example, the equation

$$D \cong \mathrm{At} + (D \to D) \tag{1}$$

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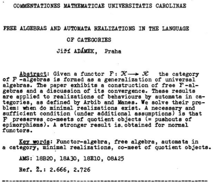


COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

FREE ALGEBRAS AND AUTOMATA REALIZATIONS IN THE LANGUAGE OF CATEGORIES

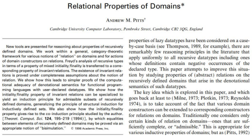
Jiří ADÁMEK, Praha

Abstract: Given a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ the category of F-algebras is formed as a generalization of universal algebras. The paper exhibits a construction of free F-algebras and a discussion of its convergence. These results are applied to realizations of behaviours by automata in categories, as defined by Arbib and Manes. We solve their problem: when do minimal realizations exist. A necessary and sufficient condition (under additional assumptions) is that F preserves co-meets of quotient objects (= pushouts of epimorphisms). A stronger result is obtained for normal functors.

Key words: Functor-algebra, free algebra, automata in a category, minimal realizations, co-meet of quotient objects.

AMS: 18B20, 18A30, 18E10, 08A25

Ref. Ž.: 2.666, 2.726



Relational Properties of Domains*

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New tools are presented for reasoning about properties of recursively defined domains. We work within a general, category-theoretic framework for various notions of "relation" on domains and for actions of domain constructors on relations. Freyd's analysis of recursive types in terms of a property of mixed initiality/finality is transferred to a corresponding property of *invariant* relations. The existence of invariant relations is proved under completeness assumptions about the notion of relation. We show how this leads to simpler proofs of the computational adequacy of denotational semantics for functional programming languages with user-declared datatypes. We show how the initiality/finality property of invariant relations can be specialized to yield an induction principle for admissible subsets of recursively defined domains, generalizing the principle of structural induction for inductively defined sets. We also show how the initiality/finality property gives rise to the co-induction principle studied by the author. [*Theoret. Comput. Sci.* **124**, 195–219 (1994)], by which equalities between elements of recursively defined domains may be proved via an appropriate notion of "bisimulation." © 1996 Academic Press, Inc.

properties of lazy datatypes have been considered on a case-by-case basis (see Thompson, 1989, for example), there are remarkably few reasoning principles in the literature that apply uniformly to all recursive datatypes including ones whose definitions contain negative occurrences of the declared type. This paper attempts to improve this situation by studying properties of (abstract) relations on the recursively defined domains that arise in the denotational semantics of such datatypes.

The key idea which is explored in this paper, and which goes back at least to (Milne, 1973; Plotkin, 1973; Reynolds 1974), is to take account of the fact that various domain constructors can be extended to corresponding constructors for relations on domains. Traditionally one considers only certain kinds of relation on domains—ones that are sufficiently complete, or "admissible." This is appropriate for various inductive properties of domains; but as (Pitts, 1992)

Categorical Semantics of Recursive Types

Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ be a functor. If μF exists, then $F(\mu F) \cong \mu F$.

We define a fixpoint of $F : \mathcal{C} \to \mathcal{C}$ to be an object of $\mathcal{C}$, denoted by $\mathrm{Fix}\,\mathcal{F}$, such that $F(\mathrm{Fix}\,\mathcal{F}) \cong \mathrm{Fix}$.

- Find other fixpoints of $1 + X : \mathrm{Set} \to \mathrm{Set}$.
- The initial algebra gives us a canonic fixpoint, i.e. the least fixpoint.

Let $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We seek a canonical object $\mathrm{Fix}\,\mathcal{F}$ such that

$$\mathcal{F}(\mathrm{Fix}\,\mathcal{F}, \mathrm{Fix}\,\mathcal{F}) \cong \mathrm{Fix}\,\mathcal{F}$$

Suppose that the fixpoint of $\mathcal{F}(X, \_) : \mathcal{C} \to \mathcal{C}$ exists for all $X : \mathcal{C}$. This induces a functor $\mathcal{F}^\star : \mathcal{C} \to \mathcal{C}$ which sends X to $\mathrm{Fix}\,\mathcal{F}(X, \_)$:

$$\mathcal{F}^\star(X) \cong \mathcal{F}(X, \mathcal{F}^\star(X))$$

Any candidate for $\mathrm{Fix}\,\mathcal{F}$?

