Basics of category theory

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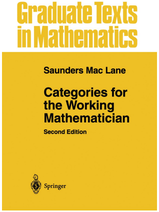
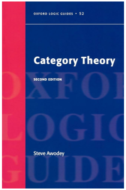
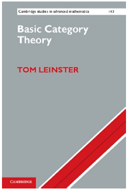
Overview

References







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nLab

category theory

See nLab for more references

Slides are here

Introduction on category theory

Presentation of properties of mathematical systems via diagrams.

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

$$h = g \circ f$$

A monoid is a set M equipped with an element $e \in M$ and a binary operation $M \times M \to M$, denoted by $\bullet$, such that

- $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in M$, and
- $e \bullet a = a \bullet e = a$ for all $a \in M$.

The element e can be consider as $\eta : \{\star\} \to M$, i.e $\eta(\star) = e$

$$\begin{tikzcd}
(a, b, c) \arrow[r, mapsto] \arrow[d, mapsto] & (a, b \bullet c) \arrow[d, mapsto] \\
(a \bullet b, c) \arrow[r, mapsto] & (a \bullet b) \bullet c = a \bullet (b \bullet c)
\end{tikzcd}$$

$$\begin{tikzcd}
(\star, a) \arrow[r, mapsto] \arrow[d, mapsto] & (\eta(\star), a) \arrow[d, mapsto] \\
a \arrow[r, equal] & e \bullet a
\end{tikzcd}
\qquad
\begin{tikzcd}
(a, \star) \arrow[r, mapsto] \arrow[d, mapsto] & (a, \eta(\star)) \arrow[d, mapsto] \\
a \arrow[r, equal] & a \bullet e
\end{tikzcd}$$

In diagrams, we do not really need to refer to <u>elements</u> explicitly:

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mu\times\mathrm{id}"'] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
\{\star\} \times M \arrow[r, "\eta\times\mathrm{id}"] \arrow[d, "\simeq"] & M \times M \arrow[dl, "\mu"] \\
M &
\end{tikzcd}
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\begin{tikzcd}
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We can do the same for groups by adding the following diagram:

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[d] & M \times M \arrow[r, "\mathrm{id}\times\alpha"] & M \times M \arrow[d, "\mu"] \\
\{\star\} \arrow[rr, "\eta"] & & M
\end{tikzcd}$$

$$e = x \bullet x^{-1}$$

No explicit mention of elements: M can be other mathematical structure.

- For example, instead of M being a set, we can say M is a topological space and arrows are continuous maps. This defines a topological group.
- Different structures can be presented by the same diagram in different categories.

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On definition of category

A *metagraph* consists of

- objects $a, b, c, \cdots$
- arrows $f, g, h, \cdots$
- dom : arrows $\to$ objects
- cod : arrows $\to$ objects

$$a_1 \longrightarrow a_2 \rightrightarrows a_3 \overset{\longrightarrow}{\rightrightarrows} \cdots$$

A *metacategory* is a metagraph such that

1. $\mathrm{id} : \text{objects} \to \text{arrows}$
2. $\circ : \text{arrows} \times \text{arrows} \to \text{arrows}$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & c \end{tikzcd}$$

3. Associativity: $k \circ (g \circ f) = (k \circ g) \circ f$

$$\begin{tikzcd} a \arrow[r, "k\circ(g\circ f)=(k\circ g)\circ f"] \arrow[d, "f"] \arrow[dr] & f \\ b \arrow[r, "g"] \arrow[ur] & c \arrow[u, "k"'] \end{tikzcd}$$

4. Unit law: $\mathrm{id}_b \circ f = f$ and $g \circ \mathrm{id}_b = g$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"] & b \arrow[d, "\mathrm{id}_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"] & c \end{tikzcd}$$