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$$\operatorname{Fix} \mathcal{F} := \operatorname{Fix}(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$$

We could ask the existence of either $\mu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$ or $\nu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$.

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The analogue of the Lambek's theorem for contravariant functor $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$:

Lemma

If $\mu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$ and $\nu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$ exist, then $\mathcal{F}$ transforms $\mu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$ to $\nu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$ and vice verca:

$$\mathcal{F}(\mu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})) \cong \nu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F}) \qquad \mathcal{F}(\nu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})) \cong \mu(\mathcal{F}^{\mathsf{op}} \circ \mathcal{F})$$

If we assume that

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All we have seen so far motivates the following definitions.

Definition

A category $\mathcal{C}$ is *algebraically complete* if $\mu\mathcal{F}$ exists for each functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$.

It is *algebraically cocomplete* if $\nu\mathcal{F}$ exists for each functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$.

Definition

A categroy $\mathcal{C}$ is *algebraically compact* if it is algebraically complete and cocomplete and $\mu\mathcal{F} \cong \nu\mathcal{F}$ canonically.

We have proven that:

Proposition

Let $\mathcal{C}$ be a algebraically compact category and $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ be a functor. Then $\mathcal{F}$ has a fixpoint, viz., $\mathsf{Fix}((\mathcal{F}^{\star})^{\mathsf{op}} \circ \mathcal{F}^{\star})$.

- There are very few examples of algebraically complete (compact) categories.
- But we do not need to ask the existence of $\mathrm{Fix}\,\mathcal{F}$ for all functor.
- We could ask the existence of $\mathrm{Fix}\,\mathcal{F}$ for $\mathcal{F}$ coming from interpretation of types (formulas).
- Thus, the question of interpreting types of fixpoint logics in a category reduces to finding a class of functors that have fixpoints and are closed under the type constructors of the given logic.

Digression

The traditional way to solve recursive equation in CPO-categories due to Scott [1]:

Let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ be a CPO-functor. The idea is to transform $\mathcal{F}$ to a CPO-endofunctor $\mathcal{F}_E : \mathcal{C}_E \to \mathcal{C}_E$ on category $\mathcal{C}_E$ of embedding in $\mathcal{C}$ such that a fixpoint of $\mathcal{F}_E$ is a fixpoint of $\mathcal{F}$.

Calculation of a fixpoint of $\mathcal{F}_E$:

- $\mathcal{C}_E$ is required to have initial object 0.
- $\mathcal{C}_E$ is required to have colimit of ω-chains.
- Limit/Colimit Coincidence Theorem.
- Relating colimit to the initial algebra.
- Lambek's lemma.

[1] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, Toposes, Algebraic Geometry and Logic, volume 274 of Lecture Notes in Mathematics, 1972.

Categorical Semantics of Fixpoint Logics

General recipe for $\mu\mathbf{L}$

Let L be an arbitrary logic and $\mu\mathbf{L}$ be the finitary system obtained by extending L with fixpoints of types.

A categorical model of μL is a pair $(\mathcal{C}, \overrightarrow{\mathcal{C}})$ where

1. $\mathcal{C}$ is categorical model of L
2. $\overrightarrow{\mathcal{C}} = (\mathcal{C})_n$ is a class of n-ary functor $\mathcal{C}^n \to \mathcal{C}$, and $\mathcal{C}_0 = \mathcal{C}$
3. Type constructors of L induces functor in $\mathcal{C}_n$ for some n
4. All $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ in $\mathcal{C}_1$ have initial algebra and final coalgebra. So, if $\mathcal{F} : \mathcal{C}_{n+1}$, there are fixpoint functors $\mu\mathcal{F}$ and $\nu\mathcal{F}$. We ask $\mu\mathcal{F} \in \mathcal{C}_n$ and $\nu\mathcal{F} \in \mathcal{C}_n$.
5. $\overrightarrow{\mathcal{C}}$ is closed under composition, identity, and projections.
6. We Sometimes need to assume that functors in $\overrightarrow{\mathcal{C}}$ are equipped with a structure, mainly, to deal with context in the interpretation of inferences rules.

The case of intuitionistic logic

STRONG FUNCTORS AND INTERLEAVING FIXPOINTS IN GAME SEMANTICS *

PIERRE CLAIRAMBAULT[1]

Abstract. We describe a sequent calculus μLJ with primitives for inductive and coinductive datatypes and equip it with reduction rules allowing a sound translation of Gödel's system T. We introduce the notion of a μ-*closed category*, relying on a uniform interpretation of open μLJ formulas as strong functors. We show that any μ-closed category is a sound model for μLJ. We then turn to the construction of a concrete μ-closed category based on Hyland-Ong game semantics. The model relies on three main ingredients: the construction of a general class of strong functors called *open functors* acting on the category of games and strategies, the solution of recursive arena equations by exploiting *cycles* in arenas, and the adaptation of the winning conditions of parity games to build initial algebras and terminal coalgebras for many open functors. We also prove a weak completeness result for this model, yielding a normalisation proof for μLJ.

The case of linear logic

Relational model

We want to see if Rel is a model of μLL.

1. Rel is categorical model of LL
2. $\overrightarrow{\mathsf{Rel}} = (\mathsf{Rel})_n$ is a class of n-ary functor $\mathsf{Rel}^n \to \mathsf{Rel}$, and $\mathsf{Rel}_0 = \mathsf{Rel}$
3. Type constructors of LL induces functor in Rel_n for some n
4. All $\mathcal{F} : \mathsf{Rel} \to \mathsf{Rel}$ in Rel_1 have initial algebra and final coalgebra. So, if $\mathcal{F} : \mathsf{Rel}_{n+1}$, there are fixpoint functors $\mu\mathcal{F}$ and $\nu\mathcal{F}$. We ask $\mu\mathcal{F} \in \mathsf{Rel}_n$ and $\nu\mathcal{F} \in \mathsf{Rel}_n$.
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5. $\overrightarrow{\mathsf{Rel}}$ is closed under composition, identity, and projections.
6. $\overrightarrow{\mathsf{Rel}}$ are strong functors.

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4. all $\mathcal{F} : \mathsf{Rel} \to \mathsf{Rel}$ in Rel_1 have initial algebra and final coalgebra. So, if $\mathcal{F} : \mathsf{Rel}_{n+1}$, there are fixpoint functors $\mu\mathcal{F}$ and $\nu\mathcal{F}$. We ask $\mu\mathcal{F} \in \mathsf{Rel}_n$ and $\nu\mathcal{F} \in \mathsf{Rel}_n$.
5. $\overrightarrow{\mathsf{Rel}}$ is closed under composition, identity, and projections.
6. $\overrightarrow{\mathsf{Rel}}$ are strong functors.

Let's define $\overrightarrow{\mathsf{Rel}}$ and prove item (4).

$\overrightarrow{\mathsf{Rel}}$

A functor $\mathcal{F} : \mathsf{Rel} \to \mathsf{Rel}$ is *continuous* if

- for all directed set of sets D, one has $\mathcal{F}(\bigcup D) = \bigcup_{X \in D} \mathcal{F}(X)$,
- for $X, Y \in \mathsf{Rel}$ and all directed set $D \subseteq \mathsf{Rel}(X, Y)$, one has $\mathcal{F}(\bigcup D) = \bigcup\{\mathcal{F}(s) \mid s \in D\}$

We take Rel_n as the collection of n-ary continuous functors.

Let $\mathcal{F}$ be a continuous functor. We define σF as $\bigcup_{n \in \mathbb{N}} F^n(\varnothing)$. By continuity, $(\sigma\mathcal{F}, \mathrm{id})$ is a $\mathcal{F}$-coalgebra.

Theorem

The coalgebra $(\sigma F, \mathrm{id})$ is final coalgebra.