Examples of metacategories

- Sets: Objects are all sets; arrows are all functions with the usual composition.
- Groups: Objects are all groups; arrows are all homomorphisms of groups.
- Topological spaces: Objects are all topological spaces; arrows are all continuous functions.
- Hausdorff spaces with continuous functions.
- All ringed spaces with their morphisms.
- $\dots$

A *directed graph* is a set Obj of objects, a set Arr of arrows, and two functions:

$$\begin{tikzcd} \mathsf{Arr} \arrow[r, bend left, "\mathsf{dom}"] \arrow[r, bend right, "\mathsf{cod}"'] & \mathsf{Obj} \end{tikzcd}$$

The set of composable pairs of arrows:

$$\mathsf{Arr} \times_O \mathsf{Arr} := \{(g, f) \mid f, g \in A \quad \text{and} \quad \mathsf{dom}\, g = \mathsf{cod}\, f\}$$

A (small) *category* is a directed graph such that

- $\mathrm{id} : \mathrm{Obj} \to \mathrm{Arr}$
- $\circ : \mathrm{Arr} \times_O \mathrm{Arr} \to \mathrm{Arr}$
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$$\begin{tikzcd}
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Exercise: Is this definition just a monoid for the product $\times_O$.

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Notation

Let $\mathcal{C}$ be a category.

- $a : C$ for $a \in O$.
- $\hom(b, c) = \{f \mid f \in A, \quad \operatorname{dom} f = b, \quad \operatorname{cod} f = c\}$
- $\mathcal{C}(b, c) = \hom(b, c) = \hom_{\mathcal{C}}(b, c)$

Examples of categories

- $\mathbf{0}$: the empty category
- $\mathbf{1}$: $\begin{tikzcd} \star \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$
- $\mathbf{2}$: $\begin{tikzcd} a \arrow[loop left, "\mathrm{id}"] \arrow[r] & b \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$
- $\mathbf{3}$:
$$\begin{tikzcd} & b \arrow[loop above] \arrow[dr] & \\ a \arrow[loop left] \arrow[ur] \arrow[rr] & & c \arrow[loop right] \end{tikzcd}$$
- Discrete categoris: let E be a set $\{x, y, z, \cdots\}$
$$\begin{tikzcd} x \arrow[loop right, "\mathrm{id}"] & y \arrow[loop right, "\mathrm{id}"] & z \arrow[loop right, "\mathrm{id}"] & \cdots \end{tikzcd}$$
- Monoids:
$$\begin{tikzcd} \star \arrow[loop left, "\ddots"] \arrow[loop above left, "a"] \arrow[loop above right, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

- Matr_K: natural numbers as set of objects, and $f : m \to n$ is a $m \times n$ matrix with entries in K.
- Ens_V: elements of the set V as a set of objects, and all functions as arrows.
- Preorders: a category P in which there is at most one arrow between any pair of objects.
- ω: $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$
- Simplicial category: finite ordinals as set of objects, and order-preserving functions as set of arrows.
- Set_ω: category of all finite ordinals and all functions.

Large Categories

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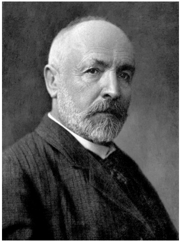
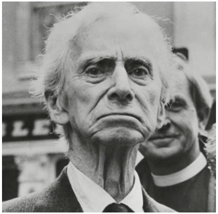
Is *Sets* really a category?  

Large Categories

Let U be a *big enough* set, the "universe". A set x is *small* if $x \in U$.

- Set: all small sets, and all functions between them.
- Pointed sets $\mathrm{Set}_\star$: all small sets with one distinguished element $\star$, and all $\star$-preserving functions.
- Mon: small monoids, and monoid morphisms.
- Grp: (small) groups, and group morphisms.
- Ab: (small) abelian groups, with morphisms of such.
- Rng: (small) rings, and ring morphisms.
- R-Mod: left modules over the ring R, and linear maps.
- Top: (small) topological spaces, and continuous maps.
- Cat: small categories, and functors.
  - A category is small, if set of its objects and the set of its arrows are both small.