Proof

Let (X, f) be a coalgebra. We define a sequence $g_n \in \mathrm{Rel}(X, \sigma F)$ as follows where $g_0 = \varnothing$:

$$g_{n+1} : X \xrightarrow{f} FX \xrightarrow{F(g_n)} F(\sigma F) = \sigma F$$

By induction, we have $g_n \subseteq g_{n+1}$ for all n using continuity. Let $g = \bigcup_{n \in \mathbb{N}} g_n$. We can then verify that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & FX \arrow[d, "Fg"] \\ \sigma\mathcal{F} \arrow[r] & \mathcal{F}(\sigma\mathcal{F}) \end{tikzcd}$$

We can end the proof by showing that g is unique.

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We can end the proof by showing that g is unique.

Let $g' \in \mathrm{Rel}(X, \sigma F)$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g'"'] & FX \arrow[d, "Fg'"] \\ \sigma\mathcal{F} \arrow[r] & \mathcal{F}(\sigma\mathcal{F}) \end{tikzcd}$$

We define a new sequence $i_n \in \mathrm{Rel}(\sigma F, \sigma F)$ by taking $i_0 = \varnothing$, and $i_{n+1} = \mathcal{F}(i_n)$.
By induction, we can show that $i_n \circ g = i_n \circ g'$.

$$g' = (\bigcup_{n \in \mathbb{N}} i_n) \circ g' = \bigcup_{n \in \mathbb{N}} (i_n \circ g') = \bigcup_{n \in \mathbb{N}} (i_n \circ g) = g$$

We can also check that $(\sigma F, \mathrm{id})$ is also an initial algebra. So, we look for a model to distinguish them.

Let $g' \in \mathrm{Rel}(X, \sigma F)$ such that

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Another example of a model of μLL: non-uniform totality spaces (Nuts)

Nuts

Given $\mathcal{T} \subseteq \mathcal{P}(X)$ we set

$$\mathcal{T}^{\perp} = \{u' \subseteq X \mid \forall u \in \mathcal{T} \ u \cap u' \neq \varnothing\}$$

$\mathcal{T}$ is a *totality candidate* for E if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

Fact

- $\mathcal{T}$ is a totality candidate on E iff $\mathcal{T} \subseteq \mathcal{P}(X)$ and $\mathcal{T} = \uparrow\mathcal{T}$.
- $\mathrm{Tot}(X)$ (The set of all totality candidates on X), ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections).

Nuts

A Nuts is a pair $X = (|X|, \mathcal{T}(X))$ where

- $|X|$ is a set
- $\mathcal{T}(X)$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$.

$t \in \mathrm{Nuts}(X, Y)$ if $t \in \mathrm{Rel}(|X|, |Y|)$ and

$$\forall u \in \mathcal{T}(X) \quad t \cdot u \in \mathcal{T}(Y)$$

Nuts

Let $\mathcal{F}$ be a continuous functor, and $\overline{\mathcal{F}}$ be a lifting of $\mathcal{F}$.

$$\begin{tikzcd}
\mathsf{Nuts} \arrow[r, "\overline{\mathcal{F}}"] \arrow[d] & \mathsf{Nuts} \arrow[d] \\
\mathsf{Rel} \arrow[r, "\mathcal{F}"'] & \mathsf{Rel}
\end{tikzcd}$$

Does $\nu\overline{F}$ exist?

Let $\mathcal{F}$ be a continuous functor, and $\overline{\mathcal{F}}$ be a lifting of $\mathcal{F}$.

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Let $X = (|X|, \mathcal{T}(X))$ be an object in Nuts.

Since $\overline{F}$ is a lifting of $\mathcal{F}$, there is an operation $\theta_{\mathcal{F}}$: $\mathrm{Tot}(|X|) \to \mathrm{Tot}(\mathcal{F}(|X|))$ such that

$$\overline{\mathcal{F}}(|X|, \mathcal{T}(X)) = (F(|X|), \theta_{\mathcal{F}}(\mathcal{T}(X)))$$

Let $|X|$ be $\sigma\mathcal{F}$. Then $\theta_{\mathcal{F}} : \mathrm{Tot}(\sigma\mathcal{F}) \to \mathrm{Tot}(\sigma\mathcal{F})$.