Foundations

- Set is not a small category: $U \in U$, and contradiction to the axiom of regularity [1].
- ZFC + the existence of U suffices for the usual purposes of category theory.
  - One needs to define the notion of universe U in such a way that its properties ensure that any standard operation of set theory applied to elements of U will produce elements of U.
- But how to have a categorical representation of certain metacategories, such as the metacategory of all sets?
  1. Change the definition of category: a class of objects, a set $\hom(b, c)$ of morphisms for each pair (b, c) plus $\cdots$
  2. We still have set-like objects that are not object of Set. So, we need a set-free foundation for category theory.


Michael Shulman

category of all sets

See here for more discussion

[1] this axiom asserts that there are no infinite chains $\cdots \in x_n \in x_{n-1} \in \cdots \in x_0$

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Functors

A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a morphism of categories:

- F assigns an object FX of $\mathcal{D}$ to each object X of $\mathcal{C}$.
- It assigns a function $F_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ to any pair (X, Y).
- $F(g \circ f) = F(g) \circ F(f)$
  $$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \quad = \quad FX \xrightarrow{\quad F(g \circ f) \quad} FZ$$
- $F(\mathrm{id}_A) = \mathrm{id}_{FA}$
  $$FX \xrightarrow{F(\mathrm{id}_A)} FX \quad = \quad FX \xrightarrow{\mathrm{id}_{FA}} FX$$

Examples

1. Power set functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$:
   - $X \mapsto \mathcal{P}(X)$
   - $(f : X \to Y) \mapsto \mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ where it sends $S \subseteq X$ to $\{a \mid \exists b \in S, f(b) = a\} \subseteq Y$
2. $\mathrm{GL}_n : \mathrm{CRng} \to \mathit{Grp}$ where $n \in \mathbb{N}$:
   - $\mathrm{GL_n}(K)$ is the group of all non-singular $n \times n$ matrices with entries in K.
3. Forgetful functor:
   - $U : \mathrm{Grp} \to \mathrm{Set}$, $U : \mathrm{Rng} \to \mathrm{Ab}$, etc.
4. The identity functor $\mathcal{I}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$
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7. $\mathrm{List} : \mathrm{Set} \to \mathrm{Set}$:
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8. Exercise: Let FG be the center of G, i.e. $\{z \in G \mid \forall g \in G, z.g = g.z\}$. Does F define a functor $\mathrm{Grp} \to \mathrm{Ab}$?

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism*, if it is a bijection, both on objects and arrows.

- Equivalently, there is a functor $F' : \mathcal{D} \to \mathcal{C}$ such that both compositions $F \circ F'$ and $F' \circ F$ are identity functors.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if the function $F_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ is surjective for each pair (X, Y).

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if the function $F_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ is injective for each pair (X, Y).

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- The forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ is faithful but not full and not a bijection on objects.

A *concrete category* is a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and U is a faithful functor $\mathcal{C} \to \mathsf{Set}$.

- each object $X : C$ comes equipped with an "underlying" set
- each arrow $f : X \to Y$ is an actual function $UX \to UY$, and
- composition of arrows is composition of functions.

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- composition of arrows is composition of functions.

Examples

1. Category CL complete lattices [2] and sup-preserving maps:
   - Forgetful functor $U : \mathsf{CL} \to \mathsf{Set}$
2. Category Cat:
   - $\mathcal{C} \mapsto \mathrm{Obj}(C)$, and functors to functions.
3. Category Rel of sets and relations:
   - Power set functor $\mathcal{P} : \mathsf{Rel} \to \mathsf{Set}$
4. Category Ban of Banach space and <u>linear contraction</u> [3]:
   - $U : V \mapsto \{v \mid \|v\| \leq 1\}$
   - U is representable by $\mathbb{R}$, i.e. $\hom(\mathbb{R}, V) \cong U(V)$