Let $\mathcal{F}$ be a continuous functor, and $\overline{\mathcal{F}}$ be a lifting of $\mathcal{F}$.

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Let $|X|$ be $\sigma\mathcal{F}$. Then $\theta_{\mathcal{F}} : \mathrm{Tot}(\sigma\mathcal{F}) \to \mathrm{Tot}(\sigma\mathcal{F})$.

Let $S_1, S_2 \in \mathrm{Tot}(\sigma\mathcal{F})$ with $S_1 \subseteq S_2$. Then $\mathrm{id} \in \mathrm{Nuts}((\sigma\mathcal{F}, S_1), (\sigma\mathcal{F}, S_2))$.

$$\begin{aligned} \overline{\mathcal{F}}(id) &= \mathrm{Nuts}(\overline{\mathcal{F}}(\sigma\mathcal{F}, S_1), \overline{\mathcal{F}}(\sigma\mathcal{F}, S_2)) \\ &= \mathrm{Nuts}((\mathcal{F}(\sigma\mathcal{F}), \theta_{\mathcal{F}} S_1), (\mathcal{F}(\sigma\mathcal{F}), \theta_{\mathcal{F}} S_2)) \\ &= \mathrm{Nuts}((\sigma\mathcal{F}, \theta_{\mathcal{F}} S_1), (\sigma\mathcal{F}, \theta_{\mathcal{F}} S_2)) = \mathrm{id} \\ &\Rightarrow \theta_{\mathcal{F}} S_1 \subseteq \theta_{\mathcal{F}} S_2 \end{aligned}$$

$$
\begin{array}{ccc}
\mathsf{Nuts} & \xrightarrow{\overline{\mathcal{F}}} & \mathsf{Nuts} \\
\downarrow & & \downarrow \\
\mathsf{Rel} & \xrightarrow[\mathcal{F}]{} & \mathsf{Rel}
\end{array}
$$

$$\theta_{\mathcal{F}} : \mathsf{Tot}(\sigma\mathcal{F}) \to \mathsf{Tot}(\sigma\mathcal{F})$$

The function $\theta_{\mathcal{F}}$ is monotone and $\mathsf{Tot}(\sigma\mathcal{F})$ is complete lattice. Hence a candidate for $\nu\overline{\mathcal{F}}$ is

$$(\sigma\mathcal{F}, \nu\theta_{\mathcal{F}})$$

Definition of $\nu\theta_{\mathcal{F}}$

Let $(\mathcal{T}_\alpha)_{\alpha\in\mathcal{O}}$, where $\mathcal{O}$ is the class of all ordinals, be defined as

- $\mathcal{T}_0 = \mathcal{P}(\sigma\mathcal{F})$
- $\mathcal{T}_{\alpha+1} = \theta_{\mathcal{F}}(\theta_\alpha)$
- $\mathcal{T}_\gamma = \bigwedge_{\alpha<\gamma} \mathcal{T}_\alpha$

The sequence is increasing, and by a cardinality argument, there is γ such that $\mathcal{T}_{\gamma+1} = \mathcal{T}_\gamma$, and we take $\nu\theta_{\mathcal{F}}$ as $\mathcal{T}_\gamma$.

By construction, $((\sigma\mathcal{F}, \mathcal{T}_\gamma), \mathrm{id})$ is a $\overline{\mathcal{F}}$ coalgebra.

Theorem

$(\sigma\mathcal{F}, \mathcal{T}_\gamma)$ is final coalgebra of $\overline{\mathcal{F}}$.