[2] A poset which has all small joins and meets

[3] A a linear map f s.t $\|f\| \leq 1$

Natural Transformations (1)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\theta : F \to G$ is:

- a collection of morphisms $\theta_X : FX \to GX$ where X is an object of $\mathcal{C}$ such that
- θ_X is *natural* in X, i.e. the following square commutes for every arrow $f : X \to Y$:

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"] & GY \end{tikzcd}$$

Translating the picture F to the picture G:

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & Y \arrow[dl, "g"] \\ Z & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} FX \arrow[rr, "\theta_X"] \arrow[dr, "Ff"] \arrow[dd, "Fh"'] & & GX \arrow[dr, "Gf"] \arrow[dd] & \\ & FY \arrow[rr, "\theta_Y"] \arrow[dl, "Fg"] & & GY \arrow[dl, "Gg"] \\ FZ \arrow[rr, "\theta_Z"] & & GZ & \end{tikzcd}$$

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& FY \arrow[rr, "\theta_Y"] \arrow[dl, "Fg"] & & GY \arrow[dl, "Gg"] \\
FZ \arrow[rr, "\theta_Z"] & & GZ &
\end{tikzcd}$$

Natural Transformations (2)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\theta : F \to G$ is:

- a function $\theta_X : FX \to GX$ for each object X such that
- θ_X is *natural* in X, i.e. the following square commutes for every arrow $f : X \to Y$:

$$\begin{array}{ccc} FX & \xrightarrow{\theta_X} & GX \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\theta_Y} & GY \end{array}$$

Definition

A natural transformation $\theta : F \to G$ is *natural isomorphism*, denoted by $\theta : F \cong G$, if θ_X is an isomorphism in $\mathcal{D}$ for any $X : C$.

Examples (1)

The determinant is a natural transformation.
$\mathrm{GL}_n : \mathrm{CRng} \to \mathrm{Grp}$ where $n \in \mathbb{N}$, and $\_^\star : \mathrm{CRng} \to \mathrm{Grp}$:

- $\mathrm{GL_n}(K)$ is the group of all non-singular $n \times n$ matrices with entries in K.
- $K^\star$ is the group of invertible elements of K.
- $\det : \mathrm{GL}_n \to \_^\star$ where $\det_K(M)$ is the determinant of M. The determinant is defined by the same formula for all rings K:

$$\begin{tikzcd}
\mathrm{GL_n}(K) \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL_n}(K)(f)"'] & K^\star \arrow[d, "f^\star"] \\
\mathrm{GL_n}(K') \arrow[r, "\det_{K'}"'] & K'^\star
\end{tikzcd}$$

Examples (2)

Let G and G' be two groups (category with one object) and $f, g : G \to G'$ two functor (groups homomorphisms) such that f and g are conjugate:

- there is $x \in G'$ s.t $ga = x(fa)x^{-1}$ for all $a \in G$.
- This defines a natural transformation $\theta : f \to g$ by taking $\theta_\star = x$:

$$\begin{tikzcd} f\star \arrow[r, "\theta_\star"] \arrow[d, "fa"] & g\star \arrow[d, "ga"] \\ f\star \arrow[r, "\theta_\star"] & g\star \end{tikzcd}$$

Examples (3)

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We say $\mathcal{C}$ is equivalence to $\mathcal{D}$, denoted by $\mathcal{C} \cong \mathcal{D}$ if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathcal{I}_{\mathcal{D}}$ and $G \circ F \cong \mathcal{I}_{\mathcal{C}}$.

Exercise: let FinVec_F be the category of all finite-dimensional vector spaces and linear maps.

- Prove $\mathrm{Matr}_F \cong \mathrm{FinVec}_F$.

End of the first part & Cheat sheet

A category $\mathcal{C}$ is given by

- a class of objects
- a set of $\hom(X, Y)$ of morphisms,
- plus composition, identity, associativity of composition, and neutral of identity.

A functor $F : \mathcal{C} \to \mathcal{D}$ is given by

- an object FX of $\mathcal{D}$ for each $X : \mathcal{C}$,
- a function $F_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ for each pair of objects (X, Y) of $\mathcal{C}$
- plus preservation of composition and identity.

A natural transformation θ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\theta_X : FX \to GX$ of the category $\mathcal{D}$ indexed by $X : \mathcal{C}$ such that θ_X is natural in X for all $X : \mathcal{C}$.

Overview

Some definitions needed for the second part of the lecture

Product of two objects

Let $\mathcal{C}$ be a category and $X, Y : \mathcal{C}$ be two objects. The *product* of X and Y, denoted by $X \times Y$, is an object $\mathcal{C}$ with two morphisms pr_l and pr_r:

$$X \xleftarrow{\mathrm{pr}_l} X \times Y \xrightarrow{\mathrm{pr}_r} Y$$

such that for any two morphsims $f : Z \to X$ and $g : Z \to Y$, there is a unique morphism $\langle f, g\rangle : Z \to X \times Y$ such that

$$\begin{tikzcd}
X & X \times Y \arrow[l, "\mathrm{pr}_l"'] \arrow[r, "\mathrm{pr}_r"] & Y \\
& Z \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g\rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Examples

Many examples that we have seen have the product of two objects:

1. Set: cartesian product of sets
   $(X \times Y = \{(a, b) \mid a \in X, b \in Y\})$
2. FinVec_F: the cartesian product of two vector spaces V and W such that
   - $(v, w) + (v', w') := (v + v', w + w')$, and
     $k.(v, w) = (k.v, k.w)$
3. Grp, Ab: direct product
   - The underlying set of $G \times G'$ is cartesian product, and
     $(g_1, g_1').(g_2, g_2') := (g_1.g_2, g_1'.g_2')$
4. CRng: cartesian product plus componentwise operations.

Arbitrary product

Let $\mathcal{C}$ be a category and $(X_i)_{i\in I}$ be a family of objects. The product of the family, denoted by $\prod_{i\in I} X_i$, is an is an object $\mathcal{C}$ equipped with morphisms $(\mathrm{pr}_i)_{i\in I}$:

$$\prod_{i\in I} X_i \xrightarrow{f_i} X_i$$

such that for any family $(f_i)_{i\in I}$ of morphisms

$$Z \xrightarrow{\mathrm{pr}_i} X_i$$

there is a unique morphism $g : Z \to \prod_{i\in I} X_i$ such that

$$\begin{tikzcd}
X_i & \prod_{i\in I} X_i \arrow[l, "\mathrm{pr}_i"'] \\
 & Z \arrow[ul, "f_i"] \arrow[u, dashed, "g"]
\end{tikzcd}$$

Terminal object

The *terminal* object of a category $\mathcal{C}$, denoted by $\top$, is just the empty product, i.e. when I is the empty set.

- In other words, $\top$ is the terminal object of $\mathcal{C}$, if for all object $X : \mathcal{C}$, there is a unique one morphism $X \to \top$ in $\mathcal{C}$.

Example

1. Set:singleton set
2. Rel: empty set is terminal
3. Grp: trivial group
4. Cat: $\mathbf{1}$

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Example

1. Set:singleton set
2. Rel: empty set is terminal
3. Grp: trivial group
4. Cat: $\mathbf{1}$

Coprodcut of two objects

Let $\mathcal{C}$ be a category and $X, Y : \mathcal{C}$ be two objects. The coproduct of X and Y, denoted by $X + Y$, is an object $\mathcal{C}$ with two morphism inj_l and inj_r:

$$\begin{tikzcd} X \arrow[r, "\mathsf{inj}_l"] & X + Y & Y \arrow[l, "\mathsf{inj}_r"'] \end{tikzcd}$$

such that for any two morphsims $f : X \to Z$ and $g : Y \to Z$, there is a unique morphism $[f, g] : X + Y \to Z$ such that

$$\begin{tikzcd} X \arrow[r, "\mathsf{inj}_l"] \arrow[dr, "f"'] & X \oplus Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\mathsf{inj}_r"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Examples