Proof

Let (Y, g) be a $\overline{\mathcal{F}}$-coalgebra. $(|Y|, g)$ is $\mathcal{F}$-coalgebra. Since $(\sigma\mathcal{F}, \mathrm{id})$ is final $\mathcal{F}$-coalgebra, there is a unique arrow $h : |Y| \to \sigma\mathcal{F}$ such that

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & FY \arrow[d, "\mathcal{F}h"] \\ \sigma\mathcal{F} \arrow[r, equal] & \mathcal{F}(\sigma\mathcal{F}) \end{tikzcd}$$

We prove that actually $h \in \mathrm{Nuts}((Y, g), ((\sigma\mathcal{F}, \mathcal{T}_\gamma), \mathrm{id}))$. One can show by induction on α that:

$$\forall v \in \mathcal{T}(Y), h.v \in \mathcal{T}_\alpha$$

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Example (1)

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\begin{tikzcd}
\mathsf{Nuts} \arrow[r, "\overline{1 \oplus X}"] \arrow[d] & \mathsf{Nuts} \arrow[d] \\
\mathsf{Rel} \arrow[r, "1 \oplus X"'] & \mathsf{Rel}
\end{tikzcd}
$$

We want to compute $\mu(\overline{1 \oplus X})$.

We have seen that $\sigma(1 \oplus X)$ is isomorphic to $\mathbb{N}$.

$\theta_{1 \oplus X}$: $\mathsf{Tot}(\mathbb{N}) \to \mathsf{Tot}(\mathbb{N})$ which send $\mathcal{T}$ to

$$\{u \subseteq \mathbb{N} \mid 0 \in u \text{ or } \{n \mid n+1 \in u\} \in \mathcal{T}\}$$

Therefore $\nu\theta_{\mathcal{F}}$ is $\{u \subseteq \mathbb{N} \mid u \neq \varnothing\}$

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The elements of $\sigma(\mathbb{N} \oplus (X \otimes X))$ can be describe as

$$\alpha, \beta, \cdots := \langle n \rangle \mid \langle \alpha, \beta \rangle$$

A similar computation shows that $\nu(\theta_{\mathbb{N}\oplus(X\otimes X)})$ is

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What is $\sigma(\mathbb{N} \otimes X)$?

The opreation $\theta_{\mathbb{N}\otimes X}$ maps $\mathcal{T}$ to

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Hence $\nu\theta_{\mathbb{N}\otimes X}$ is $\{\varnothing\}$.

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\end{tikzcd}$$

We want to compute $\nu(1 \,\&\, (\mathbb{N} \otimes X))$.

What is $\sigma(1 \,\&\, (\mathbb{N} \otimes X))$?
This functor send X to

$$\{(1, \star)\} \cup \{2\} \times \mathbb{N} \times X$$

Hence $\sigma(1 \,\&\, (\mathbb{N} \otimes X))$ is $\mathbb{N}^{<\omega}$.

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The operation $\theta_{1\&(\mathbb{N}\otimes X)}$ send $\mathcal{T}$ to

$$\{v \subseteq \mathbb{N}^{<\omega} \mid () \in v \text{ and } \exists n \in \mathbb{N}, u \in \mathcal{T}, \{n\} \times u \subseteq v\}$$

For example

$$\theta_{1\&(\mathbb{N}\otimes X)}(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid () \in u\}$$

$$\theta^3_{1\&(\mathbb{N}\otimes X)}(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid \exists n_1, n_2, () \in u, (n_1) \in u, (n_1, n_2) \in u\}$$

Example (4)

$$\begin{array}{ccc} \mathsf{Nuts} & \xrightarrow{\overline{1\&(\mathbb{N}\otimes X)}} & \mathsf{Nuts} \\ \downarrow & & \downarrow \\ \mathsf{Rel} & \xrightarrow[1\&(\mathbb{N}\otimes X)]{} & \mathsf{Rel} \end{array}$$

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Hence, $\nu(\theta_{1\&(\mathbb{N}\otimes X)})$ is

$$\{u \subseteq \mathbb{N}^{<\omega} \mid \exists f \in \mathbb{N}^{\omega}, \forall k < \omega, (f(1), \cdots, f(k)) \in u\}$$

So, a total element must contain (at least) an infinite stream of integer.

We can generalize what we did for Nuts.

$$\begin{tikzcd}
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We can generalize what we did for Nuts [2].

$$\begin{tikzcd}
\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[r, "\overline{\mathcal{F}}"] \arrow[d] & \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[d] \\
\mathcal{L} \arrow[r, "\mathcal{F}"'] & \mathcal{L}
\end{tikzcd}$$

Where $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ is a focused orthogonality category [3].