Many examples that we have seen have the coproduct of two objects:

1. Set: disjoint union of sets ($X \oplus Y := (X \times 1) \cup (Y \times 2)$)
2. Ab: direct sum of abelian groups
3. Grp: free product of groups (equivalence classes of sequence $g_1 h_1 \cdots g_k h_k$ for $g_i \in G, h_i \in H$)
   - Exercise: Show that the direct sum of groups does not define a coproduct in the category Grp.
4. FinVec$_F$: direct sum of vector spaces
5. CRng: tensor product of rings

Arbitrary coproduct

Let $\mathcal{C}$ be a category and $(X_i)_{i \in I}$ be a family of objects. The ,coproduct of the family, denoted by $\coprod_{i \in I} X_i$, is an is an object $\mathcal{C}$ equipped with morphisms $(\mathrm{inj}_i)_{i \in I}$:

$$X_i \xrightarrow{\mathrm{inj}_i} \coprod_{i \in I} X_i$$

such that for any family $(f_i)_{i \in I}$ of morphisms

$$X_i \xrightarrow{f_i} Z$$

there is a unique morphism $g : \coprod_{i \in I} X_i \to Z$ such that

$$\begin{tikzcd} X_i \arrow[r, "\mathrm{inj}_i"] \arrow[dr, "f_i"'] & \coprod_{i \in I} X_i \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Initial object

The *initial* object of a category $\mathcal{C}$, denoted by I, is just the empty coproduct, i.e. when I is the empty set.

- In other words, I is the initial object of $\mathcal{C}$, if for all object $X : \mathcal{C}$, there is a unique one morphism $I \to X$ in $\mathcal{C}$.

Example

1. Set: empty set
2. Rel: empty set
3. Grp: trivial group
4. Cat: $\mathbf{0}$

Initial object

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Example

1. Set: empty set
2. Rel: empty set
3. Grp: trivial group
4. Cat: $\mathbf{0}$

Opposite category

Let $\mathcal{C}$ be a category.

- $\mathcal{C}^{\mathrm{op}}$ is the category whose objects are the same as $\mathcal{C}$, and $f \in \mathcal{C}^{\mathrm{op}}(X, Y)$ if $f \in \mathcal{C}(Y, X)$.
- A *contravariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Examples

1. $\_^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}$ defines a (covariant) functor:
   - $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, $F : \mathcal{C} \to \mathcal{D} \mapsto F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$
   - $F^{\mathrm{op}}X := FX$. Let $f \in \mathcal{C}^{\mathrm{op}}(X, Y)$, $F^{\mathrm{op}}(f) := F(f)$ where it is in $\mathcal{D}(FY, FX) = \mathcal{D}^{\mathrm{op}}(FX, FY)$
2. Let $X : \mathcal{C}$. $\hom_{\mathcal{C}}(X, \_) : \mathcal{C} \to \mathrm{Set}$ defines a functor which sends $Y : \mathcal{C}$ to the set $\mathcal{C}(X, Y)$.
3. $\hom_{\mathcal{C}}(\_, X) : \mathcal{C} \to \mathrm{Set}$ defines a contravariant functor.

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- A *contravariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Examples

1. $\_^{\mathrm{op}}$: Cat $\to$ Cat defines a (covariant) functor:
   - $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, $F : \mathcal{C} \to \mathcal{D} \mapsto F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$
   - $F^{\mathrm{op}}X := FX$. Let $f \in \mathcal{C}^{\mathrm{op}}(X, Y)$, $F^{\mathrm{op}}(f) := F(f)$ where it is in $\mathcal{D}(FY, FX) = \mathcal{D}^{\mathrm{op}}(FX, FY)$
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