Question

How about non-focused models such as finiteness spaces?

[2] M. Fiore, Z. Galal, F. Jafarrahmani, Fixpoint constructions in focused orthogonality models of linear logic.

[3] M. Hyland and A. Schalk, Glueing and orthogonality for models of linear logic.

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Categorical semantics of non-wellfounded proofs

Interpretation proofs via approximation

We need to interpret non-wellfounded proofs.

$$[\![\pi]\!] = \cup_{\rho \in \mathsf{fin}(\pi)} [\![\rho]\!]$$

So, we need to equip our categorical model $(\mathcal{C}, \overrightarrow{\mathcal{C}})$ with enough structure to capture $\bigcup_{\rho \in \mathsf{fin}(\pi)}$.

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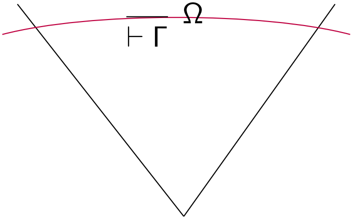

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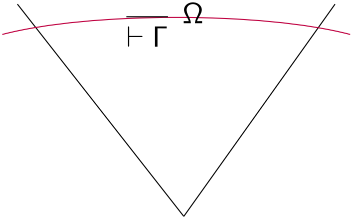

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General recipe for μ^ωL

Let L be an arbitrary logic, and μ^ωL be an infinitary system.

A categorical model of μ^ωL is a μL model $(\mathcal{C}, \overrightarrow{\mathcal{C}})$ such that $\mathcal{C}$ is a CPO-enriched category and greatest lower bounds of any non-empty set exist in each hom-set.

A drawback

Although this definition includes many examples of model of linear logic such as Rel and Coh, this does not apply necessarily to the orthogonality construction such as Nuts.

We can nevertheless prove that if $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ is built on top of a μ^ωLL model, then $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ is also a model of μ^ωLL [4].

- Semantical understanding of (syntactic) validity condition: the interpretation of valid proofs are morphism in the orthogonality category, and then are total.

[4] T. Ehrhard, F. Jafarrahmani, A. Saurin, On the denotation of circular and non-wellfounded proofs in linear logic with fixed points

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On relationship between circular proofs and inductive proofs

Any finitary linear logic proof can be transformed into a circular proof. This translation is proof-relevant:

- the semantics is preserved via this transformation.

There is a transformation in the reverse direction for a fragment of circular proofs [5].

- This is also proof-relevant [6].
- This gives us a direct and canonic interpretation of strongly valid circular proofs in a fragment of μLL model.

[5] David Baelde, Amina Doumane, Alexis Saurin: Least and Greatest Fixed Points in Ludics.

[6] T. Ehrhard, F. Jafarrahmani, A. Saurin, On the denotation of circular and non-wellfounded proofs in linear logic with fixed points

Any finitary linear logic proof can be transformed into a circular proof. This translation is proof-relevant:

- the semantics is preserved via this transformation.

There is a transformation in the reverse direction for a fragment of circular proofs [5].

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Some Questions

Semantical Brotherston-Simpson's conjecture

Let π be a circular $\mu\mathsf{LL}_\infty$ proof of $\vdash \Gamma$. There exists a $\mu\mathsf{LL}$ (finite) proof π' of $\vdash \Gamma$ such that $[\![\pi]\!] = [\![\pi']\!]$.

- A. Das and G. Curzi gave an answer to this for the case of function on natural numbers in computability (or realizability) semantics [7].

[7] Gianluca Curzi and Anupam Das, Computational expressivity of (circular) proofs with fixed points

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Interpretation of all circular proofs in a μLL model

Let π be a strongly valid proof. Then $[\![\pi^{\mathsf{fin}}]\!] = [\![\pi]\!]$ where the interpretations of proofs are in any μLL model.

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Functorial view on circular proofs

Considering proof systems as generalized CFG [10], and relating the (syntactic) validity condition to the question of unicity of gCFG models [11].

- Our result is currently for the Kleene algebra system given by D. Kuperberg, L. Pinault, and D. Pous [12].

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